Chapter 1

Preliminaries and auxiliary results

In this chapter we collect some basic tools on the main topics used throughout the thesis. We recall the basic definitions and the most important properties of semigroups theory and measure theory. These recalls are only intended to fix some notations and references and are confined to what will be useful in the sequel. For what concerns the results on semigroups and sectorial operators we refer to [31], [19] while a more deep analysis concerning results of measure theory can be found in [5] and [20].

1.1 Recall on semigroups theory

One of our aims is to prove existence, uniqueness and regularity properties for the solution of the following parabolic second order problem

$$\begin{cases} u_t(t,x) = Au(t,x) & t > 0,\ x \in \Omega \\ u(0,x) = f(x) & x \in \Omega \\ Bu(t,x) = 0 & t > 0,\ x \in \partial\Omega \end{cases}$$

where A is a linear second order operator in divergence form and B is a non-tangential first order differential operator defined on $\partial\Omega$, and the initial datum f is taken in $L^1(\Omega)$. This problem is studied as an abstract Cauchy problem in a suitable Banach space,

$$\begin{cases} u'(t) = Au(t), & t > 0 \\ u(0) = x \end{cases} \tag{1.1}$$

by looking at the semigroup generated by A with a suitable domain. Here X is a complex Banach space with norm $\|\cdot\|_X$, $A : D(A) \subset X \to X$ is a linear operator and $x \in X$. Of course the solution of (1.1) and its properties depend upon the class of operators considered.

In our case the operator A will be sectorial (see Definition 1.2.1 below). This ensures that the solution of (1.1) admits an integral representation with a complex contour integral and the solution map $t \mapsto u(t,x)$ of (1.1) is given by an analytic semigroup (see Definition 1.2.2).

1.2 Sectorial operators

Definition 1.2.1. *Let $A : D(A) \subset X \to X$ be a linear operator. We say that A is sectorial if there exist $\omega \in \mathbf{R}$, $\theta \in]\frac{\pi}{2}, \pi[$, $M > 0$ such that*

$$\rho(A) \supset \Sigma_{\theta,\omega} = \{\lambda \in \mathbf{C}; \lambda \neq \omega, |\arg(\lambda - \omega)| < \theta\} \tag{1.2}$$

$$\|R(\lambda, A)\|_{\mathcal{L}(X)} \leq \frac{M}{|\lambda - \omega|} \qquad \forall\, \lambda \in \Sigma_{\theta,\omega}. \tag{1.3}$$

Here the resolvent set $\rho(A)$ is the set $\{\lambda \in \mathbf{C} : \exists(\lambda - A)^{-1} \in \mathcal{L}(X)\}$ and for $\lambda \in \rho(A)$, $R(\lambda, A)$ denotes the resolvent operator $(\lambda - A)^{-1}$.
A sectorial operator is immediately closed since its resolvent set is not empty, hence its domain $D(A)$, endowed with the graph norm $\|x\|_{D(A)} = \|x\|_X + \|Ax\|_X$, is a Banach space. Conditions (1.2) and (1.3) guarantee that the linear operator e^{tA}, defined for $t \geq 0$ as follows

$$e^{0A} := I, \quad e^{tA} := \frac{1}{2\pi i}\int_{\omega+\gamma_{r,\eta}} e^{t\lambda} R(\lambda, A)\, d\lambda, \qquad t > 0, \tag{1.4}$$

where $r > 0$, $\eta \in (\frac{\pi}{2}, \theta)$, and

$$\gamma_{r,\eta} = \{\lambda \in \mathbf{C}; |\arg\lambda| = \eta, |\lambda| \geq r\} \cup \{\lambda \in \mathbf{C}; |\arg\lambda| \leq \eta, |\lambda| = r\}$$

oriented counterclockwise, is well defined and independent of $r > 0$ and $\eta \in (\frac{\pi}{2}, \theta)$.

Before stating the basic properties of e^{tA}, we recall when a family of operators $(T(t))_{t\geq 0} \subset \mathcal{L}(X)$ is called a semigroup.

Definition 1.2.2. *(Analytic semigroup) A family of operators $(T(t))_{t\geq 0} \subset \mathcal{L}(X)$ is called a semigroup if*

$$T(0) = I \quad \textit{and} \quad T(t+s) = T(t)T(s) \quad t, s \geq 0.$$

It is said to be strongly continuous *if for each $x \in X$ the function $t \mapsto T(t)x$ is continuous in $[0, +\infty[$. Moreover it is called an* analytic semigroup *of angle $\delta \in]0, \pi/2]$ if the function $z \mapsto T(z)$ is analytic in the sector $\Sigma_\delta = \{z \in \mathbf{C} : |\arg z| < \delta\}$ and for every $0 < \delta' < \delta$ and $x \in Y$, being Y a closed subspace of X, it holds that*

$$\lim_{\substack{z \to 0 \\ z \in \Sigma_{\delta'}}} T(z)x = x.$$

Proposition 1.2.3. *Let $A : D(A) \subset X \to X$ be a sectorial operator, and $(e^{tA})_{t\geq 0}$ defined as in (1.4). Then the following properties hold:*

(i) $e^{tA}x \in D(A^k)$ *for each* $t > 0$, $x \in X$, $k \in \mathbf{N}$. *Moreover if* $x \in D(A^k)$ *then*

$$A^k e^{tA} x = e^{tA} A^k x, \quad t \geq 0; \tag{1.5}$$

(ii) $e^{(t+s)A} = e^{tA} e^{sA}, \qquad t, s \geq 0;$

(iii) *there are constants* M_i, $i = 0, \ldots, k$, *such that*

$$\|e^{tA}\|_{\mathcal{L}(X)} \leq M_0 e^{\omega t}, \quad t > 0,$$

$$\|t^k (A - \omega I)^k e^{tA}\|_{\mathcal{L}(X)} \leq M_k e^{\omega t}, \quad t > 0, \tag{1.6}$$

where ω *is given in Definition 1.2.1*

(iv) *the function* $t \mapsto e^{tA}$ *belongs to* $C^\infty((0,\infty); \mathcal{L}(X))$ *and*

$$\frac{d^k}{dt^k} e^{tA} = A^k e^{tA}, \quad t > 0.$$

Moreover, it has an analytic extension in the sector

$$\Sigma_{\theta - \frac{\pi}{2}} = \{\lambda \in \mathbf{C} : \lambda \neq 0, |\arg \lambda| < \theta - \pi/2\}.$$

These properties motivate the following definition.

Definition 1.2.4. *Let* $A : D(A) \subset X \to X$ *be a sectorial operator. The family* $(e^{tA})_{t \geq 0}$ *defined by* (1.4) *is said to be the analytic semigroup generated by* A *in* X.

Analogously one can prove that $\{e^{tA}\}_{t \geq 0}$ is strongly continuous if and only if the domain $D(A)$ is dense in X, indeed $\lim_{t \to 0} e^{tA} x = x$ if and only if $x \in \overline{D(A)}$.

The following results solve the problem of identifying the generator of a given analytic semigroup. In the next lemma an integral representation of the resolvent of A in terms of the semigroup generated by A is given. The following proposition states that for a given analytic semigroup $\{T(t)\}_{t \geq 0}$ there exists a sectorial operator A such that $T(t) = e^{tA}$.

Lemma 1.2.5. *Let* $A : D(A) \subset X \to X$ *be as in Definition 1.2.1. Then for every* $\lambda \in \mathbf{C}$ *such that* $\operatorname{Re} \lambda > \omega$ *we have*

$$R(\lambda, A) = \int_0^\infty e^{-\lambda t} e^{tA}\, dt. \tag{1.7}$$

Proposition 1.2.6. *Let* $\{T(t)\}_{t>0}$ *be a family of linear bounded operators such that* $t \mapsto T(t)$ *is differentiable with values in* $\mathcal{L}(X)$ *and verifies*

(i) $T(t)T(s) = T(t+s)$ *for every* $t, s > 0$;

(ii) $\|T(t)\|_{\mathcal{L}(X)} \leq M_0 e^{\omega t}$, $\|t \frac{dT(t)}{dt}\|_{\mathcal{L}(X)} \leq M_1 e^{\omega t}$ *for some* $\omega \in \mathbf{R}$, $M_0, M_1 > 0$

(iii) $\lim_{t \to 0} T(t)x = x$ *for every* $x \in X$.

Then $t \mapsto T(t)$ is analytic in $(0,\infty)$ with values in $\mathcal{L}(X)$, and there exists a unique sectorial operator $A : D(A) \subset X \to X$ such that $T(t) = e^{tA}$ for every $t \geq 0$.

Let us give a sufficient condition, seemingly weaker than (1.2)-(1.3), in order that a linear operator be sectorial. It will be useful to prove that the realizations of some elliptic partial differential operators are sectorial in the usual function spaces.

Proposition 1.2.7. *Let $A : D(A) \subset X \to X$ be a linear operator such that $\rho(A)$ contains a half plane $\{\lambda \in \mathbf{C};\ \mathrm{Re}\,\lambda \geq \omega\}$, and*

$$\|\lambda R(\lambda, A)\|_{\mathcal{L}(X)} \leq M, \quad \mathrm{Re}\,\lambda \geq \omega, \tag{1.8}$$

with $\omega \in \mathbf{R}$, $M > 0$. Then A is sectorial.

Proof. By using the fact that if $\lambda_0 \in \rho(A)$ then the ball

$$\{\lambda \in \mathbf{C};\ |\lambda - \lambda_0| < \|R(\lambda_0, A)\|^{-1}_{\mathcal{L}(X)}\}$$

is contained in $\rho(A)$, we get that for every $r > 0$ the resolvent set of A contains the open ball centered at $\omega + ir$ with radius $|\omega + ir|/M$. The union of such balls contains the sector $S = \{\lambda \neq \omega :\ |\arg(\lambda - \omega)| < \pi - \arctan M\}$. Moreover, for $\lambda \in V = \{\lambda : \mathrm{Re}\,\lambda < \omega,\ |\arg(\lambda - \omega)| \leq \pi - \arctan(2M)\}$, $\lambda = \omega + ir - \theta r/M$ with $0 < \theta \leq 1/2$, we can write

$$R(\lambda, A) = \sum_{k=0}^{\infty} (-1)^k (\lambda - \omega - ir)^k R^{k+1}(\omega + ir, A)$$

therefore

$$\|R(\lambda, A)\|_{\mathcal{L}(X)} \leq \sum_{k=0}^{\infty} |\lambda - (\omega + ir)|^k \frac{M^{k+1}}{(\omega^2 + r^2)^{\frac{k+1}{2}}} \leq \frac{2M}{r}.$$

On the other hand, since $\lambda = \omega + ir - \theta r/M$, the following estimate holds

$$r \geq (1/(4M^2) + 1)^{-1/2} |\lambda - \omega|.$$

Finally

$$\|R(\lambda, A)\|_{\mathcal{L}(X)} \leq 2M(1/(4M^2) + 1)^{1/2} |\lambda - \omega|^{-1}$$

and the claim is proved. □

Thus in order to prove sectoriality for a given elliptic operator one needs to prove

(i) existence and uniqueness for the solution of a boundary value problem of the type

$$\begin{cases} \lambda u(x) - Au(x) = f(x) & \text{in } \Omega\,, \\ Bu(x) = g(x) & \text{in } \partial\Omega\,, \end{cases}$$

at least for $\mathrm{Re}\,\lambda$ large, and

(ii) the resolvent estimate (1.8).

1.2.1 Perturbation of sectorial operators

When dealing with second order partial differential operators, it is often easier to study operators with smooth coefficients or without lower order terms. Subsequently, one can try to remove the smoothness assumption by using an approximation argument and to add lower order terms with a perturbation argument. In this case it is important to know that sectoriality is preserved and this can be guaranteed by an abstract perturbation result. More specifically, let $A : D(A) \subset X \to X$ be a sectorial operator, generator of the analytic semigroup $(T(t))_{t\geq 0}$, and consider another operator $B : D(B) \subset X \to X$. The perturbation theory gives conditions under which the sum $A+B$ is a sectorial operators, too, and therefore generates itself an analytic semigroup.
If B is "small" with respect to A, in a suitable sense, we say that the operator A is *perturbed* by the operator B or that B is a *perturbation* of A. Before stating the main result we need in the sequel, we observe that the sum $A + B$ defined in the natural way

$$(A + B)x := Ax + Bx$$

and it is meaningful only for

$$x \in D(A + B) := D(A) \cap D(B),$$

a subspace that in general could reduce to $\{0\}$.
We start with a theorem of perturbation (whose proof can be found in [19]) where the simplest case, that is the case in which the perturbing operator is bounded, is considered. In this case, of course, $D(B) = X$.

Theorem 1.2.8. *Let $(A, D(A))$ be the generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$ on a Banach space X satisfying $\|T(t)\| \leq Me^{\omega t}$ for every $t \geq 0$, $\omega \in \mathbf{R}$ and $M \geq 1$. If $B \in \mathcal{L}(X)$, then*

$$A + B \quad \textit{with} \quad D(A + B) := D(A)$$

generates a strongly continuous semigroup $(S(t))_{t\geq 0}$ satisfying

$$\|S(t)\| \leq Me^{(\omega+M\|B\|)t} \quad t \geq 0.$$

Moreover if $(T(t))_{t\geq 0}$ is analytic, then so is the semigroup $(S(t))_{t\geq 0}$ generated by $A+B$.

Whereas a bounded perturbation of an operator preserves its properties, the sum of two unbounded operators raises more delicate questions since the domain $D(A) \cap D(B)$ can be too small and the good properties of single operators can be destroyed in the sum. For this reason we need a definition for perturbing operators for which this situation is avoided.

Definition 1.2.9. *Let $A : D(A) \subset X \to X$ be a linear operator on the Banach space X. An operator $B : D(B) \subset X \to X$ is called* A-bounded *if $D(A) \subseteq D(B)$ and if there exist constants $a, b \in \mathbf{R}^+$ such that*

$$\|Bx\| \leq a\|Ax\| + b\|x\| \tag{1.9}$$

for all $x \in D(A)$. The A-bound *of* B *is*

$$a_0 := \inf\{a \geq 0 : \textit{there exists } b \in \mathbf{R}_+ \textit{ such that } (1.9) \textit{ holds}\}.$$

Finally we prove a useful perturbation theorem that will be used later.

Theorem 1.2.10. *Let $A : D(A) \subset X \to X$ be a sectorial operator and let $B : D(B) \subset X \to X$ be a A-bounded operator with A-bound a_0. Then there exists a constant $\alpha > 0$ such that if $a_0 < \alpha$, then $A + B : D(A) \to X$ is sectorial.*

Proof. Let $\omega \in \mathbf{R}$ be such that $R(\lambda, A)$ exists and $\|\lambda R(\lambda, A)\| \leq M$ for $\mathrm{Re}\,\lambda \geq \omega$. We write $\lambda - A - B = (I - BR(\lambda, A))(\lambda - A)$ and we observe that

$$\|BR(\lambda, A)x\| \leq a\|AR(\lambda, A)x\| + b\|R(\lambda, A)x\| \leq \big(a(M+1) + \frac{bM}{|\lambda|}\big)\|x\| \leq \frac{1}{2}\|x\|$$

if $a(M+1) \leq 1/4$ and $bM/|\lambda| \leq 1/4$. Therefore, if $a \leq \alpha := (4(M+1))^{-1}$ and for $\mathrm{Re}\,\lambda$ sufficiently large, $\|BR(\lambda, A)\| \leq 1/2$ and

$$\|(\lambda - A - B)^{-1}\| \leq \|R(\lambda, A)\|\|(I - BR(\lambda, A))^{-1}\| \leq \frac{2M}{|\lambda|}$$

The statement now follows from Proposition 1.2.7. □

1.3 Analytic semigroups and spaces $D_A(\theta, p)$

In this section we present some results on the intermediate spaces $D_A(\theta, p)$ coming from a sectorial operator A. The classical results on interpolation between Banach spaces are collected in Appendix A. The definition of the spaces $D_A(\theta, p)$ is due to H. Berens and P. L. Butzer [9]. They can be defined in several different ways, one of them comes out from the behavior of $AT(t)x$ near $t = 0$. We have seen in Proposition 1.2.3 that, for each $x \in X$, $\|tAT(t)x\|$ is bounded in $(0, 1)$, whereas, for every $x \in D(A)$, $\|AT(t)x\|$ is bounded in $(0, 1)$. This behavior of $AT(t)$ leads to the definition of a class of intermediate spaces between X and $D(A)$. In this section we set $1/\infty = 0$.

Definition 1.3.1. *Let $0 < \theta < 1$, $1 \leq p \leq \infty$, and $(\theta, p) = (1, \infty)$, we set*

$$D_A(\theta, p) = \{x \in X : \ t \mapsto \|t^{1-\theta-1/p} AT(t)x\| \in L^p(0, 1)\}$$

endowed with the norm

$$\|x\|_{D_A(\theta,p)} = \|x\|_X + [x]_{D_A(\theta,p)},$$

where $[x]_{D_A(\theta,p)} = \|t^{1-\theta-1/p} AT(t)x\|_{L^p(0,1)}$. Define

$$D_A(\theta) = \{x \in D_A(\theta, \infty) : \ \lim_{t \to 0} t^{1-\theta} AT(t)x = 0\}.$$

Now, we state an important characterization of the space $D_A(\theta, p)$ that will be used in the sequel and whose proof can be found in [9, Theorem 3.4.2 and 3.5.3]. We denote by $(X, Y)_{\theta,p}$ the real interpolation space between X and Y.

Theorem 1.3.2. *Assume that $(A, D(A))$ generates an analytic semigroup on a Banach space X. Then for $0 < \theta < 1$ and $1 \le p \le \infty$, and for $(\theta, p) = (1, \infty)$ we have*

$$D_A(\theta, p) = (X, D(A))_{\theta,p}$$

moreover, for $0 < \theta < 1$,

$$D_A(\theta) = (X, D(A))_\theta$$

with equivalence of the respective norms.

The previous characterization provides several properties of these spaces deduced from the similar ones of the real interpolation spaces (see Appendix A). Some of these properties are recalled in the following corollary.

Corollary 1.3.3. *(i) Suppose that A and B generate bounded analytic semigroups in X. If $D(A) = D(B)$ (with equivalence of the norms) then*

$$D_A(\theta, p) = D_B(\theta, p) \quad \text{and} \quad D_A(\theta) = D_B(\theta).$$

(ii) The spaces $D_A(\theta, p)$ and $D_A(\theta)$ belong to the class J_θ between X and $D(A)$, i.e., there is a constant $c > 0$ such that

$$\|x\|_{D_A(\theta,p)} \le c\|x\|_X^{1-\theta}\|x\|_{D(A)}^\theta \quad \forall x \in D(A).$$

(iii) For $0 < \theta_1 < \theta_2 < \infty$ and $1 \le p \le \infty$ and for $(\theta_2, p) = (1, \infty)$, we have

$$D_A(\theta_2, p) \subset D_A(\theta_1, p).$$

For $0 < \theta < 1$, $1 \le p_1 \le p_2 < \infty$,

$$D_A(1, \infty) \subset D_A(\theta, p_1) \subset D_A(\theta, p_2) \subset D_A(\theta) \subset D_A(\theta, \infty) \subset \overline{D(A)}.$$

Now we give an useful estimate for the function $t \mapsto A^k T(t)$ as $t \to 0^+$ in the intermediate spaces just introduced. In the next proposition we set $D_A(0, p) = X$ for every $p \in [1, \infty]$.

Proposition 1.3.4. *Let $(\alpha, p), (\beta, p) \in (0, 1) \times [1, +\infty] \cup \{(1, \infty)\}$, and let $k \in \mathbf{N}$. Then there exists $C = C(k, p, \alpha, \beta)$ such that*

$$\|t^{k-\alpha+\beta} A^k T(t)\|_{\mathcal{L}(D_A(\alpha,p), D_A(\beta,p))} \le C \qquad 0 < t \le 1 \tag{1.10}$$

The statement holds also for $k = 0$, provided $\alpha \le \beta$.

Proof. Without loss of generality we can assume that A satisfies (1.2), (1.3) with $\omega = 0$, otherwise we consider $A - \omega I$. By (1.6), we get that

$$C_k = \sup_{0<t\leq 1} \|t^k A^k T(t)\|_{\mathcal{L}(X)} < \infty \quad \text{for all} \quad k \in \mathbf{N} \tag{1.11}$$

First we prove the estimate (1.10) for $\alpha = 0$. Let $x \in X$, $k \in \mathbf{N} \cup \{0\}$. Since $D_A(\beta, p)$ is of class J_β between X and $D(A)$, we get that

$$\|z\|_{D_A(\beta,p)} \leq c\|z\|^{\beta}_{D(A)}\|z\|^{1-\beta}_X \quad \forall\, z \in D(A).$$

Thus, using (1.11), we get

$$\|t^k A^k T(t)x\|_{D_A(\beta,p)} \leq c\|t^k A^k T(t)x\|^{\beta}_{D(A)}\|t^k A^k T(t)x\|^{1-\beta}_X \leq ct^{-\beta}\|x\|_X$$

for $0 < t \leq 1$, which is the claim for $\alpha = 0$ and $k \in \mathbf{N} \cup \{0\}$.
Now, let $k \in \mathbf{N}$, $0 < \alpha < 1$ and let $x \in D_A(\alpha, p)$ or $x \in D_A(1, \infty)$. Then, using (1.5), we get

$$\begin{aligned}
\|t^k A^k T(t)x\|_{D_A(\beta,p)} &= \|t^k A^{k-1} T(t/2) AT(t/2)x\|_{D_A(\beta,p)} \\
&\leq 2^k \|(t/2)^{k-1+\alpha} A^{k-1} T(t/2)\|_{\mathcal{L}(X, D_A(\beta,p))} \|(t/2)^{1-\alpha} AT(t/2)x\|_X \\
&\leq 2^{k+\beta-\alpha} t^{\alpha-\beta} C(k-1, p, 0, \beta)\|x\|_{D_A(\alpha,\infty)}.
\end{aligned}$$

Now, let $k = 0$, $\alpha \leq \beta$ and $x \in D_A(\alpha, p)$. Then for $0 < s \leq 1$,

$$\begin{aligned}
\|T(t)x\|_{D_A(\beta,p)} &= \|s^{1-\beta-1/p} AT(s)T(t)x\|_{L^p(0,1;X)} + \|T(t)x\|_X \\
&\leq C_0(\|s^{1-\alpha-1/p} AT(s)T(t)x\|_{L^p(0,1;X)} + \|x\|_X) = C_0\|x\|_{D_A(\alpha,p)}
\end{aligned}$$

which allows us to deduce the claim for $k = 0$ and $\alpha = \beta$. Finally, for $\beta > \alpha$, we get

$$\begin{aligned}
\|T(t)x\|_{D_A(\beta,p)} &\leq \|T(1)x\|_{D_A(\beta,p)} + \Big\| \int_t^1 AT(s)x\, ds \Big\|_{D_A(\beta,p)} \\
&\leq C(0, p, 0, \beta)\|x\|_X + C(1, p, \alpha, \beta)\|x\|_{D_A(\alpha,\infty)} \int_t^1 s^{\alpha-\beta-1}\, ds \\
&\leq C(0, p, 0, \beta)\|x\|_X + C(1, p, \alpha, \beta)\|x\|_{D_A(\alpha,\infty)} \frac{t^{-\beta+\alpha}}{\beta - \alpha}.
\end{aligned}$$

that complete the proof also for $k = 0$. □

1.4 Preliminaries of measure theory

In this section we briefly review the basic definitions and the most important properties of measure theory. The main reference for our approach is [5] and other references for related topics are [20], [21] and [37].

Let Ω be an open subset of $\mathbf{R}^n$ and let $\mathcal{B}(\Omega)$ be the σ-algebra of Borel subsets of Ω, that is, the σ- algebra generated by the open subsets of Ω. We call the pair $(\Omega, \mathcal{B}(\Omega))$ a *measure space*.

Definition 1.4.1. *Let* $(\Omega, \mathcal{B}(\Omega))$ *be a measure space and let* $m \in \mathbf{N}$, $m \geq 1$. *We say that* $\mu : \mathcal{B}(\Omega) \to \mathbf{R}^m$ *is a* measure *if*

$$\mu(\emptyset) = 0 \tag{1.12}$$

and μ *is* σ*- additive on* $\mathcal{B}(\Omega)$*, i.e., for any sequence* E_h *of pairwise disjoint elements of* $\mathcal{B}(\Omega)$

$$\mu(\bigcup_{h=0}^{\infty} E_h) = \sum_{h=0}^{\infty} \mu(E_h). \tag{1.13}$$

We denote by $[\mathcal{M}(\Omega)]^m$ *the space of* $\mathbf{R}^m$ *-valued measures. If* $m > 1$ *we say that* μ *is a* vector measure*, whereas if* $m = 1$ *we say that* μ *is a* real measure.

Definition 1.4.2. *(Positive measure) If* $\mu : \mathcal{B}(\Omega) \to [0, +\infty]$ *satisfies* (1.12) *and* (1.13) *then* μ *is called a* positive measure *or a* Borel measure.

Notice that positive measures are not a particular case of real measures since real measures must be finite according to the previous definition. In this latter case we say that μ is a *finite* measure if $\mu(\Omega) < \infty$. A positive measure μ such that $\mu(\Omega) = 1$ is also called a *probability measure*.

For a real, vector or positive measure we can define its *total variation measure*.

Definition 1.4.3. *We define the* total variation *of* μ *the set function denoted by* $|\mu| : \mathcal{B}(\Omega) \to [0, +\infty]$ *such that for every* $A \in \mathcal{B}(\Omega)$

$$|\mu|(A) := \sup \left\{ \sum_{h=0}^{\infty} |\mu(A_h)| : \; A_h \in \mathcal{B}(\Omega) \, \textit{pairwise disjoint}, A = \bigcup_{h=0}^{\infty} A_h \right\}.$$

It can be shown that if μ is a measure then $|\mu|$ is a positive finite measure.

Definition 1.4.4. *(Radon measure) If a Borel measure is finite on compact sets then it is called* positive Radon measure.
A Radon measure *on* Ω *is a real or vector valued set function* μ *that is a measure on* $(K, \mathcal{B}(K))$ *for every compact set* $K \subset \Omega$. *It is called a* finite Radon measure *if* $\mu : \mathcal{B}(\Omega) \to \mathbf{R}^m$ *is a measure in the sense specified before.*

If $m > 1$ and $B \in \mathcal{B}(\Omega)$, then $\mu(B) = (\mu_1(B), \ldots, \mu_m(B))$ and $\mu_i : \mathcal{B}(\Omega) \to \mathbf{R}$ are Radon measures.

Definition 1.4.5. *(Support of a measure) Let* μ *be a positive measure on* Ω*; we call* support of μ *the closed set of all points* $x \in \Omega$ *such that* $\mu(U) > 0$ *for every neighborhood* U *of* x *and we denote it by* supp μ. *If* μ *is a real or vector measure, we call the support of* μ *the support of* $|\mu|$.

For a positive, real or vector measure on the measure space $(\Omega, \mathcal{B}(\Omega))$ and for $E \in \mathcal{B}(\Omega)$ we denote by $\mu \llcorner E$ the restriction of μ to E so defined: $\mu \llcorner E(F) = \mu(E \cap F)$ for

every $F \in \mathcal{B}(\Omega)$; moreover, if μ is a Borel (Radon) measure and E is a Borel set, then the measure $\mu \llcorner E$ is a Borel (Radon measure), too. When $\mu \llcorner E = \mu$ we say that μ is *concentrated* on E. We say that a set E is *μ-negligible* if there exists $B \supset E$, $B \in \mathcal{B}(\Omega)$ such that $\mu(B) = 0$. Moreover a Borel set E is called *μ-measurable* if E is of the form $E \cup N$ with N μ-negligible.
We now state the classical Riesz representation theorem. Recall that we denote by $C_c(\Omega)$ the space of continuous functions with compact support and by $C_0(\Omega)$ its completion with respect the sup norm.

Theorem 1.4.6. *(Riesz Representation Theorem) Let $L : C_c(\Omega; \mathbf{R}^m) \to \mathbf{R}$ be a linear functional. Suppose that there exists $c < +\infty$ such that for all $f \in C_c(\Omega; \mathbf{R}^m)$*

$$|L(f)| \leq c \|f\|_{L^\infty(\Omega)}.$$

Then, there is a unique $\mathbf{R}^m$- valued Radon measure μ on Ω such that

$$L(f) = \int_\Omega f \, d\mu = \sum_{h=1}^{m} \int_\Omega f_h d\mu_h \qquad \forall f \in C_c(\Omega; \mathbf{R}^m).$$

Moreover

$$\sup\{L(f) : \ f \in C_c(\Omega; \mathbf{R}^m), \ \|f\|_{L^\infty(\Omega)} \leq 1\} = |\mu|(\Omega).$$

1.4.1 Weak convergence of measures

From the Riesz theorem, it follows that the space of $[\mathcal{M}(\Omega)]^m$, endowed with the norm $\|\mu\| := |\mu|(\Omega)$, is linearly isometric to the dual space of $C_c(\Omega; \mathbf{R}^m)$ and so it is a Banach space. This fact allows us to consider several topologies on $[\mathcal{M}(\Omega)]^m$. Of particular interest are the following two different kinds of convergence induced by $C_c(\Omega; \mathbf{R}^m)$ and $C_0(\Omega; \mathbf{R}^m)$, respectively.

Definition 1.4.7. *Let μ_k, μ be $\mathbf{R}^m$- valued Radon measures on Ω.*

(i) We say that μ_k converges locally weakly* *to μ and write $\mu_k \xrightarrow{w^*_{loc}} \mu$ if*

$$\int_\Omega f \, d\mu_k \longrightarrow \int_\Omega f \, d\mu \qquad \forall f \in C_c(\Omega; \mathbf{R}^m).$$

(ii) We say that μ_k converges weakly* *to μ and write $\mu_k \xrightarrow{w^*} \mu$ if*

$$\int_\Omega f \, d\mu_k \longrightarrow \int_\Omega f \, d\mu \qquad \forall f \in C_0(\Omega; \mathbf{R}^m).$$

An important connection between these two different kinds of convergence is given by the following property. Let μ_k, μ be $\mathbf{R}^m$- valued finite Radon measures. Then $\mu_k \xrightarrow{w^*} \mu$ if and only if $\mu_k \xrightarrow{w^*_{loc}} \mu$ and the norms $|\mu_k|(\Omega)$ are bounded.

Definition 1.4.8. *(Convergence in measure) We say that* (E_h) *converges to* E *in measure in* Ω *if*

$$|\Omega \cap (E_h \Delta E)| \to 0 \quad as\ h \to \infty.$$

We say that E_h *locally converges in measure to* E *if* (E_h) *converges to* E *in measure in every open set* A *with* $A \subset\subset \Omega$.

We can notice that these convergences correspond to $L^1(\Omega)$ and $L^1_{\mathrm{loc}}(\Omega)$ convergences of the characteristic functions.

1.4.2 Differentiation of measures

Two important relations between measures are presented in the following definition, the absolute continuity and the mutually singularity.

Definition 1.4.9. *(Absolute continuity and singularity) Let* μ *be a positive measure and* σ *a real or a vector measure on the measure space* $(\Omega, \mathcal{B}(\Omega))$*; we say that* σ *is* absolutely continuous *with respect to* μ*, and write* $\sigma << \mu$*, if for* $A \in \mathcal{B}(\Omega)$*,* $\mu(A) = 0$ *implies* $\sigma(A) = 0$*. If the measures* μ, σ *are both positive, we say that they are mutually singular and write* $\mu \perp \sigma$ *if there exists* $E \in \mathcal{B}(\Omega)$ *such that* $\mu(E) = 0$ *and* $\sigma(\Omega \setminus E) = 0$*.*

This latter definition can be extended also to vector measures: in that case we say that two vector measures μ and σ are mutually singular if $|\mu|$ and $|\nu|$ are so.

Theorem 1.4.10. *(Besicovitch differentiation theorem) Let* μ *be a positive Radon measure and* σ *a real or vector valued measure both defined on the same open set* Ω *of* $\mathbf{R}^n$*. Then, for* μ*- a.e.* $x \in \Omega$ *there exists the limit*

$$\lim_{\rho \to 0} \frac{\sigma(B_\rho(x))}{\mu(B_\rho(x))} = D_\mu \sigma(x)$$

and it is equal to $+\infty$ *for* $x \notin \operatorname{supp} \mu$*. The function* $D_\mu \sigma(x) \in [L^1_{loc}(\Omega, \mu)]^m$ *and for every Borel set* $B \in \mathcal{B}(\Omega)$

$$\sigma(B) = \int_B D_\mu \sigma(x)\, d\mu(x) + \sigma^s(B), \tag{1.14}$$

where $\sigma^s \perp \mu$ *and is concentrated on a Borel set* μ*-negligible.*

By the representation (1.14) of σ we can deduce that the integral part is absolutely continuous with respect to μ, and σ^s is singular.
This decomposition of σ with respect to μ is called *Lebesgue decomposition* and it is uniquely determined. The function $D_\mu \sigma$ is called the derivative of σ respect to μ and it is usually denoted by σ/μ. The proof of the Besicovitch theorem, as is stated here, can be found in [41].
An useful decomposition immediately follows from the Besicovitch theorem if we take into account that each real or vector measure μ is absolute continuous with respect to its total variation $|\mu|$.

Corollary 1.4.11. *(Polar decomposition) Let μ be a $\mathbf{R}^m$-valued measure on the measure space $(\Omega, \mathcal{B}(\Omega))$; then there exists a unique $\mathbf{S}^{m-1}$-valued function $f \in (L^1(\Omega, |\mu|))^m$ such that $\mu = f|\mu|$.*

1.4.3 Hausdorff measures and rectifiable sets

The notion that we are going to introduce is a mild regularity property of subsets of $\mathbf{R}^n$ known as *rectifiability.* First we provide the definition of Hausdorff k-dimensional measures. This class of measures is defined in terms of the diameters of suitable coverings and allows an intrinsic definition of k-dimensional area without any reference to parametrizations.

Definition 1.4.12. *(Hausdorff measures) Let $A \subset \mathbf{R}^n$, $k \in [0, \infty)$ and $\delta \in (0, \infty]$. Define*

$$\mathcal{H}^k_\delta(A) := \frac{\omega_k}{2^k} \inf \Big\{ \sum_{i \in I} [\mathrm{diam}(A_i)]^k : A \subset \bigcup_{i \in I} A_i, \, \mathrm{diam}(A_i) < \delta \Big\} \tag{1.15}$$

for finite or countable covering $\{A_i\}_{i \in I}$ (with $\mathrm{diam}\, \emptyset = 0$). Here

$$\omega_k = \frac{\pi^{k/2}}{\Gamma(1 + k/2)}$$

where $\Gamma(s) = \int_0^\infty x^{s-1} e^{-x} \, dx$ is the Euler gamma function.
For A and k as above, define

$$\mathcal{H}^k(A) := \lim_{\delta \to 0} \mathcal{H}^k_\delta(A). \tag{1.16}$$

Remark 1.4.13. We notice that the limit in (1.16) exists (finite or infinite) since $\delta \mapsto \mathcal{H}^k_\delta(A)$ is decreasing in $(0, \infty]$. It is also worth noticing that requiring $\delta \to 0$ forces the coverings to follow the local geometry of the set A.
Finally let us observe that $\mathcal{H}^0$ corresponds to the counting measure and it is not trivial to prove that $\mathcal{H}^n = \mathcal{L}^n$ on $\mathbf{R}^n$.

Definition 1.4.14. *(Countably $\mathcal{H}^{n-1}$-rectifiable sets) We say that $E \subset \mathbf{R}^n$ is countably* $\mathcal{H}^{n-1}$-rectifiable *if there exist (at most) countably many C^1 embedded hypersurfaces $\Gamma_i \subset \mathbf{R}^n$ such that*

$$\mathcal{H}^{n-1}(E \setminus \bigcup_i \Gamma_i) = 0.$$

1.5 Some further preliminaries

In this section we collect some miscellaneous classical results, which is useful to state in the form we shall use later.
Throughout this thesis, we shall consider functions defined in $\mathbf{R}^n$ or in subset of $\mathbf{R}^n$, particularly in $\mathbf{R}^n_+ = \{x = (x_1, \ldots, x_n) \in \mathbf{R}^n; \; x_n \geq 0\}$ and in domains with uniformly C^2 boundary $\partial\Omega$. Let Ω be an open set in $\mathbf{R}^n$, and $m \in \mathbf{N}$. Let us give the definition.

Definition 1.5.1. *(Uniformly C^m domain) We say that the boundary $\partial\Omega$ is uniformly C^m if there exist $r, L > 0$ and a (at most countable) collection of open balls $U_j = \{x \in \mathbf{R}^n; |x - x_j| < r\}$, $j \in \mathbf{N}$, covering $\partial\Omega$ and such that there exists an integer k with the property that $\bigcap_{j\in J} U_j = \emptyset$ for all $J \subset \mathbf{N}$ with more than k elements. Moreover there exist coordinate transformations $\varphi_j : U_j \to B(0,1)$, C^m diffeomorphisms such that*

$$\varphi_j(\overline{U_j} \cap \Omega) = B^+(0,1) = B(0,1) \cap \mathbf{R}^n_+$$

$$\varphi_j(\overline{U_j} \cap \partial\Omega) = B(0,1) \cap \{x_n = 0\}.$$

Moreover, all the coordinate transformations φ_j and their inverses are supposed to have uniformly bounded derivatives up to the order m,

$$\sup_{j\in\mathbf{N}} \sum_{1\le|\alpha|\le m} \left(\|D^\alpha \varphi_j\|_\infty + \|D^\alpha \varphi_j^{-1}\|_\infty\right) \le L$$

We shall use the classical Sobolev embedding theorems which are recalled in the next lemma. We refer to [1] for their proof.

Theorem 1.5.2. *Let Ω be either $\mathbf{R}^n$, or an open set in $\mathbf{R}^n$ with uniformly C^1 boundary. Let $p > n$ and set $\alpha = 1 - \frac{n}{p}$. Then $W^{1,p}(\Omega) \subset C^\alpha_b(\overline{\Omega})$. Moreover, there exists $C > 0$ such that for every $u \in W^{1,p}_{loc}(\Omega)$ and for every $x_0 \in \Omega$ we have*

(i) $\|u\|_{L^\infty(\Omega_{x_0,r})} \le C r^{-\frac{n}{p}} (\|u\|_{L^p(\Omega_{x_0,r})} + r\|Du\|_{L^p(\Omega_{x_0,r})})$,

(ii) $[u]_{C^\alpha(\Omega_{x_0,r})} \le C\|Du\|_{L^p(\Omega_{x_0,r})}$.

where $\Omega_{x_0,r} = \Omega \cap B(x_0, r)$ and $[u]_{C^\alpha(\Omega)} = \sup_{x,y\in\Omega} \frac{|u(x)-u(y)|}{|x-y|^\alpha}$.

Another useful tool is a classical result of functional analysis known as continuity method recalled in the next theorem.

Theorem 1.5.3. *Let X, Y be Banach spaces, L_0 and L_1 be two linear and continuous operators from X to Y. We consider the family of operators*

$$L_t = (1-t)L_0 + tL_1, \qquad t \in [0,1],$$

and we suppose that there exists a constant $C > 0$ such that

$$\|L_t x\|_Y \ge C\|x\|_X, \qquad x \in X, t \in [0,1]. \tag{1.17}$$

If L_0 is surjective, then L_1 is surjective too (hence bijective for the estimate (1.17)).

Proof. Let $V = \{t \in [0,1] : L_t \text{ is bijective}\}$. By hypothesis $V \neq \emptyset$ since $0 \in V$. If $t_0 \in V$ then L_{t_0} is bijective and $\|L_{t_0}^{-1}\| \le \frac{1}{C}$ by (1.17). Moreover, since

$$L_t = L_{t_0}(I + (t - t_0)L_{t_0}^{-1}(L_1 - L_0))$$

L_t is invertible if and only if $(I + (t - t_0)L_{t_0}^{-1}(L_1 - L_0))$ is invertible. But, if $|t - t_0| < \frac{C}{\|L_1\|+\|L_0\|}$ then $\|(t-t_0)L_{t_0}^{-1}(L_1 - L_0)\| < 1$ and L_t is invertible. Setting $\delta = \frac{C}{2(\|L_1\|+\|L_0\|)}$ we get that $[0, \delta] \subset V$. Analogous argument proves that $[\delta, 2\delta] \subset V$ and so on.
Finally, after a finite number of steps we get that $[0, 1] \subset V$. $\square$

Finally, it is useful to recall two well-known inequalities due to G. H. Hardy [25]. For the proof we use two lemmas. The first follows from the Hölder inequality and its proof can be found in [25, Theorem 191].

Lemma 1.5.4. *Let Ω be an open set of $\mathbf{R}^n$, $p > 1$ and $p' = p/(p-1)$; then $\|f\|^p_{L^p(\Omega)} \leq C_0$ if and only if $\|fg\|_{L^1(\Omega)} \leq C_0^{1/p} C_1^{1/p'}$ for all g such that $\|g\|^{p'}_{L^{p'}(\Omega)} \leq C_1$.*

We shall deduce Theorem 1.5.6 from the following more general theorem whose method of proof is due to Schur, even though in [38], it is assumed $p = 2$.

Lemma 1.5.5. *Let $p > 1$ and $p' = p/(p-1)$. Let $K(x, y)$ be a non-negative and homogeneous of degree -1 function, (i.e. $K(\lambda x, \lambda y) = \lambda^{-1}K(x, y)$) such that*

$$\int_0^\infty K(x,1)x^{-1/p}\,dx = \int_0^\infty K(1,y)y^{-1/p'}\,dy = k.$$

Then, for every non-negative functions $f \in L^p(0,\infty)$ and $g \in L^{p'}(0,\infty)$ we get

$$\int_0^\infty\int_0^\infty K(x,y)f(x)g(y)\,dx\,dy \leq k\Big(\int_0^\infty f^p(x)\,dx\Big)^{1/p}\Big(\int_0^\infty g^{p'}(y)\,dy\Big)^{1/p'}, \tag{1.18}$$

$$\int_0^\infty dy\big(\int_0^\infty K(x,y)f(x)\,dx\big)^p \leq k^p\int_0^\infty f^p(x)\,dx \tag{1.19}$$

$$\int_0^\infty dx\big(\int_0^\infty K(x,y)g(y)\,dy\big)^{p'} \leq k^{p'}\int_0^\infty g^{p'}(y)\,dy \tag{1.20}$$

Proof. We have

$$\begin{aligned}\int_0^\infty f(x)\,dx\int_0^\infty K(x,y)g(y)\,dy &= \int_0^\infty f(x)\,dx\int_0^\infty xK(x,xw)g(xw)\,dw\\ &= \int_0^\infty f(x)\,dx\int_0^\infty K(1,w)g(xw)dw\\ &= \int_0^\infty K(1,w)\,dw\int_0^\infty f(x)g(xw)\,dx\end{aligned}$$

if any of integrals are convergent. Applying Lemma 1.5.4 to the inner integral, and observing that

$$\int g^{p'}(xw)\,dx = \frac{1}{w}\int g^{p'}(y)\,dy$$

we obtain (1.18). Finally (1.19) and (1.20) can be deduced by Lemma 1.5.4, indeed by (1.18) we get that

$$\|hg\|_{L^1(0,\infty)} \leq (k^pC_0)^{1/p}C_1^{1/p'}$$

holds for all $g \in L^{p'}(\Omega)$ where $h(y) = \int_0^\infty K(x,y)f(x)dx$, $C_0 = \int_0^\infty f^p(x)\,dx$ and $C_1 = \int_0^\infty g^{p'}(y)\,dy$. Thus, Lemma 1.5.4 implies that

$$\|h\|^p_{L^p(0,\infty)} \leq k^p C_0$$

whence (1.19) is proved. The same argument can be used to prove (1.20). □

Now, an immediate application of Lemma 1.5.5 is obtained by specializing the choice of $K(x,y)$.

Theorem 1.5.6. *(Hardy's inequalities) Let $\alpha > 0$, $1 \leq p \leq \infty$. If $\psi(s)$ is a non-negative measurable function with respect to the measure ds/s on $(0,\infty)$, then*

$$\Big\{\int_0^\infty \Big(t^{-\alpha}\int_0^t \psi(s)\frac{ds}{s}\Big)^p \frac{dt}{t}\Big\}^{1/p} \leq \frac{1}{\alpha}\Big\{\int_0^\infty (s^{-\alpha}\psi(s))^p \frac{ds}{s}\Big\}^{1/p} \tag{1.21}$$

and

$$\Big\{\int_0^\infty \Big(t^{\alpha}\int_t^\infty \psi(s)\frac{ds}{s}\Big)^p \frac{dt}{t}\Big\}^{1/p} \leq \frac{1}{\alpha}\,\Big\{\int_0^\infty (s^{\alpha}\psi(s))^p \frac{ds}{s}\Big\}^{1/p} \tag{1.22}$$

Proof. Let $\alpha > 0$, $1 \leq p \leq \infty$, then the function

$$K(s,t) := \begin{cases} s^{\alpha+\frac{1}{p}-1}t^{-\alpha-\frac{1}{p}} & s < t \\ 0 & \text{elsewhere} \end{cases}$$

satisfies the assumption of Lemma 1.5.5 with $k = \frac{1}{\alpha}$. Then (1.21) can be obtained by (1.19) with $K(s,t)$ as before and $f(s) = s^{-\alpha-\frac{1}{p}}\psi(s)$. Finally (1.22) can be proved similarly choosing K and f in a suitable way. □

The next lemma is used only in Propositions 3.1.1 and 3.1.3. We omit the proof which can be considered a particular case of [26, Lemma 7.1.1].

Lemma 1.5.7. *(Gronwall's generalized inequality) Suppose $a, b \geq 0$, $0 \leq \alpha, \beta < 1$, $0 < T < \infty$. Let $u(t)$ be a nonnegative and locally integrable function on $0 \leq t \leq T$ with*

$$u(t) \leq at^{-\alpha} + b\int_0^t (t-s)^{-\beta}u(s)\;ds$$

on $(0,T)$; then there exists a constant $C(b,\beta,T) < \infty$ such that

$$u(t) \leq \frac{at^{-\alpha}}{1-\alpha}C(b,\beta,T).$$