Università del Salento

Dipartimento di Matematica "E. De Giorgi"

Doctoral Thesis in Mathematics – Mat/05

Short-time behavior of semigroups

and

functions of bounded variation

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TESI DI DOTTORATO DISCUSSA IL 3 LUGLIO 2008

Riassunto

Lo spazio delle funzioni a variazione limitata, usualmente denotato con BV, ha avuto ed ha tuttora un ruolo importante in numerosi problemi nell'ambito del Calcolo delle Variazioni. Le principali proprietà che fanno di questo spazio l'ambiente adatto in cui formulare problemi variazionali riguardano i risultati di compattezza, relativi a funzionali integrali a crescita lineare nel gradiente, e la possibilità di supporre che tali funzioni ammettano delle ipersuperfici di discontinuità, caratteristica importante in numerosi problemi fisici e di natura geometrica. Il prototipo dei funzionali integrali a crescita lineare nel gradiente è il funzionale dell'area, mentre, nell'ambito dei problemi variazionali con discontinuità, il primo successo della teoria risale alla risoluzione completa del problema isoperimetrico in $\mathbf{R}^n$. Più recentemente, sono stati oggetto di studio i problemi con discontinuità libere (introdotti da E. De Giorgi in [17]), tra cui ricordiamo il problema della segmentazione delle immagini digitali e problemi di meccanica delle fratture. L'interesse verso tali problemi è sicuramente motivato dalle applicazioni alla biologia, all'informatica ed alla fisica, in cui rispettivamente l'elaborazione delle immagini digitali e le proprietà elasto-plastiche dei materiali costituiscono elementi di notevole rilevanza. Si noti che le funzioni di Sobolev non godono di proprietà di compattezza altrettanto generali quanto le funzioni BV, nè ammettono insiemi di discontinuità $(n-1)$-dimensionali.
Lo studio vasto e accurato di questa classe di funzioni ha prodotto una teoria completa ed esauriente che comprende risultati di approssimazione, teoremi di immersione, teoremi di traccia e proprietà fini. Per un'analisi approfondita e dettagliata di tale classe di funzioni e delle relative proprietà facciamo riferimento al libro di L. Ambrosio, N. Fusco e D. Pallara [5].

L'obiettivo di questa tesi è lo studio di alcuni legami esistenti tra la teoria delle funzioni a variazione limitata e la teoria dei semigruppi generati da operatori ellittici del secondo ordine. Ricordiamo che, dati un aperto Ω di $\mathbf{R}^n$ ed $u \in L^1(\Omega)$, si dice che u è una funzione a variazione limitata (e si scrive $u \in BV(\Omega)$) se la sua derivata distribuzionale Du è rappresentabile mediante una misura di Radon la cui variazione totale così definita

$$|Du|(\Omega) = \sup\Big\{ \int_{\mathbf{R}^n} u\mathrm{div}\,\phi\; dx \;: \phi \in C^1_c(\Omega, \mathbf{R}^n), \|\phi\|_{L^\infty(\Omega)} \leq 1 \Big\} \tag{1}$$

è finita. Nel caso particolare in cui $u = \chi_E$, la funzione caratteristica di un insieme $E \subset \mathbf{R}^n$, si definisce perimetro di E in Ω la variazione totale di $D\chi_E$; in tal caso scriveremo $\mathcal{P}(E,\Omega) = |D\chi_E|(\Omega)$ e diremo che E è un insieme di perimetro finito in Ω se

$\mathcal{P}(E,\Omega) < \infty$. Quando $\Omega = \mathbf{R}^n$ scriveremo semplicemente $\mathcal{P}(E)$.
Il punto di partenza dei risultati di ricerca presentati in questa tesi è il lavoro [15] in cui De Giorgi dà una definizione di variazione totale, che risulta equivalente a (1) se $\Omega = \mathbf{R}^n$. L'interesse di De Giorgi in [15] era rivolto allo studio delle proprietà di struttura degli insiemi di perimetro finito, a possibili estensioni di disuguaglianze isoperimetriche e ad eventuali generalizzazioni della formula di Gauss-Green e perciò si limita ad approssimare funzioni $L^\infty(\mathbf{R}^n)$ tramite opportuni nuclei di convoluzione. Assegnata $f \in L^\infty(\mathbf{R}^n)$, definisce

$$W(t)f(x) := (G_t * f)(x)$$

dove $G_t(x)$ è il nucleo di Gauss-Weierstrass

$$G_t(x) = (4\pi t)^{-n/2} e^{-\frac{|x|^2}{4t}}, \qquad t > 0,\ x \in \mathbf{R}^n.$$

Questa particolare scelta fa sì che $W(t)f$ soddisfaccia una legge di semigruppo

$$W(t+s)f(x) = W(t)W(s)f(x) \qquad t,s > 0$$

e che la funzione $t \mapsto H(t) := \|DW(t)f\|_{L^1(\mathbf{R}^n)}$ risulti una funzione monotona non crescente in $(0,\infty)$; infatti per $t,s > 0$ risulta

$$\|DW(t+s)f\|_{L^1(\mathbf{R}^n)} = \|DW(t)W(s)f\|_{L^1(\mathbf{R}^n)} = \|W(t)DW(s)f\|_{L^1(\mathbf{R}^n)} \le \|DW(s)f\|_{L^1(\mathbf{R}^n)}$$

Tale monotonia garantisce l'esistenza del limite di $H(t)$ per $t \to 0$. Dato $E \subset \mathbf{R}^n$, De Giorgi definisce

$$P(E) := \lim_{t\to 0} \int_{\mathbf{R}^n} |DW(t)\chi_E|\, dx \tag{2}$$

Si osservi come la definizione (2) ha senso per una qualsiasi $f \in L^p(\mathbf{R}^n)$, $p \in [1,\infty]$. Così, analogamente a (2), si potrebbe dare la definizione di variazione totale per una funzione $f \in L^1(\mathbf{R}^n)$ che risulterà equivalente a quella data in (1), pertanto

$$|Df|(\mathbf{R}^n) = \lim_{t\to 0} \int_{\mathbf{R}^n} |DW(t)f|\, dx \tag{3}$$

e $\mathcal{P}(E) = P(E)$. D'altra parte, si noti che $W(t)f$ rappresenta la soluzione dell'equazione del calore in $\mathbf{R}^n$ con dato iniziale f, cioè $W(t)f$ risolve

$$\begin{cases} \partial_t v(t,x) = \Delta v(t,x) & t \in (0,\infty),\ x \in \mathbf{R}^n \\ v(0,x) = f(x) & x \in \mathbf{R}^n. \end{cases} \tag{4}$$

Pertanto all'uguaglianza (3) si può dare un ulteriore significato. Più precisamente, partendo dalla definizione (1), la formula (3) stabilisce un legame tra la variazione totale di una funzione $f \in L^1(\mathbf{R}^n)$ e la soluzione dell'equazione del calore in $\mathbf{R}^n$ con dato iniziale f. La definizione di perimetro (o di variazione totale) in $\mathbf{R}^n$ data da De Giorgi mette in relazione teorie apparentemente distanti tra loro, come la teoria delle funzioni a variazione limitata e la teoria delle equazioni di evoluzione.
Il problema che ci siamo posti è stato quello di vedere se tale relazione possa essere estesa al caso di domini, cioè se partendo dalla definizione (1) di variazione totale in

un dominio, sia possibile stabilire una relazione tipo (3) con la soluzione di un generico problema parabolico. D'altronde il caso del Laplaciano in $\mathbf{R}^n$ può essere considerato un caso modello e quindi il problema (4) il prototipo di tali problemi. Un elemento di novità nel nostro lavoro di ricerca è costituito dal fatto che considereremo Ω aperto generico; infatti in letteratura si trovano molti risultati riguardanti la teoria L^1, la maggior parte dei quali ambientati in $\mathbf{R}^n$ o in aperti limitati. Descriviamo brevemente le ipotesi considerate in questa tesi. Sia Ω un aperto di $\mathbf{R}^n$ con bordo uniformemente di classe C^2 e consideriamo $\mathcal{A}$ un operatore uniformemente ellittico in forma di divergenza

$$\mathcal{A}(x,D) = \sum_{i,j=1}^{n} D_i(a_{ij}(x)D_j) + \sum_{i=1}^{n} b_i(x)D_i + c(x).$$

Se $\Omega \neq \mathbf{R}^n$, associamo ad esso l'operatore al bordo $\mathcal{B}$ di tipo conormale

$$\mathcal{B}(x,D) = \sum_{i,j=1}^{n} a_{ij}(x)\nu_i(x)D_j,$$

dove ν è la normale esterna al bordo $\partial\Omega$.
Nel Capitolo 3 forniamo delle ipotesi sui coefficienti di $\mathcal{A}$ and $\mathcal{B}$ affinché il problema

$$\begin{cases} \partial_t w - \mathcal{A}w = 0 & \text{in } (0,\infty)\times\Omega \\ w(0) = u_0 & \text{in } \Omega \\ \mathcal{B}w = 0 & \text{in } (0,\infty)\times\partial\Omega \end{cases} \tag{5}$$

abbia un'unica soluzione per ogni dato $u_0 \in L^1(\Omega)$ e tale soluzione sia tale che il gradiente e le derivate seconde spaziali soddisfacciano delle stime opportune in norma L^1. La scelta di condizioni al bordo di tipo conormale sembra la più naturale ai fini di quello che vogliamo misurare. Il metodo usato per provare l'esistenza di tale soluzione consiste nel dimostrare la settorialità di $(A_1, D(A_1))$ cioè della realizzazione di $\mathcal{A}(\cdot, D)$ in $L^1(\Omega)$ con condizioni omogenee al bordo $\mathcal{B}(\cdot, D) = 0$.
Per ottenere la settorialità di $(A_1, D(A_1))$ è stato necessario provare risultati di esistenza e unicità per problemi ellittici del tipo

$$\begin{cases} \lambda w - \mathcal{A}(\cdot, D)w = f & x \in \Omega \\ \mathcal{B}(\cdot, D)w = 0 & x \in \partial\Omega \end{cases} \tag{6}$$

con dati $f \in L^1(\Omega)$, insieme con alcune stime sul risolvente. Tali risultati sono stati ottenuti per dualità dalla teoria L^∞. Gli argomenti di dualità richiedono ovviamente esistenza per il problema duale e ipotesi di maggiore regolarità per i coefficienti. Tali ipotesi sono state successivamente indebolite con argomenti di perturbazione. Nelle ipotesi

$$a_{ij} = a_{ji} \in W^{2,\infty}(\Omega) \quad \text{and} \quad b_i, c \in L^\infty(\Omega).$$

e di uniforme ellitticità per la matrice $A = (a_{ij})_{ij}$,

$$\mu^{-1}|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \leq \mu|\xi|^2, \qquad x \in \overline{\Omega},\ \xi \in \mathbf{R}^n$$

con $\mu \geq 1$, dimostriamo l'esistenza di un semigruppo analitico e fortemente continuo in $L^1(\Omega)$ che fornisce la soluzione di (5). Per il dominio $D(A_1)$ ricaviamo l'immersione $D(A_1) \hookrightarrow W^{1,1}(\Omega)$, da cui deriva la forte continuità del semigruppo in $W^{1,1}$ per funzioni $u_0 \in D(A_1)$, cioè

$$\lim_{t\to 0} \|T(t)u_0 - u_0\|_{W^{1,1}(\Omega)} = 0$$

per ogni $u_0 \in D(A_1)$, e questo fatto costituisce un risultato più forte di quello cercato, almeno per funzioni nel dominio.
La parte più importante del Capitolo 3 consiste nel provare delle stime sulla norma L^1 del gradiente del semigruppo e sulle derivate seconde spaziali. La prima stima è

$$\|DT(t)u\|_{L^1(\Omega)} \leq \frac{C}{\sqrt{t}}\|u\|_{L^1(\Omega)} \quad t > 0,$$

che viene provata usando le stime sul risolvente $R(\lambda, A_1)$ e la rappresentazione del semigruppo in termini del risolvente. La stima provata sulle derivate seconde è

$$\|D^2T(t)u\|_{L^1(\Omega)} \leq \frac{C}{t}\|u\|_{L^1(\Omega)},$$

che nel caso di un dato iniziale più regolare diventa

$$t^\delta \|D^2T(t)u\|_{L^1(\Omega)} \leq C\|u\|_{W^{1,1}(\Omega)}, \quad t \in (0,1), \tag{7}$$

con $\delta \in (1/2, 1)$. La stima (7) sarà utile nel Capitolo 4 per stabilire un risultato di tipo monotonia per la funzione

$$F(t) = \int_\Omega |DT(t)u_0|\,dx$$

In particolare nella Proposizione 4.3.3 ricaviamo la seguente disuguaglianza per funzioni nel dominio di A_1

$$\int_\Omega \eta |DT(t)v|_A\,dx \leq \int_\Omega \eta |Dv|_A\,dx + Ct^{1-\delta}\|v\|_{W^{1,1}(\Omega)} \quad t \in (0,1)$$

dove $|Dv|_A$ denota la variazione totale di v pesata con la matrice dei coefficienti A (per la definizione si veda la Sezione 4.2) e η è una qualsiasi funzione non negativa di classe $C_b^1(\overline{\Omega})$. Tale risultato di monotonia e un risultato di approssimazione in variazione per funzioni BV ci permetteranno di concludere e quindi di caratterizzare la variazione totale di una funzione in $L^1(\Omega)$ in termini della norma L^1 del gradiente della soluzione del problema (5): la relazione

$$|Du_0|(\Omega) = \lim_{t\to 0}\int_\Omega |D(T(t)u_0)|\,dx \tag{8}$$

è verificata per ogni $u_0 \in L^1(\Omega)$. Pertanto ne segue che $u_0 \in BV(\Omega)$ se e solo se il limite al secondo membro in (8) è finito. In verità si riesce a provare una caratterizzazione anche delle funzioni BV con peso continuo e limitato (vedi Teorema 4.3.4).

Nel Capitolo 4 illustriamo una seconda caratterizzazione delle funzioni BV. Tale caratterizzazione è ottenuta utilizzando in modo differente il semigruppo $(T(t))_{t\geq 0}$ ed il suo comportamento per $t \to 0$. Mediante le stime del nucleo $p(t,x,y)$ associato al semigruppo $(T(t))_{t\geq 0}$ e strumenti di teoria della misura si ottiene dapprima una completa caratterizzazione per gli insiemi di perimetro finito in Ω e successivamente mediante la formula di coarea si riesce a generalizzare i risultati di [33] ed a provare che una data funzione $u \in L^1(\Omega)$ è a variazione limitata in Ω se e solo se

$$\liminf_{t\to 0} \frac{1}{\sqrt{t}} \int_{\Omega\times\Omega} p(t,x,y)|u(x)-u(y)|\,dx\,dy < \infty$$

e in tal caso

$$|Du|_A(\Omega) = \lim_{t\to 0} \frac{\sqrt{\pi}}{2\sqrt{t}} \int_\Omega \int_\Omega p(t,x,y)|u(x)-u(y)|\,dydx,$$

Il Capitolo 1 e le Appendici A e B contribuiscono a rendere quanto più possibile autosufficiente questo lavoro di tesi. Infatti, nel primo capitolo richiamiamo le principali definizioni e qualche risultato utile relativo alla teoria dei semigruppi ed alla teoria della misura. L'Appendice A è dedicata ad una breve introduzione riguardo la teoria dell'interpolazione reale e complessa. In essa si raccolgono definizioni, qualche risultato classico e un teorema di caratterizzazione per lo spazio di interpolazione reale

$$(L^1(\Omega), W^{2,1}(\Omega) \cap W^{1,1}_{A,\nu}(\Omega))_{\theta,1},$$

dove $W^{1,1}_{A,\nu}(\Omega)$ è la chiusura di $\{u \in C^1(\overline{\Omega}) \,|\, \langle A(x)\cdot\nabla u, \nu(x)\rangle = 0 \text{ per } x \in \partial\Omega\}$, rispetto alla topologia di $W^{1,1}(\Omega)$. Questo ci permette di caratterizzare gli spazi intermedi $D_{A_1}(\theta,1)$. In verità, la trattazione poteva essere fatta in maggiore generalità, ma è stato scelto un livello più vicino ai casi concreti effettivamente utilizzati nella tesi. Nell'ultima appendice raccogliamo stime Gaussiane dall'alto e dal basso per la soluzione fondamentale dell'operatore $\partial_t - \mathcal{A}$. Per dedurre le stime dal basso, trattiamo dapprima il caso simmetrico e successivamente estendiamo le stime ottenute al caso non simmetrico, che è quello di nostro interesse.

Introduction

Functions of bounded variation, usually denoted by BV, have had and have an important role in several problems of calculus of variations. The main features that make BV functions suitable for dealing with specific variational problems are their compactness properties, in connection with integral functionals with linear growth on the gradient, and their property of allowing for discontinuities along hypersurfaces, which is important in several geometrical and physical problems. The prototype of integral functional with linear growth on the gradient is the area functional, whereas, among variational problems with discontinuities, maybe the first success of the theory has been the complete solution of the isoperimetric problem in $\mathbf{R}^n$, and more recently free discontinuity problems (a term introduced by E. De Giorgi in [17]) have been studied. These problems come from image segmentation and smoothing and fracture mechanics, motivated by biology and physics, where digital image processing and the study of elasticity properties of materials are of relevant importance. Notice that Sobolev functions do not either share compactness properties as general as BV, or allow for $(n-1)$-dimensional discontinuity sets (like boundaries).

BV functions have nowadays a satisfactory theory that regards their functional properties, including approximation, embedding theorems, smoothing, boundary trace theorems and fine properties. For a systematic and self-contained treatment of the theory of functions of bounded variation we consider as main reference the book of L. Ambrosio, N. Fusco and D. Pallara [5]. Other references are the monographs of E. Giusti [23], U. Massari and M. Miranda [32], L. C. Evans and R. F. Gariepy [20], and W. P. Ziemer [49].

Given Ω an open subset of $\mathbf{R}^n$, functions with bounded variation in Ω are defined as those $L^1(\Omega)$ functions whose distributional derivative is representable by a finite $\mathbf{R}^n$-valued Radon measure, denoted by Du, whose total variation defined as

$$|Df|(\Omega) = \sup\left\{ \int_{\mathbf{R}^n} f \operatorname{div}\phi \, dx \ : \phi \in C_c^1(\Omega, \mathbf{R}^n), \|\phi\|_{L^\infty(\Omega)} \leq 1 \right\} \tag{1}$$

is finite. A particular case of interest is when $f = \chi_E$, the characteristic function of $E \subset \mathbf{R}^n$. In this case, we set $\mathcal{P}(E, \Omega) = |D\chi_E|(\Omega)$, and E is said to be a set of finite perimeter in Ω if $\mathcal{P}(E, \Omega) < \infty$.

The theory of BV functions is closely related to that of sets with finite perimeter. The link is established by the coarea formula, that relates the variation measure of u and the perimeter of its level sets:

$$|Du|(\Omega) = \int_{\mathbf{R}} \mathcal{P}(E_t, \Omega) dt, \tag{2}$$

where $E_t = \{x \in \Omega : \; u(x) > t\}$.
One of the starting points of this thesis is the paper [15], where De Giorgi defines for the first time the perimeter of a set. At that time, it was more or less clear (see also [10]) that a class of sets enjoying good geometric and variational properties would come from an approximation procedure. De Giorgi's idea was to start from a convolution with real analytic kernels. With the aim of extending the isoperimetric inequality and the Gauss-Green formula, for a given function $f \in L^\infty(\mathbf{R}^n)$, he defines the approximating functions as

$$W(t)f(x) = (4\pi t)^{-n/2} \int_{\mathbf{R}^n} e^{-\frac{|x-y|^2}{4t}} f(y)\, dy.$$

This choice of convolution kernel $G_t(x) = (4\pi t)^{-n/2} e^{-\frac{|x|^2}{4t}}$ has an advantage with respect to the compactly supported mollifiers, i.e., the function $W(t)f$ satisfies a semigroup law:

$$W(t+s)f(x) = W(t)W(s)f(x) \qquad t, s > 0.$$

In fact, the function $u(t,x) = W(t)f(x)$ is the solution of the parabolic problem

$$\begin{cases} \partial_t w(t,x) = \Delta w(t,x) & t \in (0,\infty), \; x \in \mathbf{R}^n \\ w(0,x) = f(x) & x \in \mathbf{R}^n \end{cases} . \tag{3}$$

The heat semigroup $(W(t))_{t\ge 0}$ is contractive on $L^1(\mathbf{R}^n)$ and commutes with the spatial derivatives, so that

$$\|DW(t+s)f\|_{L^1(\mathbf{R}^n)} = \|DW(t)W(s)f\|_{L^1(\mathbf{R}^n)} = \|W(t)DW(s)f\|_{L^1(\mathbf{R}^n)} \le \|DW(s)f\|_{L^1(\mathbf{R}^n)}$$

hence the function

$$t \mapsto \int_{\mathbf{R}^n} |DW(t)f|\, dx$$

is non increasing and the existence of the limit as $t \to 0$ is guaranteed.
In particular, given $E \subset \mathbf{R}^n$, De Giorgi defines the perimeter of E through the limit

$$P(E) := \lim_{t\to 0} \int_{\mathbf{R}^n} |DW(t)\chi_E|\, dx. \tag{4}$$

Now, since definition (4) makes sense also for functions in $L^1(\mathbf{R}^n)$, one could compute the limit in the right hand side of (4) (with a generic $f \in L^1(\mathbf{R}^n)$ in place of χ_E) and prove that

$$|Df|(\mathbf{R}^n) = \lim_{t\to 0} \int_{\mathbf{R}^n} |DW(t)f|\, dx, \tag{5}$$

i.e. that the limit in (5) coincides with the supremum in (1) for every $f \in L^1(\mathbf{R}^n)$.
The aim of this thesis is to investigate if the same result is true if $|Df|$ in (1) is replaced by a more general weighted variation of f, and the heat semigroup $(W(t))_{t\ge 0}$ in (5) is replaced by the semigroup generated by a general elliptic operator of second order in an open set $\Omega \subset \mathbf{R}^n$, with suitable boundary conditions.

Let us briefly describe the problem considered.
Let Ω be a (possibly unbounded) domain in $\mathbf{R}^n$ with uniformly C^2 boundary and let $\mathcal{A}$ be a uniformly elliptic second order operator in divergence form:

$$\mathcal{A}(x,D) = \sum_{i,j=1}^n D_i(a_{ij}(x)D_j) + \sum_{i=1}^n b_i(x)D_i + c(x). \tag{6}$$

If $\Omega \neq \mathbf{R}^n$ we consider the (conormal) operator $\mathcal{B}$ acting on the boundary $\partial\Omega$

$$\mathcal{B}(x,D) = \sum_{i,j=1}^n a_{ij}(x)\nu_i(x)D_j = \langle AD, \nu\rangle, \tag{7}$$

where ν is the outward unit normal to $\partial\Omega$ and $A = (a_{ij})$. We consider the following problem

$$\begin{cases} \partial_t w - \mathcal{A}w = 0 & \text{in } (0,\infty)\times\Omega \\ w(0) = u_0 & \text{in } \Omega \\ \mathcal{B}w = 0 & \text{in } (0,\infty)\times\partial\Omega. \end{cases}, \tag{8}$$

with initial datum $u_0 \in L^1(\Omega)$. Let us briefly comment on the homogeneous boundary condition $\langle ADw, \nu\rangle = 0$. In the simplest case when $\mathcal{A} = \Delta$ and $u_0 = \chi_E$ in (8), the natural boundary condition to obtain $\mathcal{P}(E,\Omega)$ as the limit as $t \to 0$ is the Neumann condition $\frac{\partial w}{\partial \nu} = 0$, because in this way the function u_0 is not immediately modified near the boundary, and then for short times the contribution of the gradient of the solution is significant only in the interior of Ω, thus measuring only the relative boundary of E. The natural extension of $\frac{\partial w}{\partial \nu} = 0$ in $(0,\infty)\times\partial\Omega$ when we consider a generic operator $\mathcal{A}$ is $\langle ADw, \nu\rangle = 0$ in $(0,\infty)\times\partial\Omega$.
In order to study our problem, it has proved to be convenient to translate it in the language of semigroups, and exploit the relative techniques. This leads us to consider the realization $A_1 : D(A_1) \subset L^1(\Omega) \to L^1(\Omega)$ of $\mathcal{A}$ in $L^1(\Omega)$, where the domain $D(A_1)$ takes into account the boundary conditions. We shall prove that $(A_1, D(A_1))$ is sectorial in $L^1(\Omega)$, hence it is the generator of an analytic semigroup $(T(t))_{t\geq 0}$.
In order to prove that a linear operator $A : D(A) \subset X \to X$ is sectorial it is needed to prove first of all that the resolvent set $\rho(A)$ contains a sector

$$\Sigma_\theta = \{\lambda \in \mathbf{C} : \ \lambda \neq \omega, |\arg(\lambda - \omega)| < \theta\},$$

with $\omega \in \mathbf{R}$ and $\theta > \frac{\pi}{2}$; then, that there is $M > 0$ such that the resolvent operator of A, $R(\lambda, A) = (\lambda - A)^{-1}$ verifies

$$\|R(\lambda, A)\| \leq M/|\lambda - \omega| \quad \text{for } \lambda \in \Sigma_\theta. \tag{9}$$

For the first requirement one has to prove existence and uniqueness of the solution of elliptic boundary value problems in $L^1(\Omega)$.
Basically, two ways are known to show the sectoriality of $(A_1, D(A_1))$. One is based on the integral representation

$$(T(t)f)(x) := \int_\Omega p(t,x,y)f(y)\,dy, \tag{10}$$

and consists in proving the existence of the kernel p, and subsequently in deriving suitable estimates on p and its derivative. Relying on earlier ideas of R. Beals and L. Hörmander, this point of view is deeply pursued by H. Tanabe in his book [45].
The other way is based on a duality argument. There is a serious obstruction in extending to $L^1(\Omega)$ the L^p-theory $(1 < p < \infty)$, because the classical Calderón-Zygmund and Agmon-Douglis-Nirenberg estimates are known to fail for $p = 1, \infty$. A way to circumvent this difficulty for $p = \infty$ has been devised by K. Masuda and H. B. Stewart (see [42], [43] and also [31]) and consists in a clever passage to the limit as $p \to \infty$ in the L^p estimates. Then, a duality argument can be used to pass from L^∞ estimates to L^1 estimates and the sectoriality in $L^1(\Omega)$. This has been done in the case Ω bounded and Dirichlet boundary conditions by G. Di Blasio [18], H. Amann [4], A. Pazy [35], and D. Guidetti [24] for the case of elliptic systems in L^1. In the same vein, we have proved sectoriality of $(A_1, D(A_1))$ in $L^1(\Omega)$ for Ω (possibly) unbounded and homogeneous co-normal boundary conditions. After proving the existence and analyticity of the semigroup $(T(t))_{t\geq 0}$, we need precise estimates on the first and second order derivatives, in order to prove that the limit in (5) exists, and to evaluate it.

Let us come to our standing hypotheses.
We suppose that the operator $\mathcal{A}$ has real valued coefficients satisfying the following assumptions

$$a_{ij} = a_{ji} \in W^{2,\infty}(\Omega) \quad \text{and} \quad b_i, c \in L^\infty(\Omega).$$

and that the uniform ellipticity condition holds, namely there exists a positive constant $\mu \geq 1$ such that for any $x \in \overline{\Omega}$ and $\xi \in \mathbf{R}^n$

$$\mu^{-1}|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \leq \mu|\xi|^2.$$

With these assumptions on the coefficients it turns out that $(A_1, D(A_1))$, where $D(A_1)$ is the closure in the graph norm $\|\cdot\|_{L^1(\Omega)} + \|A\cdot\|_{L^1(\Omega)}$ of the space

$$\{u \in L^1(\Omega) \cap C^2(\bar{\Omega});\ \mathcal{A}u \in L^1(\Omega), \mathcal{B}u = 0 \text{ in } \partial\Omega\},$$

is a sectorial operator so it generates a bounded analytic semigroup $T(t)$ in L^1, and $T(t)u_0$ is the solution of

$$\begin{cases} \partial_t w(t,x) = \mathcal{A}w(t,x) & t \in (0,\infty),\ x \in \Omega \\ w(0,x) = u_0(x) & x \in \Omega \\ \mathcal{B}w(t,x) = 0 & t \in (0,\infty),\ x \in \partial\Omega \end{cases} \tag{11}$$

By the density of $D(A_1)$ in L^1 and the fact that $D(A_1) \hookrightarrow W^{1,1}(\Omega)$ (see Remark 3.0.6) we can also deduce that $T(t)$ is strongly continuous in $D(A_1)$ with respect to the $W^{1,1}$ norm, and that

$$\lim_{t\to 0} \|T(t)u_0 - u_0\|_{W^{1,1}(\Omega)} = 0 \tag{12}$$

for every $u_0 \in D(A_1)$. Formula (12) implies the convergence of $\|DT(t)u_0\|_{L^1(\Omega)}$ to the total variation of Du_0 as $t \to 0$.

But for general $f \in L^1(\Omega)$ the existence of the limit in the right hand side of (5), with $T(t)$ in place of $W(t)$, relies on precise estimates on the first and second order derivatives of $T(t)f$. We prove that, for every $t > 0$, the inequalities

$$\begin{aligned}\|DT(t)u\|_{L^1(\Omega)} &\leq \frac{C}{\sqrt{t}}\|u\|_{L^1(\Omega)}\\ \|D^2T(t)u\|_{L^1(\Omega)} &\leq \frac{C}{t}\|u\|_{L^1(\Omega)}\end{aligned} \tag{13}$$

hold for every $u \in L^1(\Omega)$ and some constant $C > 0$ independent of u. Estimate (13) has to be improved to go ahead, and the improvement is obtained via a characterization of the interpolation space between the domain $D(A_1)$ and $L^1(\Omega)$. As a consequence, we prove that there exists $\delta \in (1/2, 1)$ such that

$$t^\delta \|D^2T(t)u\|_{L^1(\Omega)} \leq C\|u\|_{W^{1,1}(\Omega)} \quad t \in (0,1) \tag{14}$$

holds for every $u \in D(A_1)$ and for some constant $C > 0$. Estimate (14) will be very useful to estimate the "defect of monotonicity" of the function

$$F(t) = \int_\Omega |DT(t)u_0|\, dx. \tag{15}$$

Actually, we prove that for $\delta \in (1/2, 1)$ as in (14) the inequality

$$\int_\Omega \eta |DT(t)v|_A\, dx \leq \int_\Omega \eta |Dv|_A\, dx + Ct^{1-\delta}\|v\|_{W^{1,1}(\Omega)} \qquad t \in (0,1) \tag{16}$$

holds for $v \in D(A_1)$ and for any nonnegative function $\eta \in C_b^1(\overline{\Omega})$. In (16), $|Dv|_A$ denotes the A-variation of Dv, namely the total variation weighted by the matrix of the coefficients $A = (a_{ij})_{ij}$ defined as follows

$$|Du|_A(\Omega) = \sup\left\{\int_\Omega u\mathrm{div}\psi dx : \psi \in C_c^1(\Omega, \mathbf{R}^n), \|A^{-1/2}\psi\|_\infty \leq 1\right\}.$$

Finally, using (16) and a result of approximation in variation for BV functions via functions belonging to $D(A_1)$, we get that the total variation of u_0 is the limit as $t \to 0$ of the L^1 norm of the gradient of $T(t)u_0$, that is the following equality

$$|Du_0|(\Omega) = \lim_{t\to 0} \int_\Omega |D(T(t)u_0)|\, dx \tag{17}$$

holds for every $u_0 \in L^1(\Omega)$. As a consequence we get that $u_0 \in BV(\Omega)$ if and only if the above limit is finite. Let us point out that the previous characterization holds not only for classical BV functions, but also for weighted BV functions (see Theorem 4.3.4).
The proof of estimate (14) for the derivatives is a quite long tour. Following ideas introduced by V. Vespri in [47] and [48] for Dirichlet boundary conditions, we study the semigroup $(T(t))_{t\geq 0}$ in Sobolev spaces of negative order and use a complex interpolation result. We remark that in some intermediate steps (mainly, when we deal with the adjoint operator of $\mathcal{A}$) we need to assume higher regularity on the coefficients. However, a perturbation result will allow us to come back to the initial assumptions.

We study also another connection between the short-time behavior of the semigroup $(T(t))_{t\geq 0}$ in $L^1(\Omega)$ and $BV(\Omega)$. In fact, this leads to a second characterization for BV functions. In this part, we use the integral representation (10) of the semigroup and the relative estimates quoted at the beginning of this Introduction.
More precisely we extend the results in [33], where the authors prove that a given function $u \in L^1(\mathbf{R}^n)$ is a function with bounded variation if and only if

$$\liminf_{t\to 0} \frac{1}{\sqrt{t}} \int_{\mathbf{R}^n\times\mathbf{R}^n} |u(x)-u(y)|G_t(x-y)\,dx\,dy < \infty$$

and in that case its total variation can be written as

$$|Du|(\mathbf{R}^n) = \lim_{t\to 0} \frac{\pi}{2\sqrt{t}} \int_{\mathbf{R}^n\times\mathbf{R}^n} |u(x)-u(y)|G_t(x-y)\,dx\,dy. \tag{18}$$

In order to extend (18) to functions with bounded variation in the domain Ω, we first consider the special case of the characteristic functions and we characterize sets with finite perimeter in Ω. We prove that if $E \subset \mathbf{R}^n$ is such that either E or E^c has finite measure in Ω, then E has finite perimeter in Ω if and only if

$$\liminf_{t\to 0} \frac{1}{\sqrt{t}} \int_{E^c\cap\Omega} T(t)\chi_E(x)dx < +\infty,$$

and in this case the following equality holds

$$\lim_{t\to 0} \sqrt{\frac{\pi}{t}} \int_{\Omega\cap E^c} T(t)\chi_E dx = \int_{\Omega\cap\mathcal{F}E} |A^{1/2}(x)\nu_E(x)|d\mathcal{H}^{n-1}(x), \tag{19}$$

where $\mathcal{F}E$ is the reduced boundary of E (see Definition 4.5). We remark that the right hand side of (19) reduces to the classical perimeter when $A = I$, since $\mathcal{P}(E,\Omega) = \mathcal{H}^{n-1}(\mathcal{F}E\cap\Omega)$. Then, using (19) in connection with the coarea formula (2), we prove that a given function $u \in L^1(\Omega)$ is of bounded variation if and only if

$$\liminf_{t\to 0} \frac{1}{\sqrt{t}} \int_\Omega\int_\Omega p(t,x,y)|u(x)-u(y)|\,dydx < \infty$$

and its A-variation can be written as follows

$$|Du|_A(\Omega) = \lim_{t\to 0} \frac{\sqrt{\pi}}{2\sqrt{t}} \int_{\Omega\times\Omega} p(t,x,y)|u(x)-u(y)|\,dydx. \tag{20}$$

Here, p is the kernel in (10).
Important tools for this second characterization are also the results of geometric measure theory concerning the structure of sets of finite perimeter and in particular their blow-up properties. We remark that this characterization is also in the spirit of [8], [14] and [27], where only kernels depending on $|x-y|$ are considered.

The two characterization of BV functions in terms of the short-time behavior of semigroups, described below, have been published in [6]. However we point out that the proofs in [6] rely on the kernel estimates recalled in Theorem B.1.1, whereas here we use such estimates only in Chapter 5. In fact, in this thesis the construction and the analysis of the semigroup $(T(t))_{t\geq 0}$, as well as the characterization of BV in Chapter 4, are independent of the kernel estimates and are rather based on the study of the resolvent equation. In this respect, the estimates we get are self-contained, and, even though the methods are based on previous works mainly confined to the Dirichlet problem, our presentation as a whole is original.

Let us describe the contents of the thesis. We tried to be as self-contained as possible, so we start in Chapter 1 by recalling some basic definitions and the most important properties of semigroups and a few relevant notions of measure theory. Mainly following [19] for the first part and [5] for the second one, we state (often without proof) some classical theorems that will be used throughout the thesis and fix our notation. We recall the main properties of sectorial operators and some perturbation results. Moreover analytic semigroup and intermediate spaces are mentioned in the first part. The second part consists in definitions and useful results of measure theory. Finally, Section 1.5 contains a collection of analytical tools helpful in the sequel.
Chapter 2 is devoted to results of generation of analytic semigroups in suitable Banach spaces. Since we get generation in $L^1(\Omega)$ from analogous results in L^∞ by duality and since the L^∞ theory makes use of that in L^p, $1 < p < \infty$, we start by recalling some classical result of generation in L^p spaces. Then, following [42] and [43], we deduce generation for elliptic operator with non tangential boundary conditions in the space of essentially bounded functions. Thus, using the adjoint boundary value problems in L^∞, we get existence and the estimate (9) for the solution of the elliptic boundary value problem associated with $\mathcal{A}$ and $\mathcal{B}$ in L^1. We also study elliptic boundary value problems in the dual space of some Sobolev spaces to deduce by duality estimates for the gradient of the resolvent operator $R(\lambda, A_1)$.
In Chapter 3 we derive estimates for the L^1 norm of the semigroup $T(t)$ generated by $(A_1, D(A_1))$. Other useful estimates are established for the first and the second order spatial derivatives of $T(t)$ also by mean of the characterization of some new real interpolation spaces.
After a brief introduction on the possibly weighted BV functions and sets of weighted finite perimeter we collect in Chapter 4 their main properties. In particular, a version of the classical Anzellotti-Giaquinta approximation theorem is derived, and a weighted version of the coarea formula is also shown. In the simplest case of the Laplacian defined in a convex domain with homogeneous Neumann boundary condition on $\partial\Omega$, the function F in (15) can be easily proved to be non increasing by differentiating under the integral sign. We remark that in such framework the convexity of the domain is essential: in fact a counterexample to the monotonicity is provided in [22]. In general, when we consider a generic operator like $\mathcal{A}$, the same procedure does not work as well as in the previous case as we do not get monotonicity. However estimate (16) and the approximation results allow us to conclude, without convexity assumption on Ω. The first part of Chapter 5 is devoted to collect known results concerning some connections between semigroups and perimeter. In particular we refer to [27], where Ledoux connects the L^2 norm of the heat semigroup in $\mathbf{R}^n$ with the isoperimetric inequality, and to [33] for the characterization

of the perimeter of a set $E \subset \mathbf{R}^n$ in terms of the behavior of

$$\int_{\mathbf{R}^n \setminus E} W(t)\chi_E \, dx$$

as $t \to 0$. Then we extend this latter result and we provide a second characterization for sets of finite perimeter and functions with bounded variation in Ω.
At the end of the thesis there are two appendices. The first one consists in an elementary treatment of the real and complex interpolation theory. Moreover a new characterization of a real interpolation space is given. More precisely, we prove that if $\theta \in (0, 1/2)$ the real interpolation space

$$(L^1(\Omega), W^{2,1}(\Omega) \cap W^{1,1}_{A,\nu}(\Omega))_{\theta,1},$$

where $W^{1,1}_{A,\nu}(\Omega)$ is the closure of $\{u \in C^1(\overline{\Omega}) \,|\, \langle A(x) \cdot Du, \nu(x)\rangle = 0 \text{ for } x \in \partial\Omega\}$ with respect to the topology of $W^{1,1}(\Omega)$, consists of functions that are in the fractional Sobolev space $W^{2\theta,1}(\Omega)$. This fact will be used in Chapter 3 to characterize the intermediate space $D_{A_1}(\theta, 1)$. Finally a brief recall on the complex interpolation spaces is provided in Section A.3. We present this argument in a quite general context, which still is not the most general possible, but is close to our applications.
In Appendix B we gather up some Gaussian upper and lower bounds for the integral kernel p in (10), (20). For the Gaussian lower bounds we study first the symmetric case then, the estimates are extended to the non-symmetric one. More details about this matter can be found in [34].

Lecce, 22 Maggio 2008

Acknowledgments

I would like to express my deep gratitude to my advisor, Diego Pallara, for all the mathematics and the scientific enthusiasm he has transmitted to me during these years. I also thank him for his friendship and his incomparable helpfulness.
I warmly thank Fabio Paronetto and Michele Miranda for many stimulating discussions and for having shared with me part of the research work contained in this thesis.
It is also a pleasure to thank Giorgio Metafune who, during these years, has held interesting courses which have been useful for my research studies.
Many thanks go also to all my colleagues for the real fun we have had together during these years.
I have no words to thank Massimiliano and my family who have always been present and have supported me with love especially in the hard times.

Contents

Chapter 1

Preliminaries and auxiliary results

In this chapter we collect some basic tools on the main topics used throughout the thesis. We recall the basic definitions and the most important properties of semigroups theory and measure theory. These recalls are only intended to fix some notations and references and are confined to what will be useful in the sequel. For what concerns the results on semigroups and sectorial operators we refer to [31], [19] while a more deep analysis concerning results of measure theory can be found in [5] and [20].

1.1 Recall on semigroups theory

One of our aims is to prove existence, uniqueness and regularity properties for the solution of the following parabolic second order problem

$$\begin{cases} u_t(t,x) = Au(t,x) & t>0,\, x\in\Omega \\ u(0,x) = f(x) & x\in\Omega \\ Bu(t,x) = 0 & t>0,\, x\in\partial\Omega \end{cases}$$

where A is a linear second order operator in divergence form and B is a non-tangential first order differential operator defined on $\partial\Omega$, and the initial datum f is taken in $L^1(\Omega)$. This problem is studied as an abstract Cauchy problem in a suitable Banach space,

$$\begin{cases} u'(t) = Au(t), & t>0 \\ u(0) = x \end{cases} \tag{1.1}$$

by looking at the semigroup generated by A with a suitable domain. Here X is a complex Banach space with norm $\|\cdot\|_X$, $A : D(A) \subset X \to X$ is a linear operator and $x \in X$. Of course the solution of (1.1) and its properties depend upon the class of operators considered.

In our case the operator A will be sectorial (see Definition 1.2.1 below). This ensures that the solution of (1.1) admits an integral representation with a complex contour integral and the solution map $t \mapsto u(t,x)$ of (1.1) is given by an analytic semigroup (see Definition 1.2.2).

1.2 Sectorial operators

Definition 1.2.1. *Let $A : D(A) \subset X \to X$ be a linear operator. We say that A is sectorial if there exist $\omega \in \mathbf{R}$, $\theta \in]\frac{\pi}{2}, \pi[$, $M > 0$ such that*

$$\rho(A) \supset \Sigma_{\theta,\omega} = \{\lambda \in \mathbf{C}; \lambda \neq \omega, |\arg(\lambda - \omega)| < \theta\} \tag{1.2}$$

$$\|R(\lambda, A)\|_{\mathcal{L}(X)} \leq \frac{M}{|\lambda - \omega|} \qquad \forall\, \lambda \in \Sigma_{\theta,\omega}. \tag{1.3}$$

Here the resolvent set $\rho(A)$ is the set $\{\lambda \in \mathbf{C} : \exists (\lambda - A)^{-1} \in \mathcal{L}(X)\}$ and for $\lambda \in \rho(A)$, $R(\lambda, A)$ denotes the resolvent operator $(\lambda - A)^{-1}$.
A sectorial operator is immediately closed since its resolvent set is not empty, hence its domain $D(A)$, endowed with the graph norm $\|x\|_{D(A)} = \|x\|_X + \|Ax\|_X$, is a Banach space. Conditions (1.2) and (1.3) guarantee that the linear operator e^{tA}, defined for $t \geq 0$ as follows

$$e^{0A} := I, \quad e^{tA} := \frac{1}{2\pi i} \int_{\omega + \gamma_{r,\eta}} e^{t\lambda} R(\lambda, A)\, d\lambda, \qquad t > 0, \tag{1.4}$$

where $r > 0$, $\eta \in (\frac{\pi}{2}, \theta)$, and

$$\gamma_{r,\eta} = \{\lambda \in \mathbf{C}; |\arg \lambda| = \eta, |\lambda| \geq r\} \cup \{\lambda \in \mathbf{C}; |\arg \lambda| \leq \eta, |\lambda| = r\}$$

oriented counterclockwise, is well defined and independent of $r > 0$ and $\eta \in (\frac{\pi}{2}, \theta)$.

Before stating the basic properties of e^{tA}, we recall when a family of operators $(T(t))_{t \geq 0} \subset \mathcal{L}(X)$ is called a semigroup.

Definition 1.2.2. *(Analytic semigroup) A family of operators $(T(t))_{t\geq 0} \subset \mathcal{L}(X)$ is called a semigroup if*

$$T(0) = I \quad \text{and} \quad T(t+s) = T(t)T(s) \quad t, s \geq 0.$$

It is said to be strongly continuous *if for each $x \in X$ the function $t \mapsto T(t)x$ is continuous in $[0, +\infty[$. Moreover it is called an* analytic semigroup *of angle $\delta \in]0, \pi/2]$ if the function $z \mapsto T(z)$ is analytic in the sector $\Sigma_\delta = \{z \in \mathbf{C} : |\arg z| < \delta\}$ and for every $0 < \delta' < \delta$ and $x \in Y$, being Y a closed subspace of X, it holds that*

$$\lim_{\substack{z \to 0 \\ z \in \Sigma_{\delta'}}} T(z)x = x.$$

Proposition 1.2.3. *Let $A : D(A) \subset X \to X$ be a sectorial operator, and $(e^{tA})_{t\geq 0}$ defined as in (1.4). Then the following properties hold:*

(i) $e^{tA}x \in D(A^k)$ *for each* $t > 0$, $x \in X$, $k \in \mathbf{N}$. *Moreover if* $x \in D(A^k)$ *then*

$$A^k e^{tA} x = e^{tA} A^k x, \quad t \geq 0; \tag{1.5}$$

(ii) $e^{(t+s)A} = e^{tA} e^{sA}, \qquad t, s \geq 0;$

(iii) *there are constants* M_i, $i = 0, \dots, k$, *such that*

$$\|e^{tA}\|_{\mathcal{L}(X)} \leq M_0 e^{\omega t}, \quad t > 0,$$
$$\|t^k (A - \omega I)^k e^{tA}\|_{\mathcal{L}(X)} \leq M_k e^{\omega t}, \quad t > 0, \tag{1.6}$$

where ω *is given in Definition 1.2.1*

(iv) *the function* $t \mapsto e^{tA}$ *belongs to* $C^\infty((0, \infty); \mathcal{L}(X))$ *and*

$$\frac{d^k}{dt^k} e^{tA} = A^k e^{tA}, \quad t > 0.$$

Moreover, it has an analytic extension in the sector

$$\Sigma_{\theta - \frac{\pi}{2}} = \{\lambda \in \mathbf{C} : \lambda \neq 0, |\arg \lambda| < \theta - \pi/2\}.$$

These properties motivate the following definition.

Definition 1.2.4. *Let* $A : D(A) \subset X \to X$ *be a sectorial operator. The family* $(e^{tA})_{t \geq 0}$ *defined by* (1.4) *is said to be the analytic semigroup generated by* A *in* X.

Analogously one can prove that $\{e^{tA}\}_{t \geq 0}$ is strongly continuous if and only if the domain $D(A)$ is dense in X, indeed $\lim_{t \to 0} e^{tA} x = x$ if and only if $x \in \overline{D(A)}$.

The following results solve the problem of identifying the generator of a given analytic semigroup. In the next lemma an integral representation of the resolvent of A in terms of the semigroup generated by A is given. The following proposition states that for a given analytic semigroup $\{T(t)\}_{t \geq 0}$ there exists a sectorial operator A such that $T(t) = e^{tA}$.

Lemma 1.2.5. *Let* $A : D(A) \subset X \to X$ *be as in Definition 1.2.1. Then for every* $\lambda \in \mathbf{C}$ *such that* $\mathrm{Re}\,\lambda > \omega$ *we have*

$$R(\lambda, A) = \int_0^\infty e^{-\lambda t} e^{tA}\, dt. \tag{1.7}$$

Proposition 1.2.6. *Let* $\{T(t)\}_{t > 0}$ *be a family of linear bounded operators such that* $t \mapsto T(t)$ *is differentiable with values in* $\mathcal{L}(X)$ *and verifies*

(i) $T(t)T(s) = T(t + s)$ *for every* $t, s > 0$;

(ii) $\|T(t)\|_{\mathcal{L}(X)} \leq M_0 e^{\omega t}$, $\|t \frac{dT(t)}{dt}\|_{\mathcal{L}(X)} \leq M_1 e^{\omega t}$ *for some* $\omega \in \mathbf{R}$, $M_0, M_1 > 0$

(iii) $\lim_{t \to 0} T(t) x = x$ *for every* $x \in X$.

Then $t \mapsto T(t)$ is analytic in $(0,\infty)$ with values in $\mathcal{L}(X)$, and there exists a unique sectorial operator $A : D(A) \subset X \to X$ such that $T(t) = e^{tA}$ for every $t \geq 0$.

Let us give a sufficient condition, seemingly weaker than (1.2)-(1.3), in order that a linear operator be sectorial. It will be useful to prove that the realizations of some elliptic partial differential operators are sectorial in the usual function spaces.

Proposition 1.2.7. *Let $A : D(A) \subset X \to X$ be a linear operator such that $\rho(A)$ contains a half plane $\{\lambda \in \mathbf{C};\ \mathrm{Re}\,\lambda \geq \omega\}$, and*

$$\|\lambda R(\lambda, A)\|_{\mathcal{L}(X)} \leq M, \quad \mathrm{Re}\,\lambda \geq \omega, \tag{1.8}$$

with $\omega \in \mathbf{R}$, $M > 0$. Then A is sectorial.

Proof. By using the fact that if $\lambda_0 \in \rho(A)$ then the ball

$$\{\lambda \in \mathbf{C};\ |\lambda - \lambda_0| < \|R(\lambda_0, A)\|^{-1}_{\mathcal{L}(X)}\}$$

is contained in $\rho(A)$, we get that for every $r > 0$ the resolvent set of A contains the open ball centered at $\omega + ir$ with radius $|\omega + ir|/M$. The union of such balls contains the sector $S = \{\lambda \neq \omega :\ |\arg(\lambda - \omega)| < \pi - \mathrm{arctan}M\}$. Moreover, for $\lambda \in V = \{\lambda : \mathrm{Re}\,\lambda < \omega,\ |\arg(\lambda - \omega)| \leq \pi - \arctan(2M)\}$, $\lambda = \omega + ir - \theta r/M$ with $0 < \theta \leq 1/2$, we can write

$$R(\lambda, A) = \sum_{k=0}^{\infty} (-1)^k (\lambda - \omega - ir)^k R^{k+1}(\omega + ir, A)$$

therefore

$$\|R(\lambda, A)\|_{\mathcal{L}(X)} \leq \sum_{k=0}^{\infty} |\lambda - (\omega + ir)|^k \frac{M^{k+1}}{(\omega^2 + r^2)^{\frac{k+1}{2}}} \leq \frac{2M}{r}.$$

On the other hand, since $\lambda = \omega + ir - \theta r/M$, the following estimate holds

$$r \geq (1/(4M^2) + 1)^{-1/2} |\lambda - \omega|.$$

Finally

$$\|R(\lambda, A)\|_{\mathcal{L}(X)} \leq 2M(1/(4M^2) + 1)^{1/2} |\lambda - \omega|^{-1}$$

and the claim is proved. □

Thus in order to prove sectoriality for a given elliptic operator one needs to prove

(i) existence and uniqueness for the solution of a boundary value problem of the type

$$\begin{cases} \lambda u(x) - Au(x) = f(x) & \text{in } \Omega\,, \\ Bu(x) = g(x) & \text{in } \partial\Omega\,, \end{cases}$$

at least for $\mathrm{Re}\,\lambda$ large, and

(ii) the resolvent estimate (1.8).

1.2.1 Perturbation of sectorial operators

When dealing with second order partial differential operators, it is often easier to study operators with smooth coefficients or without lower order terms. Subsequently, one can try to remove the smoothness assumption by using an approximation argument and to add lower order terms with a perturbation argument. In this case it is important to know that sectoriality is preserved and this can be guaranteed by an abstract perturbation result. More specifically, let $A : D(A) \subset X \to X$ be a sectorial operator, generator of the analytic semigroup $(T(t))_{t\geq 0}$, and consider another operator $B : D(B) \subset X \to X$. The perturbation theory gives conditions under which the sum $A+B$ is a sectorial operators, too, and therefore generates itself an analytic semigroup.
If B is "small" with respect to A, in a suitable sense, we say that the operator A is *perturbed* by the operator B or that B is a *perturbation* of A. Before stating the main result we need in the sequel, we observe that the sum $A+B$ defined in the natural way

$$(A+B)x := Ax + Bx$$

and it is meaningful only for

$$x \in D(A+B) := D(A) \cap D(B),$$

a subspace that in general could reduce to $\{0\}$.
We start with a theorem of perturbation (whose proof can be found in [19]) where the simplest case, that is the case in which the perturbing operator is bounded, is considered. In this case, of course, $D(B) = X$.

Theorem 1.2.8. *Let $(A, D(A))$ be the generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$ on a Banach space X satisfying $\|T(t)\| \leq Me^{\omega t}$ for every $t \geq 0$, $\omega \in \mathbf{R}$ and $M \geq 1$. If $B \in \mathcal{L}(X)$, then*

$$A + B \quad \textit{with} \quad D(A+B) := D(A)$$

generates a strongly continuous semigroup $(S(t))_{t\geq 0}$ satisfying

$$\|S(t)\| \leq Me^{(\omega + M\|B\|)t} \quad t \geq 0.$$

Moreover if $(T(t))_{t\geq 0}$ is analytic, then so is the semigroup $(S(t))_{t\geq 0}$ generated by $A+B$.

Whereas a bounded perturbation of an operator preserves its properties, the sum of two unbounded operators raises more delicate questions since the domain $D(A) \cap D(B)$ can be too small and the good properties of single operators can be destroyed in the sum. For this reason we need a definition for perturbing operators for which this situation is avoided.

Definition 1.2.9. *Let $A : D(A) \subset X \to X$ be a linear operator on the Banach space X. An operator $B : D(B) \subset X \to X$ is called* A-bounded *if $D(A) \subseteq D(B)$ and if there exist constants $a, b \in \mathbf{R}^+$ such that*

$$\|Bx\| \leq a\|Ax\| + b\|x\| \tag{1.9}$$

for all $x \in D(A)$. The A-bound *of B is*

$$a_0 := \inf\{a \geq 0 : \textit{there exists } b \in \mathbf{R}_+ \textit{ such that } (1.9) \textit{ holds}\}.$$

Finally we prove a useful perturbation theorem that will be used later.

Theorem 1.2.10. *Let $A : D(A) \subset X \to X$ be a sectorial operator and let $B : D(B) \subset X \to X$ be a A-bounded operator with A-bound a_0. Then there exists a constant $\alpha > 0$ such that if $a_0 < \alpha$, then $A + B : D(A) \to X$ is sectorial.*

Proof. Let $\omega \in \mathbf{R}$ be such that $R(\lambda, A)$ exists and $\|\lambda R(\lambda, A)\| \leq M$ for $\operatorname{Re}\lambda \geq \omega$. We write $\lambda - A - B = (I - BR(\lambda, A))(\lambda - A)$ and we observe that

$$\|BR(\lambda, A)x\| \leq a\|AR(\lambda, A)x\| + b\|R(\lambda, A)x\| \leq \big(a(M+1) + \frac{bM}{|\lambda|}\big)\|x\| \leq \frac{1}{2}\|x\|$$

if $a(M+1) \leq 1/4$ and $bM/|\lambda| \leq 1/4$. Therefore, if $a \leq \alpha := (4(M+1))^{-1}$ and for $\operatorname{Re}\lambda$ sufficiently large, $\|BR(\lambda, A)\| \leq 1/2$ and

$$\|(\lambda - A - B)^{-1}\| \leq \|R(\lambda, A)\|\|(I - BR(\lambda, A))^{-1}\| \leq \frac{2M}{|\lambda|}$$

The statement now follows from Proposition 1.2.7. □

1.3 Analytic semigroups and spaces $D_A(\theta, p)$

In this section we present some results on the intermediate spaces $D_A(\theta, p)$ coming from a sectorial operator A. The classical results on interpolation between Banach spaces are collected in Appendix A. The definition of the spaces $D_A(\theta, p)$ is due to H. Berens and P. L. Butzer [9]. They can be defined in several different ways, one of them comes out from the behavior of $AT(t)x$ near $t = 0$. We have seen in Proposition 1.2.3 that, for each $x \in X$, $\|tAT(t)x\|$ is bounded in $(0, 1)$, whereas, for every $x \in D(A)$, $\|AT(t)x\|$ is bounded in $(0, 1)$. This behavior of $AT(t)$ leads to the definition of a class of intermediate spaces between X and $D(A)$. In this section we set $1/\infty = 0$.

Definition 1.3.1. *Let $0 < \theta < 1$, $1 \leq p \leq \infty$, and $(\theta, p) = (1, \infty)$, we set*

$$D_A(\theta, p) = \{x \in X : \ t \mapsto \|t^{1-\theta-1/p}AT(t)x\| \in L^p(0, 1)\}$$

endowed with the norm

$$\|x\|_{D_A(\theta,p)} = \|x\|_X + [x]_{D_A(\theta,p)},$$

where $[x]_{D_A(\theta,p)} = \|t^{1-\theta-1/p}AT(t)x\|_{L^p(0,1)}$. Define

$$D_A(\theta) = \{x \in D_A(\theta, \infty) : \ \lim_{t \to 0} t^{1-\theta}AT(t)x = 0\}.$$

Now, we state an important characterization of the space $D_A(\theta, p)$ that will be used in the sequel and whose proof can be found in [9, Theorem 3.4.2 and 3.5.3]. We denote by $(X, Y)_{\theta,p}$ the real interpolation space between X and Y.

Theorem 1.3.2. *Assume that $(A, D(A))$ generates an analytic semigroup on a Banach space X. Then for $0 < \theta < 1$ and $1 \le p \le \infty$, and for $(\theta, p) = (1, \infty)$ we have*

$$D_A(\theta, p) = (X, D(A))_{\theta,p}$$

moreover, for $0 < \theta < 1$,

$$D_A(\theta) = (X, D(A))_\theta$$

with equivalence of the respective norms.

The previous characterization provides several properties of these spaces deduced from the similar ones of the real interpolation spaces (see Appendix A). Some of these properties are recalled in the following corollary.

Corollary 1.3.3. *(i) Suppose that A and B generate bounded analytic semigroups in X. If $D(A) = D(B)$ (with equivalence of the norms) then*

$$D_A(\theta, p) = D_B(\theta, p) \quad \text{and} \quad D_A(\theta) = D_B(\theta).$$

(ii) The spaces $D_A(\theta, p)$ and $D_A(\theta)$ belong to the class J_θ between X and $D(A)$, i.e., there is a constant $c > 0$ such that

$$\|x\|_{D_A(\theta,p)} \le c\|x\|_X^{1-\theta}\|x\|_{D(A)}^\theta \quad \forall\, x \in D(A).$$

(iii) For $0 < \theta_1 < \theta_2 < \infty$ and $1 \le p \le \infty$ and for $(\theta_2, p) = (1, \infty)$, we have

$$D_A(\theta_2, p) \subset D_A(\theta_1, p).$$

For $0 < \theta < 1$, $1 \le p_1 \le p_2 < \infty$,

$$D_A(1, \infty) \subset D_A(\theta, p_1) \subset D_A(\theta, p_2) \subset D_A(\theta) \subset D_A(\theta, \infty) \subset \overline{D(A)}.$$

Now we give an useful estimate for the function $t \mapsto A^kT(t)$ as $t \to 0^+$ in the intermediate spaces just introduced. In the next proposition we set $D_A(0, p) = X$ for every $p \in [1, \infty]$.

Proposition 1.3.4. *Let $(\alpha, p), (\beta, p) \in (0, 1) \times [1, +\infty] \cup \{(1, \infty)\}$, and let $k \in \mathbf{N}$. Then there exists $C = C(k, p, \alpha, \beta)$ such that*

$$\|t^{k-\alpha+\beta}A^kT(t)\|_{\mathcal{L}(D_A(\alpha,p),D_A(\beta,p))} \le C \qquad 0 < t \le 1 \tag{1.10}$$

The statement holds also for $k = 0$, provided $\alpha \le \beta$.

Proof. Without loss of generality we can assume that A satisfies (1.2), (1.3) with $\omega = 0$, otherwise we consider $A - \omega I$. By (1.6), we get that

$$C_k = \sup_{0<t\le 1} \|t^k A^k T(t)\|_{\mathcal{L}(X)} < \infty \quad \text{for all } \ k \in \mathbf{N} \tag{1.11}$$

First we prove the estimate (1.10) for $\alpha = 0$. Let $x \in X$, $k \in \mathbf{N} \cup \{0\}$. Since $D_A(\beta, p)$ is of class J_β between X and $D(A)$, we get that

$$\|z\|_{D_A(\beta,p)} \le c\|z\|^{\beta}_{D(A)}\|z\|^{1-\beta}_X \quad \forall\, z \in D(A).$$

Thus, using (1.11), we get

$$\|t^k A^k T(t)x\|_{D_A(\beta,p)} \le c\|t^k A^k T(t)x\|^{\beta}_{D(A)}\|t^k A^k T(t)x\|^{1-\beta}_X \le ct^{-\beta}\|x\|_X$$

for $0 < t \le 1$, which is the claim for $\alpha = 0$ and $k \in \mathbf{N} \cup \{0\}$.
Now, let $k \in \mathbf{N}$, $0 < \alpha < 1$ and let $x \in D_A(\alpha, p)$ or $x \in D_A(1, \infty)$. Then, using (1.5), we get

$$\begin{aligned}
\|t^k A^k T(t)x\|_{D_A(\beta,p)} &= \|t^k A^{k-1} T(t/2) A T(t/2)x\|_{D_A(\beta,p)} \\
&\le 2^k \|(t/2)^{k-1+\alpha} A^{k-1} T(t/2)\|_{\mathcal{L}(X, D_A(\beta,p))} \|(t/2)^{1-\alpha} A T(t/2)x\|_X \\
&\le 2^{k+\beta-\alpha} t^{\alpha-\beta} C(k-1, p, 0, \beta)\|x\|_{D_A(\alpha,\infty)}.
\end{aligned}$$

Now, let $k = 0$, $\alpha \le \beta$ and $x \in D_A(\alpha, p)$. Then for $0 < s \le 1$,

$$\begin{aligned}
\|T(t)x\|_{D_A(\beta,p)} &= \|s^{1-\beta-1/p} A T(s) T(t)x\|_{L^p(0,1;X)} + \|T(t)x\|_X \\
&\le C_0(\|s^{1-\alpha-1/p} A T(s) T(t)x\|_{L^p(0,1;X)} + \|x\|_X) = C_0\|x\|_{D_A(\alpha,p)}
\end{aligned}$$

which allows us to deduce the claim for $k = 0$ and $\alpha = \beta$. Finally, for $\beta > \alpha$, we get

$$\begin{aligned}
\|T(t)x\|_{D_A(\beta,p)} &\le \|T(1)x\|_{D_A(\beta,p)} + \Big\| \int_t^1 A T(s)x\, ds \Big\|_{D_A(\beta,p)} \\
&\le C(0, p, 0, \beta)\|x\|_X + C(1, p, \alpha, \beta)\|x\|_{D_A(\alpha,\infty)} \int_t^1 s^{\alpha-\beta-1}\, ds \\
&\le C(0, p, 0, \beta)\|x\|_X + C(1, p, \alpha, \beta)\|x\|_{D_A(\alpha,\infty)} \frac{t^{-\beta+\alpha}}{\beta-\alpha}.
\end{aligned}$$

that complete the proof also for $k = 0$. □

1.4 Preliminaries of measure theory

In this section we briefly review the basic definitions and the most important properties of measure theory. The main reference for our approach is [5] and other references for related topics are [20], [21] and [37].

Let Ω be an open subset of $\mathbf{R}^n$ and let $\mathcal{B}(\Omega)$ be the σ-algebra of Borel subsets of Ω, that is, the σ- algebra generated by the open subsets of Ω. We call the pair $(\Omega, \mathcal{B}(\Omega))$ a *measure space*.

Definition 1.4.1. *Let $(\Omega, \mathcal{B}(\Omega))$ be a measure space and let $m \in \mathbf{N}$, $m \geq 1$. We say that $\mu : \mathcal{B}(\Omega) \to \mathbf{R}^m$ is a* measure *if*

$$\mu(\emptyset) = 0 \tag{1.12}$$

and μ is σ- additive on $\mathcal{B}(\Omega)$, i.e., for any sequence E_h of pairwise disjoint elements of $\mathcal{B}(\Omega)$

$$\mu(\bigcup_{h=0}^{\infty} E_h) = \sum_{h=0}^{\infty} \mu(E_h). \tag{1.13}$$

We denote by $[\mathcal{M}(\Omega)]^m$ the space of $\mathbf{R}^m$ -valued measures. If $m > 1$ we say that μ is a vector measure*, whereas if $m = 1$ we say that μ is a* real measure.

Definition 1.4.2. *(Positive measure) If $\mu : \mathcal{B}(\Omega) \to [0, +\infty]$ satisfies* (1.12) *and* (1.13) *then μ is called a* positive measure *or a* Borel measure.

Notice that positive measures are not a particular case of real measures since real measures must be finite according to the previous definition. In this latter case we say that μ is a *finite* measure if $\mu(\Omega) < \infty$. A positive measure μ such that $\mu(\Omega) = 1$ is also called a *probability measure.*

For a real, vector or positive measure we can define its *total variation measure.*

Definition 1.4.3. *We define the* total variation *of μ the set function denoted by $|\mu| : \mathcal{B}(\Omega) \to [0, +\infty]$ such that for every $A \in \mathcal{B}(\Omega)$*

$$|\mu|(A) := \sup \left\{ \sum_{h=0}^{\infty} |\mu(A_h)| : \; A_h \in \mathcal{B}(\Omega) \, \textit{pairwise disjoint}, A = \bigcup_{h=0}^{\infty} A_h \right\}.$$

It can be shown that if μ is a measure then $|\mu|$ is a positive finite measure.

Definition 1.4.4. *(Radon measure) If a Borel measure is finite on compact sets then it is called* positive Radon measure.
A Radon measure *on Ω is a real or vector valued set function μ that is a measure on $(K, \mathcal{B}(K))$ for every compact set $K \subset \Omega$. It is called a* finite Radon measure *if $\mu : \mathcal{B}(\Omega) \to \mathbf{R}^m$ is a measure in the sense specified before.*

If $m > 1$ and $B \in \mathcal{B}(\Omega)$, then $\mu(B) = (\mu_1(B), \ldots, \mu_m(B))$ and $\mu_i : \mathcal{B}(\Omega) \to \mathbf{R}$ are Radon measures.

Definition 1.4.5. *(Support of a measure) Let μ be a positive measure on Ω; we call* support of μ *the closed set of all points $x \in \Omega$ such that $\mu(U) > 0$ for every neighborhood U of x and we denote it by* supp μ. *If μ is a real or vector measure, we call the support of μ the support of $|\mu|$.*

For a positive, real or vector measure on the measure space $(\Omega, \mathcal{B}(\Omega))$ and for $E \in \mathcal{B}(\Omega)$ we denote by $\mu \llcorner E$ the restriction of μ to E so defined: $\mu \llcorner E(F) = \mu(E \cap F)$ for

every $F \in \mathcal{B}(\Omega)$; moreover, if μ is a Borel (Radon) measure and E is a Borel set, then the measure $\mu \llcorner E$ is a Borel (Radon measure), too. When $\mu \llcorner E = \mu$ we say that μ is *concentrated* on E. We say that a set E is *μ-negligible* if there exists $B \supset E$, $B \in \mathcal{B}(\Omega)$ such that $\mu(B) = 0$. Moreover a Borel set E is called *μ-measurable* if E is of the form $E \cup N$ with N μ-negligible.
We now state the classical Riesz representation theorem. Recall that we denote by $C_c(\Omega)$ the space of continuous functions with compact support and by $C_0(\Omega)$ its completion with respect the sup norm.

Theorem 1.4.6. *(Riesz Representation Theorem) Let $L : C_c(\Omega; \mathbf{R}^m) \to \mathbf{R}$ be a linear functional. Suppose that there exists $c < +\infty$ such that for all $f \in C_c(\Omega; \mathbf{R}^m)$*

$$|L(f)| \leq c\|f\|_{L^\infty(\Omega)}.$$

Then, there is a unique $\mathbf{R}^m$- valued Radon measure μ on Ω such that

$$L(f) = \int_\Omega f\,d\mu = \sum_{h=1}^m \int_\Omega f_h d\mu_h \qquad \forall f \in C_c(\Omega; \mathbf{R}^m).$$

Moreover

$$\sup\{L(f) : \ f \in C_c(\Omega; \mathbf{R}^m), \ \|f\|_{L^\infty(\Omega)} \leq 1\} = |\mu|(\Omega).$$

1.4.1 Weak convergence of measures

From the Riesz theorem, it follows that the space of $[\mathcal{M}(\Omega)]^m$, endowed with the norm $\|\mu\| := |\mu|(\Omega)$, is linearly isometric to the dual space of $C_c(\Omega; \mathbf{R}^m)$ and so it is a Banach space. This fact allows us to consider several topologies on $[\mathcal{M}(\Omega)]^m$. Of particular interest are the following two different kinds of convergence induced by $C_c(\Omega; \mathbf{R}^m)$ and $C_0(\Omega; \mathbf{R}^m)$, respectively.

Definition 1.4.7. *Let μ_k, μ be $\mathbf{R}^m$- valued Radon measures on Ω.*

(i) We say that μ_k converges locally weakly* *to μ and write $\mu_k \stackrel{w^*_{loc}}{\longrightarrow} \mu$ if*

$$\int_\Omega f\,d\mu_k \longrightarrow \int_\Omega f\,d\mu \qquad \forall f \in C_c(\Omega; \mathbf{R}^m).$$

(ii) We say that μ_k converges weakly* *to μ and write $\mu_k \stackrel{w^*}{\longrightarrow} \mu$ if*

$$\int_\Omega f\,d\mu_k \longrightarrow \int_\Omega f\,d\mu \qquad \forall f \in C_0(\Omega; \mathbf{R}^m).$$

An important connection between these two different kinds of convergence is given by the following property. Let μ_k, μ be $\mathbf{R}^m$- valued finite Radon measures. Then $\mu_k \stackrel{w^*}{\longrightarrow} \mu$ if and only if $\mu_k \stackrel{w^*_{loc}}{\longrightarrow} \mu$ and the norms $|\mu_k|(\Omega)$ are bounded.

Definition 1.4.8. *(Convergence in measure) We say that* (E_h) *converges to* E *in measure in* Ω *if*

$$|\Omega \cap (E_h \Delta E)| \to 0 \quad \text{as } h \to \infty.$$

We say that E_h *locally converges in measure to* E *if* (E_h) *converges to* E *in measure in every open set* A *with* $A \subset\subset \Omega$.

We can notice that these convergences correspond to $L^1(\Omega)$ and $L^1_{\text{loc}}(\Omega)$ convergences of the characteristic functions.

1.4.2 Differentiation of measures

Two important relations between measures are presented in the following definition, the absolute continuity and the mutually singularity.

Definition 1.4.9. *(Absolute continuity and singularity) Let* μ *be a positive measure and* σ *a real or a vector measure on the measure space* $(\Omega, \mathcal{B}(\Omega))$*; we say that* σ *is* absolutely continuous *with respect to* μ*, and write* $\sigma << \mu$*, if for* $A \in \mathcal{B}(\Omega)$*,* $\mu(A) = 0$ *implies* $\sigma(A) = 0$*. If the measures* μ, σ *are both positive, we say that they are mutually singular and write* $\mu \perp \sigma$ *if there exists* $E \in \mathcal{B}(\Omega)$ *such that* $\mu(E) = 0$ *and* $\sigma(\Omega \setminus E) = 0$.

This latter definition can be extended also to vector measures: in that case we say that two vector measures μ and σ are mutually singular if $|\mu|$ and $|\nu|$ are so.

Theorem 1.4.10. *(Besicovitch differentiation theorem) Let* μ *be a positive Radon measure and* σ *a real or vector valued measure both defined on the same open set* Ω *of* $\mathbf{R}^n$*. Then, for* μ*- a.e.* $x \in \Omega$ *there exists the limit*

$$\lim_{\rho \to 0} \frac{\sigma(B_\rho(x))}{\mu(B_\rho(x))} = D_\mu \sigma(x)$$

and it is equal to $+\infty$ *for* $x \notin \operatorname{supp} \mu$*. The function* $D_\mu \sigma(x) \in [L^1_{loc}(\Omega, \mu)]^m$ *and for every Borel set* $B \in \mathcal{B}(\Omega)$

$$\sigma(B) = \int_B D_\mu \sigma(x)\, d\mu(x) + \sigma^s(B), \tag{1.14}$$

where $\sigma^s \perp \mu$ *and is concentrated on a Borel set* μ*-negligible.*

By the representation (1.14) of σ we can deduce that the integral part is absolutely continuous with respect to μ, and σ^s is singular.
This decomposition of σ with respect to μ is called *Lebesgue decomposition* and it is uniquely determined. The function $D_\mu \sigma$ is called the derivative of σ respect to μ and it is usually denoted by σ/μ. The proof of the Besicovitch theorem, as is stated here, can be found in [41].
An useful decomposition immediately follows from the Besicovitch theorem if we take into account that each real or vector measure μ is absolute continuous with respect to its total variation $|\mu|$.

Corollary 1.4.11. *(Polar decomposition) Let μ be a $\mathbf{R}^m$-valued measure on the measure space $(\Omega, \mathcal{B}(\Omega))$; then there exists a unique $\mathbf{S}^{m-1}$-valued function $f \in (L^1(\Omega, |\mu|))^m$ such that $\mu = f|\mu|$.*

1.4.3 Hausdorff measures and rectifiable sets

The notion that we are going to introduce is a mild regularity property of subsets of $\mathbf{R}^n$ known as *rectifiability.* First we provide the definition of Hausdorff k-dimensional measures. This class of measures is defined in terms of the diameters of suitable coverings and allows an intrinsic definition of k-dimensional area without any reference to parametrizations.

Definition 1.4.12. *(Hausdorff measures) Let $A \subset \mathbf{R}^n$, $k \in [0, \infty)$ and $\delta \in (0, \infty]$. Define*

$$\mathcal{H}^k_\delta(A) := \frac{\omega_k}{2^k} \inf \Big\{ \sum_{i \in I} [\mathrm{diam}(A_i)]^k : A \subset \bigcup_{i \in I} A_i, \ \mathrm{diam}(A_i) < \delta \Big\} \tag{1.15}$$

for finite or countable covering $\{A_i\}_{i \in I}$ (with $\mathrm{diam}\, \emptyset = 0$). Here

$$\omega_k = \frac{\pi^{k/2}}{\Gamma(1 + k/2)}$$

where $\Gamma(s) = \int_0^\infty x^{s-1} e^{-x}\, dx$ is the Euler gamma function.
For A and k as above, define

$$\mathcal{H}^k(A) := \lim_{\delta \to 0} \mathcal{H}^k_\delta(A). \tag{1.16}$$

Remark 1.4.13. We notice that the limit in (1.16) exists (finite or infinite) since $\delta \mapsto \mathcal{H}^k_\delta(A)$ is decreasing in $(0, \infty]$. It is also worth noticing that requiring $\delta \to 0$ forces the coverings to follow the local geometry of the set A.
Finally let us observe that $\mathcal{H}^0$ corresponds to the counting measure and it is not trivial to prove that $\mathcal{H}^n = \mathcal{L}^n$ on $\mathbf{R}^n$.

Definition 1.4.14. *(Countably $\mathcal{H}^{n-1}$-rectifiable sets) We say that $E \subset \mathbf{R}^n$ is countably* $\mathcal{H}^{n-1}$-rectifiable *if there exist (at most) countably many C^1 embedded hypersurfaces $\Gamma_i \subset \mathbf{R}^n$ such that*

$$\mathcal{H}^{n-1}(E \setminus \bigcup_i \Gamma_i) = 0.$$

1.5 Some further preliminaries

In this section we collect some miscellaneous classical results, which is useful to state in the form we shall use later.
Throughout this thesis, we shall consider functions defined in $\mathbf{R}^n$ or in subset of $\mathbf{R}^n$, particularly in $\mathbf{R}^n_+ = \{x = (x_1, \dots, x_n) \in \mathbf{R}^n;\ x_n \geq 0\}$ and in domains with uniformly C^2 boundary $\partial\Omega$. Let Ω be an open set in $\mathbf{R}^n$, and $m \in \mathbf{N}$. Let us give the definition.

Definition 1.5.1. *(Uniformly C^m domain) We say that the boundary $\partial\Omega$ is uniformly C^m if there exist $r, L > 0$ and a (at most countable) collection of open balls $U_j = \{x \in \mathbf{R}^n; |x - x_j| < r\}$, $j \in \mathbf{N}$, covering $\partial\Omega$ and such that there exists an integer k with the property that $\bigcap_{j\in J} U_j = \emptyset$ for all $J \subset \mathbf{N}$ with more than k elements. Moreover there exist coordinate transformations $\varphi_j : U_j \to B(0,1)$, C^m diffeomorphisms such that*

$$\varphi_j(\overline{U_j} \cap \Omega) = B^+(0,1) = B(0,1) \cap \mathbf{R}^n_+$$

$$\varphi_j(\overline{U_j} \cap \partial\Omega) = B(0,1) \cap \{x_n = 0\}.$$

Moreover, all the coordinate transformations φ_j and their inverses are supposed to have uniformly bounded derivatives up to the order m,

$$\sup_{j\in\mathbf{N}} \sum_{1\le|\alpha|\le m} \left(\|D^\alpha \varphi_j\|_\infty + \|D^\alpha \varphi_j^{-1}\|_\infty\right) \le L$$

We shall use the classical Sobolev embedding theorems which are recalled in the next lemma. We refer to [1] for their proof.

Theorem 1.5.2. *Let Ω be either $\mathbf{R}^n$, or an open set in $\mathbf{R}^n$ with uniformly C^1 boundary. Let $p > n$ and set $\alpha = 1 - \frac{n}{p}$. Then $W^{1,p}(\Omega) \subset C^\alpha_b(\overline{\Omega})$. Moreover, there exists $C > 0$ such that for every $u \in W^{1,p}_{loc}(\Omega)$ and for every $x_0 \in \Omega$ we have*

(i) $\|u\|_{L^\infty(\Omega_{x_0,r})} \le Cr^{-\frac{n}{p}}(\|u\|_{L^p(\Omega_{x_0,r})} + r\|Du\|_{L^p(\Omega_{x_0,r})})$,

(ii) $[u]_{C^\alpha(\Omega_{x_0,r})} \le C\|Du\|_{L^p(\Omega_{x_0,r})}$.

where $\Omega_{x_0,r} = \Omega \cap B(x_0, r)$ *and* $[u]_{C^\alpha(\Omega)} = \sup_{x,y\in\Omega} \frac{|u(x)-u(y)|}{|x-y|^\alpha}$.

Another useful tool is a classical result of functional analysis known as continuity method recalled in the next theorem.

Theorem 1.5.3. *Let X, Y be Banach spaces, L_0 and L_1 be two linear and continuous operators from X to Y. We consider the family of operators*

$$L_t = (1-t)L_0 + tL_1, \qquad t \in [0,1],$$

and we suppose that there exists a constant $C > 0$ such that

$$\|L_t x\|_Y \ge C\|x\|_X, \qquad x \in X, t \in [0,1]. \tag{1.17}$$

If L_0 is surjective, then L_1 is surjective too (hence bijective for the estimate (1.17)*).*

Proof. Let $V = \{t \in [0,1] : L_t \text{ is bijective}\}$. By hypothesis $V \ne \emptyset$ since $0 \in V$. If $t_0 \in V$ then L_{t_0} is bijective and $\|L_{t_0}^{-1}\| \le \frac{1}{C}$ by (1.17). Moreover, since

$$L_t = L_{t_0}(I + (t - t_0)L_{t_0}^{-1}(L_1 - L_0))$$

L_t is invertible if and only if $(I + (t - t_0)L_{t_0}^{-1}(L_1 - L_0))$ is invertible. But, if $|t - t_0| < \frac{C}{\|L_1\|+\|L_0\|}$ then $\|(t-t_0)L_{t_0}^{-1}(L_1 - L_0)\| < 1$ and L_t is invertible. Setting $\delta = \frac{C}{2(\|L_1\|+\|L_0\|)}$ we get that $[0, \delta] \subset V$. Analogous argument proves that $[\delta, 2\delta] \subset V$ and so on. Finally, after a finite number of steps we get that $[0, 1] \subset V$. $\square$

Finally, it is useful to recall two well-known inequalities due to G. H. Hardy [25]. For the proof we use two lemmas. The first follows from the Hölder inequality and its proof can be found in [25, Theorem 191].

Lemma 1.5.4. *Let Ω be an open set of $\mathbf{R}^n$, $p > 1$ and $p' = p/(p-1)$; then $\|f\|^p_{L^p(\Omega)} \leq C_0$ if and only if $\|fg\|_{L^1(\Omega)} \leq C_0^{1/p} C_1^{1/p'}$ for all g such that $\|g\|^{p'}_{L^{p'}(\Omega)} \leq C_1$.*

We shall deduce Theorem 1.5.6 from the following more general theorem whose method of proof is due to Schur, even though in [38], it is assumed $p = 2$.

Lemma 1.5.5. *Let $p > 1$ and $p' = p/(p-1)$. Let $K(x, y)$ be a non-negative and homogeneous of degree -1 function, (i.e. $K(\lambda x, \lambda y) = \lambda^{-1} K(x, y)$) such that*

$$\int_0^\infty K(x,1)x^{-1/p}\,dx = \int_0^\infty K(1,y)y^{-1/p'}\,dy = k.$$

Then, for every non-negative functions $f \in L^p(0,\infty)$ and $g \in L^{p'}(0,\infty)$ we get

$$\int_0^\infty \int_0^\infty K(x,y)f(x)g(y)\,dx\,dy \leq k\Big(\int_0^\infty f^p(x)\,dx\Big)^{1/p}\Big(\int_0^\infty g^{p'}(y)\,dy\Big)^{1/p'}, \tag{1.18}$$

$$\int_0^\infty dy\Big(\int_0^\infty K(x,y)f(x)\,dx\Big)^p \leq k^p \int_0^\infty f^p(x)\,dx \tag{1.19}$$

$$\int_0^\infty dx\Big(\int_0^\infty K(x,y)g(y)\,dy\Big)^{p'} \leq k^{p'} \int_0^\infty g^{p'}(y)\,dy \tag{1.20}$$

Proof. We have

$$\begin{aligned}\int_0^\infty f(x)\,dx \int_0^\infty K(x,y)g(y)\,dy &= \int_0^\infty f(x)\,dx \int_0^\infty xK(x,xw)g(xw)\,dw \\ &= \int_0^\infty f(x)\,dx \int_0^\infty K(1,w)g(xw)dw \\ &= \int_0^\infty K(1,w)\,dw \int_0^\infty f(x)g(xw)\,dx\end{aligned}$$

if any of integrals are convergent. Applying Lemma 1.5.4 to the inner integral, and observing that

$$\int g^{p'}(xw)\,dx = \frac{1}{w}\int g^{p'}(y)\,dy$$

we obtain (1.18). Finally (1.19) and (1.20) can be deduced by Lemma 1.5.4, indeed by (1.18) we get that

$$\|hg\|_{L^1(0,\infty)} \leq (k^p C_0)^{1/p} C_1^{1/p'}$$

holds for all $g \in L^{p'}(\Omega)$ where $h(y) = \displaystyle\int_0^\infty K(x,y)f(x)dx$, $C_0 = \displaystyle\int_0^\infty f^p(x)\,dx$ and $C_1 = \displaystyle\int_0^\infty g^{p'}(y)\,dy$. Thus, Lemma 1.5.4 implies that

$$\|h\|^p_{L^p(0,\infty)} \leq k^p C_0$$

whence (1.19) is proved. The same argument can be used to prove (1.20). $\square$

Now, an immediate application of Lemma 1.5.5 is obtained by specializing the choice of $K(x,y)$.

Theorem 1.5.6. *(Hardy's inequalities) Let $\alpha > 0$, $1 \leq p \leq \infty$. If $\psi(s)$ is a non-negative measurable function with respect to the measure ds/s on $(0,\infty)$, then*

$$\Big\{\int_0^\infty \Big(t^{-\alpha}\int_0^t \psi(s)\frac{ds}{s}\Big)^p \frac{dt}{t}\Big\}^{1/p} \leq \frac{1}{\alpha}\Big\{\int_0^\infty (s^{-\alpha}\psi(s))^p \frac{ds}{s}\Big\}^{1/p} \tag{1.21}$$

and

$$\Big\{\int_0^\infty \Big(t^{\alpha}\int_t^\infty \psi(s)\frac{ds}{s}\Big)^p \frac{dt}{t}\Big\}^{1/p} \leq \frac{1}{\alpha}\Big\{\int_0^\infty (s^{\alpha}\psi(s))^p \frac{ds}{s}\Big\}^{1/p} \tag{1.22}$$

Proof. Let $\alpha > 0$, $1 \leq p \leq \infty$, then the function

$$K(s,t) := \begin{cases} s^{\alpha+\frac{1}{p}-1}t^{-\alpha-\frac{1}{p}} & s < t \\ 0 & \text{elsewhere} \end{cases}$$

satisfies the assumption of Lemma 1.5.5 with $k = \frac{1}{\alpha}$. Then (1.21) can be obtained by (1.19) with $K(s,t)$ as before and $f(s) = s^{-\alpha-\frac{1}{p}}\psi(s)$. Finally (1.22) can be proved similarly choosing K and f in a suitable way. $\square$

The next lemma is used only in Propositions 3.1.1 and 3.1.3. We omit the proof which can be considered a particular case of [26, Lemma 7.1.1].

Lemma 1.5.7. *(Gronwall's generalized inequality) Suppose $a, b \geq 0$, $0 \leq \alpha, \beta < 1$, $0 < T < \infty$. Let $u(t)$ be a nonnegative and locally integrable function on $0 \leq t \leq T$ with*

$$u(t) \leq at^{-\alpha} + b\int_0^t (t-s)^{-\beta}u(s)\,ds$$

on $(0,T)$; then there exists a constant $C(b,\beta,T) < \infty$ such that

$$u(t) \leq \frac{at^{-\alpha}}{1-\alpha}C(b,\beta,T).$$

Chapter 2

Generation of analytic semigroups by elliptic operators

2.1 Assumptions and formulation of the boundary value problem

In this chapter Ω will denote either $\mathbf{R}^n$ or an open subset of $\mathbf{R}^n$ ($n \geq 2$) with sufficiently smooth boundary $\partial\Omega$. For any $x \in \partial\Omega$ we denote by $\nu(x)$ the exterior unit normal vector to $\partial\Omega$ at $x \in \partial\Omega$.
We shall consider the linear second order differential operator $\mathcal{A}(x, D)$ with real coefficients operating on complex valued functions $u(x)$ defined in the domain Ω

$$\begin{aligned}\mathcal{A}(x, D) &= \sum_{i,j=1}^{n} D_i(a_{ij}(x)D_j) + \sum_{i=1}^{n} b_i(x)D_i + c(x) \\ &= \operatorname{div}(A \cdot D) + B \cdot D + c. \end{aligned} \tag{2.1}$$

The leading part of $\mathcal{A}(x, D)$ is denoted by $\mathcal{A}^0(x, D)$:

$$\mathcal{A}^0(x, D) = \sum_{i,j} a_{ij}(x) D_i D_j.$$

In what follows we assume the following conditions.

SMOOTHNESS CONDITION ON Ω: Ω is uniformly regular of class C^2. (2.2)

SMOOTHNESS CONDITION ON $\mathcal{A}$:

$$a_{ij} = a_{ji} \in C^1_b(\overline{\Omega}) \quad \text{and} \quad b_i, c \in L^\infty(\Omega). \tag{2.3}$$

ELLIPTICITY CONDITION ON Ω: $\mathcal{A}$ is uniformly μ-elliptic in Ω, i.e., there exists a constant $\mu \geq 1$ such that for any $x \in \overline{\Omega}$ and $\xi \in \mathbf{R}^n$

$$\mu^{-1}|\xi|^2 \leq \mathcal{A}^0(x,\xi) \leq \mu|\xi|^2, \tag{2.4}$$

Moreover if $\Omega \neq \mathbf{R}^n$, we consider some boundary conditions. These conditions are expressed by a linear first order differential operator with real coefficients defined for $x \in \partial\Omega$:

$$\mathcal{B}(x,D) = \sum_{i=1}^{n} \beta_i(x) D_i + \gamma(x) \tag{2.5}$$

We assume the following.
SMOOTHNESS CONDITION ON $\mathcal{B}$:

$$\beta_i, \gamma \in UC_b^1(\overline{\Omega}), \tag{2.6}$$

i.e., β, γ are differentiable on $\partial\Omega$ and the derivatives are all bounded and uniformly continuous on $\partial\Omega$ and the uniform nontangentiality condition

$$\inf_{x \in \partial\Omega} \left| \sum_{i=1}^{n} \beta_i(x) \nu_i(x) \right| > 0 \tag{2.7}$$

holds.

In the sequel the Agmon-Douglis-Nirenberg a priori estimates will be very useful. They hold for operators with complex valued coefficients for which (2.3) holds and uniform ellipticity consists in requiring that there exists a constant $\mu \geq 1$ such that for any $x \in \overline{\Omega}$ and $\xi \in \mathbf{R}^n$

$$\mu^{-1}|\xi|^2 \leq |\mathcal{A}^0(x,\xi)| \leq \mu|\xi|^2, \tag{2.8}$$

Due to the ellipticity of $\mathcal{A}$, (2.8), we get that for every real vector $\xi = (\xi_1, \ldots, \xi_n) \neq 0$ and for every point $x \in \overline{\Omega}$ there holds $\mathcal{A}^0(x,\xi) \neq 0$. Hence in particular for every linearly independent real vectors ξ and η, the polynomial $\mathcal{A}^0(x, \xi + \tau\eta)$ of the variable τ has no real roots. We assume the following.
ROOT CONDITION: For every pair of linearly independent real vectors ξ, η the polynomial $\mathcal{A}^0(x, \xi + \tau\eta)$ of the variable τ has a unique root τ_1^+ with positive imaginary part.

It is easy to verify that if $n \geq 3$ all elliptic operators satisfy the Root Condition. Indeed in the case $\xi \perp \eta$, if we take for simplicity $\eta = e_n$, then $\mathcal{A}^0(x, \xi + \tau\eta) = \mathcal{A}^0(x, \xi', \tau\eta)$ with $\xi' = (\xi_1, \ldots, \xi_{n-1})$, $\xi' \neq 0$. We define the constant functions $f_\eta, g_\eta : \mathbf{R}^{n-1} \setminus \{0\} \to \mathbf{N}$ as follows

$$f_\eta(\xi') = \#\{\tau \in \mathbf{C} : \ \mathcal{A}^0(x, \xi + \tau\eta) = 0, \, \mathrm{Im}\, \tau > 0\}$$
$$g_\eta(\xi') = \#\{\tau \in \mathbf{C} : \ \mathcal{A}^0(x, \xi + \tau\eta) = 0), \, \mathrm{Im}\, \tau < 0\},$$

and we observe that since if τ is a root for ξ, η then $-\tau$ is a root for $-\xi$, $-\eta$ we deduce $f_\eta(\xi') = g_\eta(-\xi')$. Moreover, if $n \geq 3$ then $g_\eta(-\xi') = g_\eta(\xi')$. In fact, the points ξ' and $-\xi'$ can be joined by a smooth simple curve γ in $\mathbf{R}^{n-1} \setminus \{0\}$ (which is a connected set) and the roots of the polynomial $\tau \mapsto \mathcal{A}^0(x, \gamma(\cdot) + \tau\eta)$ are continuous functions along γ. If

g_η were not constant along γ, the imaginary part of some roots would change sign, hence it would vanish and give a real root, which is impossible. Therefore, $f_\eta(\xi')$, $g_\eta(\xi')$ and $g_\eta(-\xi')$ coincide everywhere on $\mathbf{R}^{n-1}\setminus\{0\}$ if $n \geq 3$. The general case can be recovered by the previous one. Indeed let $\xi,\eta \in \mathbf{R}^n\setminus\{0\}$ with ξ and η linearly independent; we can write $\xi = \xi' + \xi''\hat{\eta}$ with $\hat{\eta} = \frac{\eta}{|\eta|}$, $\xi' \neq 0$ and $\xi' \perp \hat{\eta}$, then $\mathcal{A}^0(\xi + \tau\eta) = \mathcal{A}^0(\xi' + \tau'\hat{\eta})$ with $\tau' = \xi'' + \tau|\eta|$ and $\xi' \perp \hat{\eta}$. Finally we observe that $f_\eta(\xi') = f_{\hat{\eta}}(\xi')$ and $g_\eta(\xi') = g_{\hat{\eta}}(\xi')$; thus repeating the argument above we conclude for two arbitrary linearly independent vectors ξ,η.
Moreover, we require that the boundary conditions are expressed as before by (2.5) with complex coefficients

$$\beta_i, \gamma \in UC_b^1(\overline{\Omega};\mathbf{C}) \tag{2.9}$$

that must "complement" the differential equation. This condition called *complementing boundary condition* consists of an algebraic criterion involving the leading parts of $\mathcal{A}$ and $\mathcal{B}$.

COMPLEMENTING CONDITION (2.10)

Let x be an arbitrary point on $\partial\Omega$ and ν be the outward normal unit vector to $\partial\Omega$ at x. For each vector $\xi \neq 0$ tangential to $\partial\Omega$ at x, let τ_1^+ be the root of the polynomial $\mathcal{A}^0(x,\xi+\tau\nu)$ with positive imaginary part. Then the polynomial $\mathcal{B}^0(x,\xi+\tau\nu) = \langle\beta(x),\xi+\tau\nu\rangle$ has to be linearly independent modulo the polynomial $(\tau-\tau_1^+)$. This means that τ_1^+ cannot be solution of $\mathcal{B}^0(\xi+\tau\nu) = 0$ and it is obviously satisfied if (2.7) holds.

We notice that if the coefficients of $\mathcal{A}$ are real and satisfy

$$\sum_{i,j} a_{ij}(x)\xi_i\xi_j \geq \mu|\xi|^2 \qquad x \in \bar{\Omega},\ \xi \in \mathbf{R}^n$$

for some $\mu > 0$, then the Root Condition is immediately satisfied. Indeed in that case the polynomial in τ, $\mathcal{A}^0(\xi+\tau\nu)$ has not real roots, therefore it has two conjugate complex solutions.

Remark 2.1.1. The reason why we have considered complex valued coefficients and introduced assumption (2.8) is the fact that we shall use the Agmon-Douglis-Nirenberg estimates (2.13) and (2.14) with $\mathcal{A}$ replaced by the operator $\mathcal{A}+e^{i\theta}D_{tt}$ in $n+1$ variables (x,t), with $\theta \in [-\pi/2,\pi/2]$, which satisfies (2.8) and the Root Condition too.

2.2 Basic estimates for elliptic equations

The aim of this chapter is to prove that under the assumptions listed in Section 2.1, the realizations of $\mathcal{A}$ with homogeneous boundary conditions $\mathcal{B}u = 0$ in $\partial\Omega$, are sectorial operators in suitable Banach spaces. As a result they generate analytic semigroups in those spaces (see Proposition 1.2.3).

A sufficient condition for the sectoriality of an operator is given in Proposition 1.2.7. Here we first need some existence and uniqueness results for elliptic boundary value problems of the type

$$\begin{cases} \lambda u - \mathcal{A}(\cdot, D)u = f & \text{in } \Omega \\ \mathcal{B}(\cdot, D)u = 0 & \text{in } \partial\Omega \end{cases} \tag{2.11}$$

and then some resolvent estimate like (1.8).
Now we recall the a priori estimates due to Agmon, Douglis and Nirenberg that hold for operators with complex coefficients satisfying hypothesis of Section 2.1 in $\mathbf{R}^n$ as well as in regular domains. For a complete analysis of these estimates we refer to [2] and [3]. We recall them in the following theorem in a way that will be used later. We set

$$M = \max\{\|a_{ij}\|_{1,\infty}, \|b_i\|_\infty, \|c\|_\infty\}. \tag{2.12}$$

Theorem 2.2.1. *(Agmon-Douglis-Nirenberg)*

(i) Let $\mathcal{A}(x, D)$ be defined as in (2.1)*. Suppose that $a_{ij}, b_i, c : \mathbf{R}^n \to \mathbf{C}$ satisfy hypotheses* (2.3)*,* (2.8) *and the Root Condition. Then for every $p \in (1, +\infty)$ there exists a strictly positive constant C depending only on p, n, μ and M such that for every $u \in W^{2,p}(\mathbf{R}^n)$*

$$\|u\|_{W^{2,p}(\mathbf{R}^n)} \le C\left(\|u\|_{L^p(\mathbf{R}^n)} + \|\mathcal{A}(\cdot, D)u\|_{L^p(\mathbf{R}^n)}\right). \tag{2.13}$$

(ii) Let Ω be an open set in $\mathbf{R}^n$ with uniformly C^2 boundary, and $\mathcal{A}(x, D)$ defined by (2.1)*. Suppose that $a_{ij}, b_i, c : \Omega \to \mathbf{C}$ satisfy hypotheses* (2.3)*,* (2.8) *and the Root Condition. Let in addition β_i, γ satisfy* (2.9) *and the complementing condition. For every $u \in W^{2,p}(\Omega)$, with $1 < p < \infty$, set $f = \mathcal{A}(\cdot, D)u$, $g = \mathcal{B}(\cdot, D)u_{|\partial\Omega}$. Then there is $C_1 = C_1(p, n, \mu, M, \Omega) > 0$ such that*

$$\|u\|_{W^{2,p}(\Omega)} \le C_1\left(\|u\|_{L^p(\Omega)} + \|f\|_{L^p(\Omega)} + \|g_1\|_{W^{1,p}(\Omega)}\right). \tag{2.14}$$

where g_1 is any $W^{1,p}$ extension of g to Ω.

Observe that the estimates in Theorem 2.2.1 are not true for $p = 1$ and $p = \infty$. For this reason the theory of $L^p(\Omega)$, $1 < p < \infty$ cannot be rearranged to the cases L^1 or L^∞. For $p = \infty$ this difficulty has been overcome by K. Masuda and H.B. Stewart (see [42], [43]) using the classical L^p theory and by passing to the limit in the L^p estimates in a suitable way.
One of the ways to solve the case $p = 1$ consists in using duality from L^∞.
This chapter is organized as follows: in Section 2.2.1 we discuss the generation in L^p, $1 < p < \infty$ for an elliptic operator of second order with homogeneous non tangential boundary conditions. Using these results we study the same problem in $L^\infty(\Omega)$. Finally in Section 2.5 we confine our attention to a particular boundary operator and we prove sectoriality for the realization in L^1 of the operator $\mathcal{A}$ with the homogeneous boundary condition there specified.

2.2.1 Analytic semigroups in $L^p(\mathbf{R}^n)$, $1 < p < \infty$

First suppose $\Omega = \mathbf{R}^n$ and consider the realization of $\mathcal{A}$ in $L^p(\mathbf{R}^n)$. Define

$$D(A_p) = W^{2,p}(\mathbf{R}^n), \quad A_p u = \mathcal{A}(\cdot, D)u, \quad u \in D(A_p), \tag{2.15}$$

We start by the simplest case when $a_{ij} = \delta_{ij}$ $b_i, c = 0$. In this way the operator in (2.1) reduces to the Laplace operator:

$$\Delta = \sum_{i=1}^{n} D_{ii}.$$

By (i) of the Theorem 2.2.1, it follows that the operator Δ with domain $W^{2,p}(\mathbf{R}^n)$ is closed.

Theorem 2.2.2. *Let $1 < p < \infty$ and consider the operator Δ with domain given by $W^{2,p}(\mathbf{R}^n)$. Then, there exist $\frac{\pi}{2} < \vartheta_0 < \pi$ and $M_\vartheta > 0$ depending on p such that $\rho(\Delta) \supset \Sigma_\vartheta = \{\lambda \in \mathbf{C};\ \lambda \neq 0,\ |\arg\lambda| < \vartheta\}$ and the estimate*

$$\|(\lambda - \Delta)^{-1}\|_{\mathcal{L}(L^p(\mathbf{R}^n))} \leq \frac{M_\vartheta}{|\lambda|} \tag{2.16}$$

holds for $\lambda \in \Sigma_\vartheta$ for any $\vartheta < \vartheta_0$.

Proof. First we consider the case $p \geq 2$. For $u \in C_0^\infty(\mathbf{R}^n)$, we put $u^* := \bar{u}|u|^{p-2}$ where $\bar{u}$ denotes the complex conjugate of u. Since the function $f(z) = \bar{z}|z|^{p-2}$ is continuously differentiable, $u^* \in C_0^1(\mathbf{R}^n)$. By the chain rule we obtain

$$D_h u^* = |u|^{p-2} D_h \bar{u} + (p-2)|u|^{p-4}\bar{u}\mathrm{Re}\,(\bar{u} D_h u).$$

Integration by parts yields

$$\begin{aligned}
-\int_{\mathbf{R}^n} \Delta u \cdot u^* &= -\int_{\mathbf{R}^n} \sum_{h=1}^{n} (D_{hh}u)\bar{u}|u|^{p-2} \\
&= \int_{\mathbf{R}^n} \sum_{h=1}^{n} D_h u D_h(\bar{u}|u|^{p-2}) \\
&= \int_{\mathbf{R}^n} \sum_{h=1}^{n} (|u|^{p-2} D_h u D_h \bar{u} \\
&\quad + (p-2)|u|^{p-4}\bar{u} D_h u \mathrm{Re}\,(\bar{u} D_h u)).
\end{aligned}$$

Since

$$\mathrm{Re}\,(|u|^2 D_h u D_h \bar{u}) = (\mathrm{Re}\,(\bar{u} D_h u))^2 + (\mathrm{Im}\,(\bar{u} D_h u))^2,$$

then

$$\begin{aligned}
-\mathrm{Re} \int_{\mathbf{R}^n} \Delta u \cdot u^* &= (p-1)\int_{\mathbf{R}^n} |u|^{p-4} \sum_{h=1}^{n} (\mathrm{Re}\,(\bar{u} D_h u))^2 \\
&\quad + \int_{\mathbf{R}^n} |u|^{p-4} \sum_{h=1}^{n} \mathrm{Im}\,(\bar{u} D_h u))^2 =: (p-1)A^2 + B^2 \geq 0
\end{aligned} \tag{2.17}$$

and

$$-\mathrm{Im}\int_{\mathbf{R}^n}\Delta u\cdot u^* = (p-2)\int_{\mathbf{R}^n}|u|^{p-4}\sum_{h=1}^n \mathrm{Im}\,(\bar{u}D_h u)\mathrm{Re}\,(\bar{u}D_h u).$$

Now, using the Cauchy- Schwartz inequality we obtain

$$\begin{aligned}
&\int_{\mathbf{R}^n}|u|^{p-4}\Big|\sum_{h=1}^n \mathrm{Im}\,(\bar{u}D_h u)\mathrm{Re}\,(\bar{u}D_h u)\Big| \le \\
&\int_{\mathbf{R}^n}|u|^{\frac{p-4}{2}}\Big|\mathrm{Re}\,(\bar{u}Du))\Big|\ |u|^{\frac{p-4}{2}}\Big|\mathrm{Im}\,(\bar{u}Du))\Big| \le \\
&\Big(\int_{\mathbf{R}^n}|u|^{p-4}\sum_{h=1}^n(\mathrm{Re}\,(\bar{u}D_h u)^2\Big)^{\frac12}\Big(\int_{\mathbf{R}^n}|u|^{p-4}\sum_{h=1}^n(\mathrm{Im}\,(\bar{u}D_h u)^2\Big)^{\frac12} = AB
\end{aligned}$$

and so

$$\Big|\mathrm{Im}\int_{\mathbf{R}^n}\Delta u\cdot u^*\Big| \le |p-2|AB. \tag{2.18}$$

If $1<p<2$, we get the same estimates (2.17) and (2.18) by approximation, using the functions $u^* = \bar{u}(|u|^2+\delta)^{\frac{p-2}{2}}$ and letting $\delta\to 0$.
Now we look for the smallest positive γ_0 such that

$$|p-2|AB \le \gamma_0[(p-1)A^2+B^2]$$

for all A, B. Since for such γ_0 we have that

$$\gamma_0(p-1)\frac{A^2}{B^2} - |p-2|\frac{A}{B} + \gamma_0 \ge 0$$

for all A, B, then $(p-2)^2 - 4(p-1)\gamma_0^2 \le 0$ and so

$$\gamma_0 \ge \frac{|p-2|}{2\sqrt{p-1}}.$$

Setting $\displaystyle\int_{\mathbf{R}^n}\Delta u\cdot u^*\,dx =: x+iy$, we have obtained

$$\begin{cases} x\le 0 \\ |y|\le \gamma|x| \end{cases} \tag{2.19}$$

for $\gamma\ge\gamma_0(p)$. Define $\vartheta_0 = \pi - \arctan\gamma_0$, $\vartheta<\vartheta_0$ and prove that $\rho(\Delta)\supset\Sigma_\vartheta$.
Let $\vartheta<\vartheta_0$ and consider $\lambda\in\Sigma_\vartheta$ and $u\in C_0^\infty(\mathbf{R}^n)$, with $\|u\|_{L^p(\mathbf{R}^n)}=1$, so that $\|u^*\|_{L^{p'}(\mathbf{R}^n)} = 1 = \langle u,u^*\rangle_{L^p,L^{p'}}$. Then, by (2.19) we get $\langle\Delta u,u^*\rangle_{L^p,L^{p'}}\in\mathbf{C}\setminus\Sigma_{\vartheta_0}$, hence

$$\begin{aligned}
\|\lambda u-\Delta u\|_{L^p(\mathbf{R}^n)} &\ge |\langle\lambda u-\Delta u,u^*\rangle_{L^p,L^{p'}}| = |\lambda - \langle\Delta u,u^*\rangle_{L^p,L^{p'}}| \\
&\ge \mathrm{dist}(\lambda,\mathbf{C}\setminus\Sigma_{\vartheta_0}) \ge C_\vartheta|\lambda|.
\end{aligned}$$

By density, we deduce

$$\|\lambda u-\Delta u\|_{L^p(\mathbf{R}^n)} \ge C_\vartheta|\lambda|\|u\|_{L^p(\mathbf{R}^n)} \tag{2.20}$$

for all $u \in W^{2,p}(\mathbf{R}^n)$. Now, using the Fourier transform we prove that $\lambda \in \rho(\Delta)$. The injectivity of $\lambda - \Delta$ follows from (2.20). By (2.13) and using inequality (2.20) we have

$$\begin{aligned}\|u\|_{W^{2,p}(\mathbf{R}^n)} &\le c(\|u\|_{L^p(\mathbf{R}^n)} + \|\Delta u\|_{L^p(\mathbf{R}^n)}) \\ &\le c(\|u\|_{L^p(\mathbf{R}^n)} + |\lambda|\|u\|_{L^p(\mathbf{R}^n)} + \|\lambda u - \Delta u\|_{L^p(\mathbf{R}^n)}) \\ &= c((1+|\lambda|)\|u\|_{L^p(\mathbf{R}^n)} + \|\lambda u - \Delta u\|_{L^p(\mathbf{R}^n)}) \\ &\le C\|\lambda u - \Delta u\|_{L^p(\mathbf{R}^n)} \qquad (2.21)\end{aligned}$$

where the constant C depends on p, ϑ, λ. Now, inequality (2.21) and the closedness of Δ in $W^{2,p}(\mathbf{R}^n)$ imply that $(\lambda - \Delta)(W^{2,p}(\mathbf{R}^n))$ is closed in $L^p(\mathbf{R}^n)$. We have only to prove that $(\lambda - \Delta)(W^{2,p}(\mathbf{R}^n))$ is dense in $L^p(\mathbf{R}^n)$.
Consider the space $\mathcal{S}(\mathbf{R}^n)$ which is dense in $L^p(\mathbf{R}^n)$ and prove that

$$\forall f \in \mathcal{S}(\mathbf{R}^n)\ \exists u \in W^{2,p}(\mathbf{R}^n) \text{ such that } (\lambda - \Delta)u = f$$

Now, the solution in $W^{2,p}(\mathbf{R}^n)$ of $\lambda u - \Delta u = f$ is the function $u \in \mathcal{S}(\mathbf{R}^n)$ whose Fourier transform is

$$\hat{u} = \frac{\hat{f}}{\lambda + |\xi|^2}$$

This shows that

$$(\lambda - \Delta)(W^{2,p}(\mathbf{R}^n)) \supseteq \mathcal{S}(\mathbf{R}^n)$$

hence it is dense in $L^p(\mathbf{R}^n)$. □

The previous theorem implies that the realization of Δ in $L^p(\mathbf{R}^n)$ is a sectorial operator.

Corollary 2.2.3. *Let $1 < p < \infty$ and $\lambda \in \mathbf{C}$ with $\operatorname{Re}\lambda > 0$. Then for every $f \in L^p(\mathbf{R}^n)$ there exists a unique $u \in W^{2,p}(\mathbf{R}^n)$ such that*

$$(\lambda - \Delta)u = f. \qquad (2.22)$$

Moreover

$$|\lambda|\|u\|_{L^p(\mathbf{R}^n)} + |\lambda|^{\frac{1}{2}}\|Du\|_{L^p(\mathbf{R}^n)} + \|D^2u\|_{L^p(\mathbf{R}^n)} \le c\|f\|_{L^p(\mathbf{R}^n)} \qquad (2.23)$$

where c depends on n, p.

Proof. The result can be easily obtained from the previous one. By the estimate (2.20) and (2.21) we deduce

$$|\lambda|\|u\|_{L^p(\mathbf{R}^n)} \le C_\theta^{-1}\|f\|_{L^p(\mathbf{R}^n)} \qquad (2.24)$$

$$\|D^2u\|_{L^p(\mathbf{R}^n)} \le C\|f\|_{L^p(\mathbf{R}^n)} \qquad (2.25)$$

and finally using the interpolation estimate (A.1)

$$\|\nabla u\|_{L^p(\mathbf{R}^n)} \le c\|D^2u\|^{\frac{1}{2}}_{L^p(\mathbf{R}^n)}\|u\|^{\frac{1}{2}}_{L^p(\mathbf{R}^n)} \le C|\lambda|^{-1/2}\|f\|_{L^p(\mathbf{R}^n)}. \qquad (2.26)$$

Summing up (2.24),(2.26), (2.25) and redefining the constant we get the claim. □
Actually for what concerns the existence and the uniqueness of the solution of (2.22) in $\mathbf{R}^n$ we state the following theorem (see for example [44] for details).

Theorem 2.2.4. *Let $f \in L^p(\mathbf{R}^n)$, then for every $\lambda \notin (-\infty, 0]$ there exists $u \in W^{2,p}(\mathbf{R}^n)$ such that $\lambda u - \Delta u = f$ and the estimate*

$$\|u\|_{W^{2,p}(\mathbf{R}^n)} \leq c(n, \lambda) \|f\|_{L^p(\mathbf{R}^n)}$$

holds.

In the following proposition we extend (2.23) to a more general operator than the Laplacian.

Proposition 2.2.5. *Let $1 < p < \infty$. Then, there exist $\omega_0 \in \mathbf{R}$, $M_p > 0$ depending on n, p, μ, M such that if $\operatorname{Re} \lambda \geq \omega_0$, then for every $u \in W^{2,p}(\mathbf{R}^n)$ we have*

$$|\lambda| \|u\|_{L^p(\mathbf{R}^n)} + |\lambda|^{\frac{1}{2}} \|Du\|_{L^p(\mathbf{R}^n)} + \|D^2 u\|_{L^p(\mathbf{R}^n)} \leq M_p \|\lambda u - \mathcal{A}(\cdot, D) u\|_{L^p(\mathbf{R}^n)} \tag{2.27}$$

Proof. Let $\mathcal{E}$ the operator in $n+1$ variables defined by

$$\mathcal{E}(x, t, D) = \mathcal{A}(x, D) + e^{i\theta} D_{tt} \tag{2.28}$$

with $\theta \in [-\pi/2, \pi/2]$. It satisfies the uniform ellipticity condition (2.8) with constant $\mu_{\mathcal{E}} = \mu\sqrt{2}$. Indeed, it is obvious that $|\mathcal{A}^0(x,\xi) + e^{i\theta}\eta^2| \leq \mu(|\xi|^2 + \eta^2) \leq \mu\sqrt{2}(|\xi|^2 + \eta^2)$; for the converse inequality, we look for $\mu_{\mathcal{E}} > 1$ such that

$$|\mathcal{A}^0(x, \xi) + e^{i\theta}\eta^2| \geq \mu_{\mathcal{E}}^{-1}(|\xi|^2 + \eta^2) \tag{2.29}$$

for all $x \in \bar{\Omega}$, $(\xi, \eta) \in \mathbf{R}^n \times \mathbf{R}$ and for every $\theta \in [-\pi/2, \pi/2]$. We observe that

$$\begin{aligned} |\mathcal{A}^0(x,\xi) + e^{i\theta}\eta^2| &= \left[(\langle A\xi, \xi\rangle + \eta^2 \cos\theta)^2 + \eta^4 \sin^2\theta\right]^{1/2} \\ &= \left[(\langle A\xi, \xi\rangle)^2 + \eta^4 + 2\eta^2 \langle A\xi, \xi\rangle \cos\theta\right]^{1/2} \\ &\geq (\frac{1}{\mu^2}|\xi|^4 + \eta^4)^{1/2} \end{aligned}$$

Since we look for a $\mu_{\mathcal{E}}$ such that (2.29) holds, if

$$(\frac{1}{\mu^2}|\xi|^4 + \eta^4)^{1/2} \geq \mu_{\mathcal{E}}^{-1}(|\xi|^2 + \eta^2)$$

or equivalently using that $2|\xi|^2\eta^2 \leq |\xi|^4 + \eta^4$

$$\frac{2}{\mu_{\mathcal{E}}^2}(|\xi|^4 + \eta^4) \leq \frac{1}{\mu^2}|\xi|^4 + \eta^4 \tag{2.30}$$

holds for all $(\xi, \eta) \in \mathbf{R}^n \times \mathbf{R}$ we conclude. Now, it is easy to see that if $\mu_{\mathcal{E}}$ satisfies

$$\begin{cases} \dfrac{2}{\mu_{\mathcal{E}}^2} - \dfrac{1}{\mu^2} \leq 0 \\ \dfrac{2}{\mu_{\mathcal{E}}^2} - 1 \leq 0 \end{cases}$$

that is if $\mu_{\mathcal{E}} \geq \mu\sqrt{2}$, then (2.30) is proved.
Let $\eta \in C_c^\infty(\mathbf{R})$ be such that $\eta \equiv 1$ in $[-\frac{1}{2}, \frac{1}{2}]$ and $\operatorname{supp} \eta \subseteq [-1, 1]$. For every $u \in W^{2,p}(\mathbf{R}^n)$ and $r > 0$ we set

$$v(t,x) = \eta(t)e^{irt}u(x) \qquad t \in \mathbf{R},\, x \in \mathbf{R}^n. \tag{2.31}$$

Then

$$\mathcal{E}\, v = \eta(t)e^{irt}(\mathcal{A} - e^{i\theta}r^2)u + e^{i(\theta+rt)}(\eta'' + 2ir\eta')u.$$

Now, we can prove (2.27). Estimate (2.13), applied to the function v implies that there exists $C = C(n, p, \mu, M)$ such that

$$\begin{aligned}
\|v\|_{W^{2,p}(\mathbf{R}^{n+1})} &\leq C\left[\|v\|_{L^p(\mathbf{R}^{n+1})} + \|\mathcal{E}v\|_{L^p(\mathbf{R}^{n+1})}\right] \\
&\leq C\big[\|u\|_{L^p(\mathbf{R}^n)} \\
&\quad + \|\eta e^{irt}(\mathcal{A} - e^{i\theta}r^2)u + e^{i(\theta+rt)}(\eta'' + 2ir\eta')u\|_{L^p(\mathbf{R}^{n+1})}\big] \\
&\leq C\left[\|u\|_{L^p(\mathbf{R}^n)} + \|(\mathcal{A} - e^{i\theta}r^2)u\|_{L^p(\mathbf{R}^n)} + (1+2r)\|u\|_{L^p(\mathbf{R}^n)}\right] \\
&\leq C\left[(1+r)\|u\|_{L^p(\mathbf{R}^n)} + \|(\mathcal{A} - e^{i\theta}r^2)u\|_{L^p(\mathbf{R}^n)}\right].
\end{aligned} \tag{2.32}$$

On the other hand, since $\eta \equiv 1$ in $[-\frac{1}{2}, \frac{1}{2}]$, then

$$\begin{aligned}
\|v\|^p_{W^{2,p}(\mathbf{R}^{n+1})} \geq &\|v\|^p_{W^{2,p}(\mathbf{R}^n \times]-\frac{1}{2}, \frac{1}{2}[)} = \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{\mathbf{R}^n} \sum_{|\alpha| \leq 2} |D^\alpha(e^{irt}u(x))|^p dx dt = \\
&= \int_{\mathbf{R}^n} \Big[(1 + r^p + r^{2p})|u|^p + (1 + 2r^p)\sum_{j=1}^n |D_j u|^p + \sum_{j,k=1}^n |D_{jk}u|^p\Big] dx \\
&\geq r^{2p}\|u\|^p_{L^p(\mathbf{R}^n)} + r^p\|Du\|^p_{L^p(\mathbf{R}^n)} + \|D^2u\|^p_{L^p(\mathbf{R}^n)}.
\end{aligned} \tag{2.33}$$

Taking into account (2.32), it follows

$$\begin{aligned}
&r^2\|u\|_{L^p(\mathbf{R}^n)} + r\|Du\|_{L^p(\mathbf{R}^n)} + \|D^2u\|_{L^p(\mathbf{R}^n)} \\
&\quad \leq 3\|v\|_{W^{2,p}(\mathbf{R}^{\mathbf{n+1}})} \leq 3C\left[(1+r)\|u\|_{L^p(\mathbf{R}^n)} + \|(\mathcal{A} - e^{i\theta}r^2)u\|_{L^p(\mathbf{R}^n)}\right]
\end{aligned} \tag{2.34}$$

where C is like in (2.32). We can select r sufficiently large such that $r^2 - 3C(1+r) \geq \frac{r^2}{2}$ we get

$$\frac{1}{2}r^2\|u\|_{L^p(\mathbf{R}^n)} + r\|Du\|_{L^p(\mathbf{R}^n)} + \|D^2u\|_{L^p(\mathbf{R}^n)} \leq C\|(\mathcal{A} - e^{i\theta}r^2)u\|_{L^p(\mathbf{R}^n)} \tag{2.35}$$

Taking $\lambda = e^{i\theta}r^2$ we get (2.27) with $M_p = 6C$. $\square$

Now, by using the continuity method (see Theorem 1.5.3) we are able to prove existence and uniqueness for the solution of

$$\lambda u - \mathcal{A}u = f \in L^p(\mathbf{R}^n)$$

for $\lambda \in \mathbf{C}$ with $\operatorname{Re}\lambda$ large enough.

Theorem 2.2.6. *Let $1 < p < \infty$. There exist $\tilde{\omega}_0 \in \mathbf{R}$, $C > 0$ depending on n, p, μ, M such that if* $\operatorname{Re}\lambda \geq \tilde{\omega}_0$*, then for every $f \in L^p(\mathbf{R}^n)$*

$$\lambda u - \mathcal{A}u = f$$

has a unique solution $u \in W^{2,p}(\mathbf{R}^n)$ and the following estimates hold

$$\|(\lambda - A_p)^{-1}\|_{\mathcal{L}(L^p(\mathbf{R}^n))} \leq \frac{C}{|\lambda|}; \tag{2.36}$$

$$\|\nabla(\lambda - A_p)^{-1}\|_{\mathcal{L}(L^p(\mathbf{R}^n))} \leq \frac{C}{|\lambda|^{\frac{1}{2}}}; \tag{2.37}$$

$$\|D^2(\lambda - A_p)^{-1}\|_{\mathcal{L}(L^p(\mathbf{R}^n))} \leq C. \tag{2.38}$$

Proof. We consider the Banach spaces

$$X = W^{2,p}(\mathbf{R}^n), \qquad Y = L^p(\mathbf{R}^n)$$

and the operators

$$L_0 = \lambda - \Delta, \quad L_1 = \lambda - \mathcal{A}, \quad L_t = \lambda - \mathcal{A}_t := \lambda - [(1-t)\Delta + t\mathcal{A}].$$

We can observe that $\mathcal{A}_t$ satisfies (2.4) with $\mu_t \geq \mu$ and the constant in (2.12) for $\mathcal{A}_t$, $M_t \leq (1 \vee M)$.
Moreover, by Corollary 2.2.3 we know that the operator L_0 is invertible for $\operatorname{Re}\lambda > 0$, and by the Proposition 2.2.5 applied to the operator $\mathcal{A}_t := (1-t)\Delta + t\mathcal{A}$ we get that there exist $\omega_0 \in \mathbf{R}$ and M_p depending only on n, p, μ, M, λ such that for every $\operatorname{Re}\lambda \geq \omega_0$ and $t \in [0,1]$,

$$\|u\|_{W^{2,p}(\mathbf{R}^n)} \leq M_p \|(\lambda - \mathcal{A}_t)u\|_{L^p(\mathbf{R}^n)}.$$

Since the hypotheses of Theorem 1.5.3 are satisfied we get the invertibility of the operator $L_1 = \lambda - \mathcal{A}$ for $\operatorname{Re}\lambda \geq \tilde{\omega}_0 := \sup\{\omega_0, 0\}$.
The estimates (2.36), (2.37) and (2.38), are immediate consequences of Proposition 2.2.5. □

In view of Theorem 2.2.6 and Proposition 1.2.7 we have shown that the operator A_p defined in (2.15) is sectorial.

2.2.2 L^p-estimates on domains

In this section Ω will be either a smooth open subset of $\mathbf{R}^n$ or the half space $\mathbf{R}^n_+$. We suppose that $\mathcal{A}$, $\mathcal{B}$ satisfy assumption of Section 2.1. In this case we define

$$\begin{aligned} D(A_p^B) =& \{u \in W^{2,p}(\Omega);\, \mathcal{B}(\cdot, D)u = 0 \text{ in } \partial\Omega\}, \\ A_p^B u =& \mathcal{A}(\cdot, D)u, \ u \in D(A_p^B). \end{aligned} \tag{2.39}$$

A_p^B is the realization in $L^p(\Omega)$ of $\mathcal{A}(\cdot, D)$ with homogeneous oblique boundary condition. In order to prove that A_p^B is sectorial we prove that its resolvent set contains a complex half plane and the resolvent estimate (1.3) holds.
Here also we start with the simplest case of the Laplacian in the half space $\mathbf{R}^n_+$. The crucial points are

(i) to show an a-priori estimate for A_p^B,

(ii) to solve the Neumann problem in $\mathbf{R}^n_+$.

By means of the continuity method we deduce existence and uniqueness in $\mathbf{R}^n_+$ for the problem related to $\mathcal{A}$ with a boundary operator like $\mathcal{B}$. Finally, using the regularity of the boundary $\partial\Omega$, we deduce an analogous result in the domain Ω.

We need to prove an estimate like (2.27) for the resolvent of the operator A_p^B as next proposition states.

Proposition 2.2.7. *Let Ω be an open set with uniformly C^2 boundary. Then there exist $\omega_1 \in \mathbf{R}$, $M_p > 0$, depending on n, p, μ, M, Ω, such that if $\operatorname{Re}\lambda \geq \omega_1$, then for every $u \in W^{2,p}(\Omega)$ we have, setting $g = \mathcal{B}(\cdot, D)u_{|\partial\Omega}$,*

$$\begin{aligned} &|\lambda|\|u\|_{L^p(\Omega)} + |\lambda|^{\frac{1}{2}}\|Du\|_{L^p(\Omega)} + \|D^2u\|_{L^p(\Omega)} \leq \\ &\qquad M_p\|\lambda u - \mathcal{A}(\cdot, D)\|_{L^p(\Omega)} + |\lambda|^{1/2}\|g_1\|_{L^p(\Omega)} + \|Dg_1\|_{L^p(\Omega)} \end{aligned} \tag{2.40}$$

where g_1 is any extension of g belonging to $W^{1,p}(\Omega)$.

Proof. The proof of this result can be obtained as in Proposition 2.2.5, using now estimate (2.14) instead of (2.13) in $\Omega \times \mathbf{R}$. Let g_1 be any regular extension to Ω of the trace $(\mathcal{B}(\cdot, D)u)_{|\partial\Omega}$. Then (2.32) has to be replaced by

$$\begin{aligned} \|v\|_{W^{2,p}(\Omega\times\mathbf{R})} &\leq C_1\left(\|v\|_{L^p(\Omega\times\mathbf{R})} + \|\mathcal{E}v\|_{L^p(\Omega\times\mathbf{R})} + \|\eta e^{irt}g_1\|_{W^{1,p}(\Omega\times\mathbf{R})}\right) \\ &\leq C\big((r+1)\|u\|_{L^p(\Omega)} + \|(\mathcal{A} - e^{i\theta}r^2)u\|_{L^p(\Omega)} \\ &\quad + (r+1)\|g_1\|_{L^p(\Omega)} + \|Dg_1\|_{L^p(\Omega)}\big), \end{aligned} \tag{2.41}$$

where $C = C(n, p, \mu, M)$. Accordingly, (2.34) has to be replaced by

$$\begin{aligned} &r^2\|u\|_{L^p(\Omega)} + r\|Du\|_{L^p(\Omega)} + \|D^2u\|_{L^p(\Omega)} \\ &\quad \leq 3\|v\|_{W^{2,p}(\Omega\times\mathbf{R})} \leq 3C\big[(1+r)\|u\|_{L^p(\Omega)} + \|(A - e^{i\theta}r^2)u\|_{L^p(\Omega)} \\ &\quad + (r+1)\|g_1\|_{L^p(\Omega)} + \|Dg_1\|_{L^p(\Omega)}\big] \end{aligned} \tag{2.42}$$

As before taking $\lambda = e^{i\theta}r^2$ with r sufficiently large such that $3C(1+r) \leq \frac{r^2}{2}$ we get (2.40). $\square$

Proposition 2.2.8. *Let $1 < p < \infty$. Then there exists $\omega_2 \in \mathbf{R}$ depending on n, p, such that if $\operatorname{Re}\lambda > \omega_2$ and $f \in L^p(\mathbf{R}^n_+)$ the problem*

$$\begin{cases} \lambda u - \Delta u = f & \text{in } \mathbf{R}^n_+ \\ \dfrac{\partial u}{\partial x_n} = 0 & \text{in } \partial\mathbf{R}^n_+ \end{cases} \tag{2.43}$$

has a unique solution $u \in W^{2,p}(\mathbf{R}^n_+)$. *Moreover there exists a constant* $c(\lambda) = c(n,p,\lambda)$ *such that*

$$\|u\|_{W^{2,p}(\mathbf{R}^n_+)} \le c(\lambda)\|f\|_{L^p(\mathbf{R}^n_+)}. \tag{2.44}$$

Proof. Uniqueness and (2.44) are consequences of Proposition 2.2.7. Concerning the existence, we consider the even extension of f with respect to the last variable

$$\tilde{f}(x', x_n) = \begin{cases} f(x', x_n) & x_n \ge 0 \\ f(x', -x_n) & x_n < 0 \end{cases}$$

By Theorem 2.2.2, for $\operatorname{Re}\lambda > 0$ there exists a unique solution $\tilde{u} \in W^{2,p}(\mathbf{R}^n)$ such that $\lambda\tilde{u} - \Delta\tilde{u} = \tilde{f}$. Now, it is easy to verify that the function $u(x', x_n) := \tilde{u}(x', -x_n)$ solves the elliptic problem $\lambda u - \Delta u = \tilde{f}$ in $\mathbf{R}^n$, and, by uniqueness, $u = \tilde{u}$, that is, $\tilde{u}$ is even in x_n and so $\frac{\partial \tilde{u}}{\partial x_n}(x', 0) = 0$. Therefore for $\operatorname{Re}\lambda > \sup\{\omega_1, 0\} =: \omega_2$, the restriction of $\tilde{u}$ in $\mathbf{R}^n_+$ is the unique solution of (2.43).

□

The following theorem extends results of existence and uniqueness of problem (2.43) to a problem where $\mathcal{A}$ replaces the Laplacian and more general oblique boundary conditions are considered.

Theorem 2.2.9. *Let* $1 < p < \infty$. *We assume that* $\beta_i, \gamma \in UC^1_b(\mathbf{R}^n_+)$ *and that the uniform non tangentiality condition*

$$\inf_{x \in \partial\mathbf{R}^n_+} |\langle \beta(x), e_n \rangle| > 0 \tag{2.45}$$

holds. Then there exists $\omega_3 \in \mathbf{R}$ *depending on* n, p, μ *such that for every* $f \in L^p(\mathbf{R}^n_+)$ *and* $\operatorname{Re}\lambda > \omega_3$ *the problem*

$$\begin{cases} \lambda u - \mathcal{A}u = f & \text{in } \mathbf{R}^n_+ \\ \dfrac{\partial u}{\partial \beta} + \gamma u = 0 & \text{in } \partial\mathbf{R}^n_+ \end{cases} \tag{2.46}$$

has a unique solution $u \in W^{2,p}(\mathbf{R}^n_+)$.

Proof. We set

$$X = W^{2,p}(\mathbf{R}^n_+) \qquad Y = L^p(\mathbf{R}^n_+) \times W^{1,p}(\mathbf{R}^{n-1})$$

and consider the operators $L_s : X \to Y$ so defined

$$L_s u := \Big(\lambda u - [(1-s)\Delta u + s\mathcal{A}u], (1-s)\frac{\partial u}{\partial \nu} + s(\gamma u + \frac{\partial u}{\partial \beta})\Big), \quad s \in [0,1],$$

where ν is the exterior unit normal vector to the domain, that is $\nu = -e_n$. We notice that

$$(1-s)\frac{\partial u}{\partial \nu} + s\frac{\partial u}{\partial \beta} = \frac{\partial u}{\partial \tau}$$

with $\tau = (1-s)\nu + s\beta$ satisfies (2.45) independently of s. Moreover $A_s = (1-s)\Delta u + s\mathcal{A}u$ satisfies (2.4) with $\mu_s \geq \mu$ and $M_s \leq (1 \vee M)$, therefore we can ignore the dependence of those constants by s. Hence in (2.40) the constant M_p can be chosen independently by s and the estimate

$$\|L_s u\|_Y \geq M_p^{-1}\|u\|_X$$

holds for every $s \in [0,1]$. By Proposition 2.2.8, L_0 is surjective, therefore by Theorem 1.5.3, L_1 is surjective too. □

The hypothesis of smoothness of the domain suggests to go back by means of local charts to balls or half balls of $\mathbf{R}^n$ and to apply the results obtained before in order to get the same result in Ω as the next theorem states.

Theorem 2.2.10. *Let Ω, $\mathcal{A}$ and $\mathcal{B}$ be as in* (2.1)-(2.7). *Then there exists ω_4 depending on n, p, μ, Ω such that if $\operatorname{Re}\lambda \geq \omega_4$ and $f \in L^p(\Omega)$, the problem*

$$\begin{cases} \lambda u - \mathcal{A}(\cdot, D)u = f & \text{in } \Omega \\ \mathcal{B}(\cdot, D)u = 0 & \text{in } \partial\Omega \end{cases} \tag{2.47}$$

has a unique solution $u \in W^{2,p}(\Omega)$. Moreover there exists $C = C(n,p,\mu,M,\Omega) > 0$ such that

$$|\lambda|\|u\|_{L^p(\Omega)} + |\lambda|^{\frac{1}{2}}\|Du\|_{L^p(\Omega)} + \|D^2u\|_{L^p(\Omega)} \leq C\|f\|_{L^p(\Omega)}. \tag{2.48}$$

Proof. Observe that if we prove the existence of a solution of (2.47) then uniqueness and estimate (2.48) follow immediately from Proposition 2.2.7. Indeed the estimate

$$|\lambda|\|u\|_{L^p(\Omega)} \leq M_1\|\lambda u - \mathcal{A}u\|_{L^p(\Omega)}$$

yields the injectivity of $\lambda - A_p^B$. Thus, we have only to prove the surjectivity of the operator $\lambda - A_p^B$.
By the regularity of the boundary $\partial\Omega$ we can consider a partition of unity $\{(\eta_h^2, U_h)\}_{h\in\mathbf{N}}$ such that $\operatorname{supp}\eta_h \subset U_h$, $\sum_{h=0}^{\infty}\eta_h^2(x) = 1$ for every $x \in \overline{\Omega}$, $0 \leq \eta_h \leq 1$ and $\|\eta_h\|_{W^{2,\infty}} \leq c_\eta$ for every $h \in \mathbf{N}$. Moreover let $(U_h)_{h\in\mathbf{N}}$ be such that $U_0 \subset\subset \Omega$, U_h for $h \geq 1$ is a ball such that $\{U_h\}_{h\geq 1}$ is a covering of $\partial\Omega$ and $\{U_h\}_{h\in\mathbf{N}}$ is a covering of Ω with bounded overlapping, that is, there is $\kappa > 0$ such that

$$\sum_{h\in\mathbf{N}} \chi_{U_h}(x) \leq \kappa, \quad \forall x \in \overline{\Omega}. \tag{2.49}$$

Moreover there exist coordinate transformations $\varphi_h : U_h \to B(0,1)$, C^2 diffeomorphisms, such that

$$\varphi_h(\overline{U_h} \cap \Omega) = B^+(0,1)$$
$$\varphi_h(\overline{U_h} \cap \partial\Omega) = B(0,1) \cap \{x_n = 0\}.$$

Moreover, all the coordinate transformations φ_h and their inverses are supposed to have uniformly bounded derivatives up to the second order,

$$\sup_{h\in\mathbf{N}} \sum_{1\leq|\alpha|\leq 2} \left(\|D^\alpha\varphi_h\|_\infty + \|D^\alpha\varphi_h^{-1}\|_\infty\right) \leq c \tag{2.50}$$

Let $f \in L^p(\Omega)$; then we can write $f = \sum_{h=0}^{\infty} \eta_h^2 f$. We notice that $\eta_0 f \in L^p(\mathbf{R}^n)$, $\mathrm{supp}\,(\eta_0 f) \subseteq U_0$. Thus if we extend a_{ij}, b_i and c to the whole space $\mathbf{R}^n$ in such a way that their qualitative properties are preserved, to the extension $\tilde{\mathcal{A}}$ we can apply the Theorem 2.2.6. Hence there exists $\tilde{\omega}_0 \in \mathbf{R}$ such that for $\mathrm{Re}\,\lambda \geq \tilde{\omega}_0$ the operator $\lambda - \tilde{\mathcal{A}}$ is invertible in $L^p(\mathbf{R}^n)$. Therefore if $R(\lambda) : L^p(\mathbf{R}^n) \to W^{2,p}(\mathbf{R}^n)$ denotes the resolvent of the operator $\tilde{A}_p$ in $\mathbf{R}^n$, we can define

$$R_0(\lambda)f := \eta_0 R(\lambda)(\eta_0 f).$$

Then supp $R_0(\lambda)f \subseteq U_0$ and $R_0(\lambda) : L^p(\Omega) \to W^{2,p}(\Omega)$ and

$$\begin{aligned}
(\lambda - \mathcal{A})R_0(\lambda)f &= (\lambda - \mathcal{A})(\eta_0 R(\lambda)(\eta_0 f)) \\
&= \eta_0(\lambda - \mathcal{A})R(\lambda)(\eta_0 f) + ((\lambda - \mathcal{A})\eta_0 I + \eta_0(\lambda - \mathcal{A}))\,(R(\lambda)(\eta_0 f)) \\
&= \eta_0^2 f + [\lambda - \mathcal{A}, \eta_0]R(\lambda)(\eta_0 f)
\end{aligned}$$

where $[X, Y] = XY - YX$ is the commutator of X and Y. Letting

$$S_{\eta_0}(\lambda) := [\lambda - \mathcal{A}, \eta_0 I]R(\lambda)\eta_0$$

we can write

$$(\lambda - \mathcal{A})R_0(\lambda)f = \eta_0^2 f + S_{\eta_0}(\lambda)f.$$

It is immediate to verify that $[\lambda - \mathcal{A}, \eta_0 I]g = -[\mathcal{A}, \eta_0 I]g$. Moreover

$$-[\mathcal{A}, \eta_0 I]g = -2\sum_{h,k=1}^{N} a_{hk} D_h\eta_0 D_k g - g(\sum_{i,j=1}^{n}(D_i(a_{ij}D_j\eta_0) + b_i D_i\eta_0)$$

If we define $B_0 = [\lambda - \mathcal{A}, \eta_0 I]$, we observe that B_0 is at most a first order differential operator whose coefficients depend on those of $\mathcal{A}$ and the function η_0. We have

$$\|B_0 g\|_{L^p(\Omega)} \leq C(M, c_\eta)\|g\|_{W^{1,p}(\Omega)}. \tag{2.51}$$

Hence, using (2.51) and estimates (2.36), (2.37), we get

$$\begin{aligned}
\|S_{\eta_0}(\lambda)f\|_{L^p(\Omega)} &= \|B_0(\lambda - \mathcal{A})^{-1}(\eta_0 f)\|_{L^p(\Omega)} \\
&\leq C(M, c_\eta)\|(\lambda - \mathcal{A})^{-1}(\eta_0 f)\|_{W^{1,p}(\Omega)} \\
&\leq \frac{C}{\sqrt{|\lambda|}}\|\eta_0 f\|_{L^p(\Omega)}
\end{aligned} \tag{2.52}$$

where $C = C(n, p, \mu, M, c_\eta, \Omega)$ e $\mathrm{Re}\,\lambda \geq \tilde{\omega}_0$. So for $S_{\eta_0}(\lambda)$ we get the following estimate

$$\|S_{\eta_0}(\lambda)\|_{\mathcal{L}(L^p(\Omega))} \leq C|\lambda|^{-1/2}.$$

Now, we consider the case $h \geq 1$. Let

$$v_h(y) := (\eta_h f)(\varphi_h^{-1}(y)) =: T_h(\eta_h f)(y)$$

then $v_h \in W^{2,p}(\mathbf{R}^n_+)$. We denote by $\hat{\mathcal{A}}_h$ the operator in $\mathbf{R}^n_+$ determined by the change of variables given by φ_h

$$\hat{\mathcal{A}}_h w := \mathrm{div}(\hat{A}_h\, Dw) + \langle \hat{B}_h, Dw\rangle + \hat{c}_h w \tag{2.53}$$

defined by the coefficients (here for $\hat{\mathcal{A}}$ and its coefficients we omit the index h to simplify the notations)

$$\begin{aligned}
\hat{A}_h(y) :=& \big(D\varphi_h\big)(\varphi_h^{-1}(y)) \cdot A(\varphi_h^{-1}(y)) \cdot \big(D\varphi_h\big)^t(\varphi_h^{-1}(y)) \\
(\hat{B}_h(y))_l :=& \mathrm{Tr}\Big[\big(D\varphi_h\big)(\varphi_h^{-1}(y)) \cdot A(\varphi_h^{-1}(y)) \cdot H^l(\varphi_h^{-1}(y)) \cdot \big(D\varphi_h^{-1}\big)^t(y)\Big] \\
&+ \mathrm{Tr}\Big[\big(D\varphi_h\big)(\varphi_h^{-1}(y)) \cdot G^j(y)\Big]\big(D\varphi_h\big)^t_{jl}(\varphi_h^{-1}(y)) - \frac{\partial}{\partial y_j}\Big[\hat{a}_{jl}(y)\Big] \\
&+ \Big[\big(D\varphi_h\big)(\varphi_h^{-1}(y)) \cdot B(\varphi_h^{-1}(y))\Big]_l \\
\hat{c}_h(y) :=& c(\varphi_h^{-1}(y))
\end{aligned}$$

where $H^l_{ki} = D^2_{ki}(\varphi_h)_l$ and $G^j_{ki} = D_k a_{ij}(\varphi_h^{-1}(y))$. We remark that $\mathcal{A}(\eta_h u)(x) = \hat{\mathcal{A}}_h v_h(y)$. For what concerns the boundary condition we get

$$\begin{aligned}
\mathcal{B}(\eta_h u)(x) &= \beta(x) \cdot D(\eta_h u)(x) + \gamma(x)(\eta_h u)(x) \\
&= \Big[\big(D\varphi_h\big)(\varphi_h^{-1}(y)) \cdot \beta(\varphi_h^{-1}(y))\Big](Dv_h)(y) \cdot D(\eta_h u)(x) + \gamma(\varphi_h^{-1}(y))v_h(y) \\
&= \frac{\partial v_h}{\partial \hat{\beta}}(y) + \hat{\gamma} v_h(y) = \hat{\mathcal{B}}_h v_h(y)
\end{aligned}$$

where $\hat{\beta}(y) = \Big[\big(D\varphi_h\big)(\varphi_h^{-1}(y)) \cdot B(\varphi_h^{-1}(y))\Big]$ and $D\varphi_h$ denotes the Jacobian matrix of φ_h and $\hat{\gamma}(y) = \gamma(\varphi_h^{-1}(y))$. Now, since β is not tangent to $\partial\Omega$, $\hat{\beta}$ is not tangent to $\mathbf{R}^n_+$. We define

$$R_h(\lambda)f := T_h^{-1}\Big(T_h(\eta_h)(\lambda - \hat{\mathcal{A}}_h)^{-1}T_h(\eta_h f)\Big),$$

where $(\lambda - \hat{\mathcal{A}}_h)^{-1}$ is the resolvent of $\hat{\mathcal{A}}_h$ in $\mathbf{R}^n_+$ with the boundary condition $\hat{\mathcal{B}}_h v_h = 0$. Then $R_h(\lambda) : L^p(\Omega) \to W^{2,p}(\Omega)$ with $\mathcal{B}R_h(\lambda)f = 0$ in $\partial\Omega$ and $\operatorname{supp}(R_h(\lambda)f) \subset U_h$. We get

$$(\lambda - \mathcal{A})R_h(\lambda)f = \eta_h^2 f + S_{\eta_h}(\lambda)f$$

where $S_{\eta_h}(\lambda) = T_h^{-1}\Big([\lambda - \hat{\mathcal{A}}_h, T_h(\eta_h)](\lambda - \hat{\mathcal{A}}_h)^{-1}(T_h(\eta_h f))\Big)$.
As before for $\operatorname{Re}\lambda$ sufficiently large

$$\|S_{\eta_h}(\lambda)f\|_{L^p(\Omega)} \le c(n,p,\mu,M,\Omega,c_\eta)|\lambda|^{-1/2}\|\eta_h f\|_{L^p(\Omega)} \tag{2.54}$$

Finally, letting

$$V(\lambda) = \sum_{h\in\mathbf{N}\cup\{0\}} R_h(\lambda) : L^p(\Omega) \to W^{2,p}(\Omega)$$

observe that $\mathcal{B}V(\lambda)f = 0$ in $\partial\Omega$ and

$$(\lambda - \mathcal{A})V(\lambda)f = \sum_{h=0}^{\infty}\eta_h^2 f + \sum_{h=0}^{\infty} S_{\eta_h}(\lambda)f = f + \sum_{h=0}^{\infty} S_{\eta_h}(\lambda)f.$$

Hence

$$(\lambda - \mathcal{A})V(\lambda) : L^p(\Omega) \to L^p(\Omega) \qquad \text{and} \qquad (\lambda - \mathcal{A})V(\lambda) = I + \sum_{h=0}^{\infty} S_{\eta_h}(\lambda)$$

Now, let observe that we can select λ with $\operatorname{Re}\lambda$ sufficiently large such that

$$\|\sum_{h=0}^{\infty} S_{\eta_h}(\lambda)\|_{\mathcal{L}(L^p(\Omega))} \le \frac{1}{2}, \tag{2.55}$$

indeed, since each S_{η_h} has support contained in U_h and the covering $\{U_i\}_i$ has bounded overlapping (2.49), then

$$\begin{aligned}\|\sum_{h=0}^{\infty} S_{\eta_h}(\lambda)f\|_{L^p(\Omega)} &\le \sum_{i=0}^{\infty}\int_{U_i}|\sum_{h=0}^{\infty} S_{\eta_h}(\lambda)f|^p\,dx\\ &\le \frac{c}{\sqrt{|\lambda|}}\sum_{i=0}^{\infty}\int_{U_i}|f|^p\,dx \le \frac{c}{\sqrt{|\lambda|}}\|f\|_{L^p(\Omega)}\end{aligned}$$

where $c = c(M, c_\eta, \kappa, \Omega)$. Then, (2.55) ensures that for $\operatorname{Re}\lambda$ sufficiently large, the operator $I + \sum_{h=0}^{\infty} S_{\eta_h}(\lambda)$ is invertible in $L^p(\Omega)$ with inverse $W(\lambda) : L^p(\Omega) \to L^p(\Omega)$. Hence, since $(\lambda - \mathcal{A})V(\lambda)W(\lambda) = I$ in $L^p(\Omega)$ and $u = V(\lambda)W(\lambda)f \in W^{2,p}(\Omega)$ is the solution of (2.47) for $\operatorname{Re}\lambda$ large enough. □

2.3 Generation of analytic semigroup in $L^\infty(\Omega)$ and in the space $C(\overline{\Omega})$

Henceforth Ω will be a domain with uniformly C^2 boundary and we set, for $x_0 \in \mathbf{R}^n$ and $r > 0$,

$$\Omega_{x_0,r} = \Omega \cap B(x_0, r). \tag{2.56}$$

Our aim is to prove that the realization A_∞^B of $\mathcal{A}$ in L^∞ with homogeneous oblique boundary conditions as in (2.5)-(2.7) is a sectorial operator. In order to reach this we need that $\rho(A_\infty^B)$ contains an half plane and that an estimate like $|\lambda|\|u\|_{L^\infty(\Omega)} \le c\|\lambda u - \mathcal{A}u\|_{L^\infty(\Omega)}$ hold for $\operatorname{Re}\lambda$ large, $\lambda \in \rho(A_\infty^B)$. An important tool for the proof of the resolvent estimate in L^∞ is given by the following lemma in which a Caccioppoli type inequality in the L^p norm is stated.

Lemma 2.3.1. *Let $p > 1$ and $u \in W^{2,p}_{loc}(\Omega)$. For every λ with $\operatorname{Re}\lambda \ge \omega_1$ (ω_1 is given in Proposition (2.2.7)), set $f = \lambda u - \mathcal{A}u$ and $g = \mathcal{B}u_{|\partial\Omega}$. Then there exists C_1 depending only by n, p, μ, M and Ω such that for every $x_0 \in \overline{\Omega}$, $r \le 1$, $\alpha \ge 1$,*

$$\begin{aligned}&|\lambda|\|u\|_{L^p(\Omega_{x_0,r})} + |\lambda|^{\frac{1}{2}}\|Du\|_{L^p(\Omega_{x_0,r})} + \|D^2u\|_{L^p(\Omega_{x_0,r})}\\ &\le C_1\Bigg\{\|f\|_{L^p(\Omega_{x_0,(\alpha+1)r})} + (|\lambda|^{1/2} + \frac{1}{\alpha r})\|g_1\|_{L^p(\Omega_{x_0,(\alpha+1)r})} + \|Dg_1\|_{L^p(\Omega_{x_0,(\alpha+1)r})}\\ &\quad + \frac{1}{\alpha}\left[(\frac{1}{r^2} + \frac{|\lambda|^{1/2}}{r})\|u\|_{L^p(\Omega_{x_0,(\alpha+1)r})} + r^{-1}\|Du\|_{L^p(\Omega_{x_0,(\alpha+1)r})}\right]\Bigg\}\end{aligned} \tag{2.57}$$

where g_1 is any extension to $\overline{\Omega}$ of $\mathcal{B}u_{|\partial\Omega}$ of class $W^{1,p}_{loc}$.

Proof. Let $\theta_0 : \mathbf{R}^n \to \mathbf{R}$ be a smooth function such that $\theta_0 = 1$ in $B(0,r)$, $\operatorname{supp}\theta_0 \subset B(0,(\alpha+1)r)$ with

$$\|\theta_0\|_{L^\infty(\mathbf{R}^n)} + \alpha r\|D\theta_0\|_{L^\infty(\mathbf{R}^n)} + \alpha^2 r^2\|D^2\theta_0\|_{L^\infty(\mathbf{R}^n)} \le K$$

where K does not depend on α and r. We fix $x_0 \in \overline{\Omega}$, we set $\theta(x) = \theta_0(x - x_0)$. Define

$$v(x) = \theta(x)u(x), \quad x \in \Omega.$$

then v satisfies the following equation

$$\lambda v - \mathcal{A}(\cdot,D)v = \theta f - 2\sum_{i,j} a_{ij}D_i\theta D_j u - u\Big(\sum_{ij} D_i(a_{ij}D_j\theta) - \sum_{i=1}^n b_i D_i\theta\Big) =: f' \tag{2.58}$$

and the following boundary condition

$$\mathcal{B}v = \theta g + u\sum_{i=1}^n \beta_i D_i\theta \quad \text{in } \partial\Omega$$

Now, since $\operatorname{Re}\lambda \ge \omega_1$ and u and v coincide in $\Omega_{x_0,r}$, using Proposition 2.2.7 we get

$$\begin{aligned}
&|\lambda|\|u\|_{L^p(\Omega_{x_0,r})} + |\lambda|^{\frac12}\|Du\|_{L^p(\Omega_{x_0,r})} + \|D^2u\|_{L^p(\Omega_{x_0,r})}\\
&\quad\le |\lambda|\|v\|_{L^p(\Omega)}|\lambda|^{\frac12}\|Dv\|_{L^p(\Omega)} + \|D^2u\|_{L^p(\Omega)}\\
&\quad\le M_p(\|f'\|_{L^p(\Omega)} + |\lambda|^{1/2}\|\theta g_1 + u\sum_{i=1}^n \beta_i D_i\theta\|_{L^p(\Omega)}\\
&\qquad + \|D(\theta g_1) + D(u\sum_{i=1}^n \beta_i D_i\theta)\|_{L^p(\Omega)}).
\end{aligned} \tag{2.59}$$

Set $C_0 = \max_{i,j}\|a_{ij}\|_{W^{1,\infty}(\Omega)} + \max_i\|b_i\|_{L^\infty(\Omega)}$.
Then

$$\begin{aligned}
\|f'\|_{L^p(\Omega)} &\le \|f\|_{L^p(\Omega_{x_0,(\alpha+1)r})} + C_0K\Big(\frac{2}{\alpha r}\|Du\|_{L^p(\Omega_{x_0,(\alpha+1)r})}\\
&\quad + \frac{1}{\alpha^2r^2}\|u\|_{L^p(\Omega_{(x_0,(\alpha+1)r)})} + \frac{1}{\alpha r}\|u\|_{L^p(\Omega_{x_0,(\alpha+1)r})}\Big).
\end{aligned} \tag{2.60}$$

Moreover

$$\begin{aligned}
&|\lambda|^{1/2}\|u\sum_{i=1}^n \beta_i D_i\theta\|_{L^p(\Omega)} + \|D(u\sum_{i=1}^n \beta_i D_i\theta)\|_{L^p(\Omega)}\\
&\quad\le |\lambda|^{1/2}\sum_{i=1}^n \|\beta_i\|_\infty \frac{K}{\alpha r}\|u\|_{L^p(\Omega_{x_0,(\alpha+1)r})}\\
&\qquad + \sum_{i=1}^N \Big(\|D\beta_i\|_\infty\frac{K}{\alpha r} + \|\beta_i\|_\infty\frac{K}{\alpha^2r^2}\Big)\|u\|_{L^p(\Omega_{x_0,(\alpha+1)r})}\\
&\qquad + \sum_{i=1}^n \|\beta_i\|_\infty\frac{K}{\alpha r}\|Du\|_{L^p(\Omega_{x_0,(\alpha+1)r})}\\
&\quad\le \frac{CK}{\alpha}\Big[\Big(\frac{|\lambda|^{1/2}}{r} + \frac{2}{r^2}\Big)\|u\|_{L^p(\Omega_{x_0,(\alpha+1)r})} + \frac{1}{r}\|Du\|_{L^p(\Omega_{x_0,(\alpha+1)r})}\Big]
\end{aligned} \tag{2.61}$$

where $C = \sum_{i=1}^n \|\beta_i\|_{C^1(\overline{\Omega})}$, and

$$
\begin{aligned}
&|\lambda|^{1/2}\|\theta g_1\|_{L^p(\Omega)} + \|D(\theta g_1)\|_{L^p(\Omega)} \\
&\qquad \le |\lambda|^{1/2}\|g_1\|_{L^p(\Omega_{x_0,(\alpha+1)r})} + \frac{K}{\alpha r}\|g_1\|_{L^p(\Omega_{x_0,(\alpha+1)r})} + \|Dg_1\|_{L^p(\Omega_{x_0,(\alpha+1)r})} \qquad (2.62)
\end{aligned}
$$

Taking into account that $r \le 1$ and $\alpha \ge 1$, replacing (2.60), (2.61) and (2.62) in (2.59) we get the claim. $\square$

As a consequence we get the resolvent estimate as the following theorem states.

Theorem 2.3.2. *Let $p > n$. Then there exists $K > 0$ depending on n, p, μ, M, Ω, such that for every $\lambda \in \mathbf{C}$ with $\operatorname{Re}\lambda \ge \Lambda_p^1 = \omega_1 \vee 1$ (ω_1 is given in Proposition 2.2.7) and for every $u \in C_b^1(\overline{\Omega}) \cap W^{2,p}_{loc}(\Omega)$*

$$
\begin{aligned}
&|\lambda|\|u\|_{L^\infty(\Omega)} + |\lambda|^{1/2}\|Du\|_{L^\infty(\Omega)} + |\lambda|^{n/2p}\sup_{x_0\in\overline{\Omega}}\|D^2u\|_{L^p(\Omega_{x_0,|\lambda|^{-1/2}})} \\
&\quad \le K\Big(|\lambda|^{n/2p}\sup_{x_0\in\overline{\Omega}}\|\lambda u - \mathcal{A}u\|_{L^p(\Omega_{x_0,|\lambda|^{-1/2}})} \\
&\qquad + |\lambda|^{1/2}\|g_1\|_{L^\infty(\Omega)} + |\lambda|^{n/2p}\sup_{x_0\in\overline{\Omega}}\|Dg_1\|_{L^p(\Omega_{x_0,|\lambda|^{-1/2}})}\Big), \qquad (2.63)
\end{aligned}
$$

where g_1 is any extension of $g = \mathcal{B}u_{|\partial\Omega}$ belonging to $W^{1,p}_{loc}$. Moreover, there is $\tilde{K} > 0$ such that if $\mathcal{A}u \in L^\infty(\Omega)$ and $\mathcal{B}u_{|\partial\Omega} \in C^1(\partial\Omega)$, then

$$
\begin{aligned}
&|\lambda|\|u\|_{L^\infty(\Omega)} + |\lambda|^{1/2}\|Du\|_{L^\infty(\Omega)} + |\lambda|^{n/2p}\sup_{x_0\in\overline{\Omega}}\|D^2u\|_{L^p(\Omega_{x_0,|\lambda|^{-1/2}})} \\
&\quad \le \tilde{K}\Big(\|\lambda u - \mathcal{A}u\|_{L^\infty(\Omega)} + |\lambda|^{1/2}\|\mathcal{B}u\|_{C(\partial\Omega)} + \|\mathcal{B}u\|_{C^1(\partial\Omega)}\Big). \qquad (2.64)
\end{aligned}
$$

Proof. Let $x_0 \in \overline{\Omega}$, $|\lambda| \ge 1$, $\operatorname{Re}\lambda \ge \omega_1$ and $r = |\lambda|^{-\frac{1}{2}}$; then using the Sobolev inequality (i) of Theorem 1.5.2 we get

$$
\begin{aligned}
&|\lambda|\|u\|_{L^\infty(\Omega_{x_0,r})} + |\lambda|^{\frac{1}{2}}\|Du\|_{L^\infty(\Omega_{x_0,r})} + |\lambda|^{\frac{n}{2p}}\|D^2u\|_{L^p(\Omega_{x_0,r})} \\
&\quad \le (2C+1)|\lambda|^{\frac{n}{2p}}\Big(|\lambda|\|u\|_{L^p(\Omega_{x_0,r})} + |\lambda|^{\frac{1}{2}}\|Du\|_{L^p(\Omega_{x_0,r})} + \|D^2u\|_{L^p(\Omega_{x_0,r})}\Big).
\end{aligned}
$$

Now, using Lemma 2.3.1, we get, for every $\alpha \ge 1$,

$$
\begin{aligned}
&|\lambda|^{\frac{n}{2p}}\Big(|\lambda|\|u\|_{L^p(\Omega_{x_0,r})} + |\lambda|^{\frac{1}{2}}\|Du\|_{L^p(\Omega_{x_0,r})} + \|D^2u\|_{L^p(\Omega_{x_0,r})}\Big) \\
&\quad \le C_1|\lambda|^{\frac{n}{2p}}[\|f\|_{L^p(\Omega_\alpha)} + |\lambda|^{1/2}\big(1+\frac{1}{\alpha}\big)\|g_1\|_{L^p(\Omega_\alpha)} \\
&\quad + \|Dg_1\|_{L^p(\Omega_\alpha)} + \frac{2}{\alpha}\big(|\lambda|\|u\|_{L^p(\Omega_\alpha)} + |\lambda|^{1/2}\|Du\|_{L^p(\Omega_\alpha)}\big) \\
&\quad \le C(|\lambda|^{\frac{n}{2p}}\|f\|_{L^p(\Omega_\alpha)} + \omega_n^{1/p}(\alpha+1)^{n/p}|\lambda|^{1/2}\|g_1\|_{L^\infty(\Omega_\alpha)} \\
&\quad + |\lambda|^{\frac{n}{2p}}\|Dg_1\|_{L^p(\Omega_\alpha)} + (\frac{\omega_n^{1/p}(\alpha+1)^{n/p}}{\alpha})\big(|\lambda|\|u\|_{L^\infty}(\Omega) + |\lambda|^{1/2}\|Du\|_{L^\infty(\Omega)}\big))
\end{aligned}
$$

$$(2.65)$$

where $\Omega_\alpha = \Omega \cap B_\alpha(x_0) = \Omega \cap B(x_0, (\alpha+1)|\lambda|^{-1/2})$. Therefore

$$
\begin{aligned}
&|\lambda|\|u\|_{L^\infty(\Omega_{x_0,r})} + |\lambda|^{\frac{1}{2}}\|Du\|_{L^\infty(\Omega_{x_0,r})} + |\lambda|^{\frac{n}{2p}}\|D^2u\|_{L^p(\Omega_{x_0,r})} \\
&\quad \le C[|\lambda|^{\frac{n}{2p}}\|f\|_{L^p(\Omega_\alpha)} + \omega_n^{1/p}(\alpha+1)^{n/p}|\lambda|^{1/2}\|g_1\|_{L^\infty(\Omega_\alpha)} \\
&\quad + |\lambda|^{\frac{n}{2p}}\|Dg_1\|_{L^p(\Omega_\alpha)} + (\frac{\omega_n^{1/p}(\alpha+1)^{n/p}}{\alpha})\big(|\lambda|\|u\|_{L^\infty(\Omega_\alpha)} + |\lambda|^{1/2}\|Du\|_{L^\infty(\Omega_\alpha)}\big)
\end{aligned}
\tag{2.66}
$$

where C is a constant depending on p, n, μ, Ω. Taking the supremum over $x_0 \in \overline{\Omega}$ of the three addenda on the left hand side of (2.66) and summing up we get

$$
\begin{aligned}
&|\lambda|\|u\|_{L^\infty(\Omega)} + |\lambda|^{1/2}\|Du\|_{L^\infty(\Omega)} + |\lambda|^{\frac{n}{2p}} \sup_{x_0\in\overline{\Omega}} \|D^2u\|_{L^p(\Omega_{x_0,|\lambda|^{-1/2}})} \\
&\quad \le C\Big(|\lambda|^{\frac{n}{2p}} \sup_{x_0\in\overline{\Omega}} \|f\|_{L^p(\Omega_\alpha)} + \omega_n^{\frac{1}{p}}\frac{(\alpha+1)^{\frac{n}{p}}}{\alpha}(|\lambda|\|u\|_{L^\infty(\Omega)} + |\lambda|^{\frac{1}{2}}\|Du\|_{L^\infty(\Omega)}) \\
&\quad + \omega_n^{1/p}(\alpha+1)^{n/p}|\lambda|^{1/2}\|g_1\|_{L^\infty(\Omega)} + |\lambda|^{\frac{n}{2p}} \sup_{x_0\in\overline{\Omega}} \|Dg_1\|_{L^p(\Omega_\alpha)}\Big)
\end{aligned}
$$

Taking α sufficiently large in such a way that

$$
C\omega_n^{\frac{1}{p}}\frac{(\alpha+1)^{\frac{n}{p}}}{\alpha} \le \frac{1}{2},
$$

we obtain

$$
\begin{aligned}
&|\lambda|\|u\|_{L^\infty(\Omega)} + |\lambda|^{\frac{1}{2}}\|Du\|_{L^\infty(\Omega)} + |\lambda|^{\frac{n}{2p}} \sup_{x_0\in\overline{\Omega}} \|D^2u\|_{L^p(\Omega_{x_0,|\lambda|^{-1/2}})} \\
&\quad \le 2C(|\lambda|^{\frac{n}{2p}} \sup_{x_0\in\overline{\Omega}} \|f\|_{L^p(\Omega_\alpha)} + |\lambda|^{1/2}\|g_1\|_{L^\infty(\Omega)} + |\lambda|^{\frac{n}{2p}} \sup_{x_0\in\overline{\Omega}} \|Dg_1\|_{L^p(\Omega_\alpha)})
\end{aligned}
$$

Finally we can obtain (2.63) covering each ball $B_\alpha(x_0)$ with a finite number of balls with radius $|\lambda|^{-\frac{1}{2}}$.
To prove (2.64) we use (2.63), which implies

$$
\begin{aligned}
&|\lambda|\|u\|_{L^\infty(\Omega)} + |\lambda|^{1/2}\|Du\|_{L^\infty(\Omega)} + |\lambda|^{n/2p} \sup_{x_0\in\overline{\Omega}} \|D^2u\|_{L^p(\Omega_{x_0,|\lambda|^{-1/2}})} \\
&\quad \le K\big[\omega_n^{1/p}\Big(\|\lambda u - \mathcal{A}u\|_{L^\infty(\Omega)} + \|Dg_1\|_{L^\infty(\Omega)}\Big) + |\lambda|^{1/2}\|g_1\|_{L^\infty(\Omega)}\big]
\end{aligned}
$$

Finally, choosing $g_1 = E(\mathcal{B}u_{\partial\Omega})$, where $E \in \mathcal{L}(C(\partial\Omega), C(\overline{\Omega})) \cap \mathcal{L}(C^1(\partial\Omega), C^1(\overline{\Omega}))$ is an extension operator we get the claim. $\square$

Next theorem, together with the resolvent estimate (2.64), is sufficient to prove the sectoriality of the realization of $\mathcal{A}$ in $L^\infty(\Omega)$ so defined

$$
\begin{cases}
D(A_\infty^B) = \{u \in \bigcap_{p\ge 1} W^{2,p}_{loc}(\Omega); \quad u,\, \mathcal{A}u \in L^\infty(\Omega),\ \mathcal{B}u_{|\partial\Omega} = 0\}, \\
A_\infty^B u = \mathcal{A}u.
\end{cases}
$$

Theorem 2.3.3. *The operator $A_\infty^B : D(A_\infty^B) \to L^\infty(\Omega)$ is sectorial. Moreover, $D(A_\infty^B) \subset C^{1,\alpha}(\overline{\Omega})$, for every $\alpha \in]0,1[$.*

Proof. Fix $p > n$. Let $\Lambda_0 = \inf_{p>n} \Lambda_p^1$; then we prove that the resolvent set of A_∞^B contains the half plane $\{\lambda \in \mathbf{C};\ \mathrm{Re}\,\lambda > \Lambda_0\}$. First we show that the $\rho(A_\infty^B)$ contains the half plane $\{\mathrm{Re}\,\lambda \geq \Lambda_p^1\}$. For any $f \in L^\infty(\Omega)$ and $k \in \mathbf{N}$, let ψ_k be a cut-off function such that

$$0 \leq \psi_k \leq 1, \quad \psi_k \equiv 1 \quad \text{in } B(0,k), \quad \psi_k \equiv 0 \quad \text{outside } B(0,2k).$$

We consider $f_k = \psi_k f$. Now, if $\mathrm{Re}\,\lambda > \Lambda_p^1$, then, by Theorem 2.2.10, the problem

$$\begin{cases} \lambda u_k - \mathcal{A} u_k = f_k & \text{in } \Omega \\ \mathcal{B} u_k = 0 & \text{in } \partial\Omega \end{cases} \tag{2.67}$$

has a unique solution $u_k \in W^{2,p}(\Omega)$ and $\|u_k\|_{W^{2,p}(\Omega)} \leq C\|f_k\|_{L^p(\Omega)}$ where C is a constant depending on λ, n, p, M, Ω and μ. In particular, by the Sobolev embedding theorem (see Theorem 1.5.2), $u_k \in C_b^1(\overline{\Omega})$, therefore using (2.64) we get

$$\|u_k\|_{C^1(\Omega)} + \sup_{x_0 \in \overline{\Omega}} \|D^2 u_k\|_{L^p(\Omega_{x_0,|\lambda|^{-1/2}})} \leq K(\lambda)\|f_k\|_{L^\infty(\Omega)} \leq K(\lambda)\|f\|_{L^\infty(\Omega)}. \tag{2.68}$$

Therefore, $\{u_k\}_k$ is bounded in $C^1(\Omega)$, so that there exists a subsequence converging uniformly on each compact subset of $\overline{\Omega}$ to a function $u \in C(\overline{\Omega}) \cap Lip(\Omega)$ such that

$$\|u\|_{L^\infty(\Omega)} + [u]_{Lip(\Omega)} \leq K(\lambda)\|f\|_{L^\infty(\Omega)}. \tag{2.69}$$

Now, we show that $u \in W_{loc}^{2,p}(\Omega)$ and that it solves

$$\begin{cases} \lambda u - \mathcal{A} u = f & \text{in } \Omega \\ \mathcal{B} u = 0 & \text{in } \partial\Omega \end{cases}$$

Let $B(x_0, R)$ be the closed ball with $x_0 \in \overline{\Omega}$ and $R \geq 4|\lambda|^{-1/2}$, then by (2.68) we know that $\{u_k\}_k$ is bounded in $W^{2,p}(\Omega_{x_0,R})$, so that the limit function u is in $W^{2,p}(\Omega_{x_0,R})$. Since x_0 and R are arbitrary, $u \in W_{loc}^{2,p}(\Omega)$. Moreover there exists a subsequence $\{u_{\phi(k)}\}_k$ converging to u in $W^{1,p}(\Omega_{x_0,R})$, and for h, k sufficiently large

$$\begin{cases} \lambda(u_{\phi(h)} - u_{\phi(k)}) - \mathcal{A}(u_{\phi(h)} - u_{\phi(k)}) = 0 & \text{in } \Omega_{x_0,R} \\ \mathcal{B}(u_{\phi(h)} - u_{\phi(k)}) = 0 & \text{in } \partial\Omega \cap B_{x_0,R} \end{cases}$$

Now, applying Lemma 2.3.1 to the function $u_{\phi(h)} - u_{\phi(k)}$, we get

$$\begin{aligned} \|u_{\phi(h)} - u_{\phi(k)}\|_{W^{2,p}(\Omega_{x_0,|\lambda|^{-1/2}})} &\leq C(\lambda)\|u_{\phi(h)} - u_{\phi(k)}\|_{W^{1,p}(\Omega_{x_0,2|\lambda|^{-1/2}})} \\ &\leq C(\lambda)\|u_{\phi(h)} - u_{\phi(k)}\|_{W^{1,p}(\Omega_{x_0,R})} \to 0 \qquad \text{as } h, k \to \infty. \end{aligned}$$

Covering $B(x_0, R/2)$ by a finite number of balls with radius $|\lambda|^{-1/2}$ we get that $\{u_{\phi(k)}\}_k$ converges in $W^{2,p}(\Omega_{x_0,R/2})$, so that, letting $k \to \infty$ in (2.67) we get $\lambda u - Au = f$ in $\Omega_{x_0,R/2}$.

Moreover since the trace operator $u \to u_{\partial\Gamma}$ is continuous from $W^{1,p}(\Gamma)$ to $L^p(\partial\Gamma, d\mathcal{H}^{n-1})$ for every open subset Γ of $\mathbf{R}^n$ with bounded Lipschitz boundary, then $\mathcal{B}$ is a linear and continuous operator from $W^{2,p}(\Omega_{x_0,R/2})$ to $L^p(\partial\Omega_{x_0,R/2})$, hence we get

$$\|\mathcal{B}(u_k - u)\|_{L^p(\partial\Omega \cap B(x_0,R/2))} \leq c_1\|u_k - u\|_{W^{2,p}(\Omega_{x_0,R/2})},$$

where c_1 is a constant depending on Ω, R and by $\|\beta_i\|_{L^\infty(\Omega)}, \|\gamma\|_{L^\infty(\Omega)}$. Therefore we get $\mathcal{B}u = 0$ in $\partial\Omega \cap B(x_0, R/2)$. Since x_0 and R are arbitrary, then

$$\begin{cases} \lambda u - \mathcal{A}u = f & \text{in } \Omega \\ \mathcal{B}u = 0 & \text{in } \partial\Omega \end{cases}$$

Now, fixed any $q > n$ we can write (2.67) as follows

$$\Lambda_q u_k - \mathcal{A}u_k = (\Lambda_q - \lambda)u_k + f_k.$$

We observe that the right hand side is in $L^\infty(\Omega)$, and its sup norm is bounded by a constant independent of k. Repeating the above arguments we conclude that $u \in W^{2,q}_{\rm loc}(\Omega)$ for all $q > n$, so that $u \in D(A^B_\infty)$. Therefore $\rho(A^B_\infty) \supset \{\lambda \in \mathbf{C} :\ \mathrm{Re}\,\lambda > \Lambda^1_p\}$ for every $p > n$. Thus, from estimate (2.64) and Proposition 1.2.7 we conclude that A^B_∞ is sectorial. Now, let $u \in D(A^B_\infty)$, then by the Sobolev embedding u is continuously differentiable and its gradient is bounded: indeed, fixed $p > n$ and $f = \Lambda_p u - \mathcal{A}u$, by estimate (2.69) we get

$$\|Du\|_{L^\infty(\Omega)} \le c(\|u\|_{L^\infty(\Omega)} + \|Au\|_{L^\infty(\Omega)})$$

Moreover, choosing $p = n/(1-\alpha)$, using Theorem 1.5.2 (inequality (ii)) and (2.64) with $\lambda = \Lambda^1_p$ we get, for $i = 1, \ldots, n$,

$$|D_i u(x) - D_i u(y)| \le c|x-y|^\alpha (\|u\|_{L^\infty(\Omega)} + \|Au\|_{L^\infty(\Omega)})$$

for all $x, y \in \mathbf{R}^n$ such that $|x-y| \le (\Lambda^1_p)^{-1/2}$. On the other hand, if $|x-y| \ge (\Lambda^1_p)^{-1/2}$ then

$$\begin{aligned} \frac{|D_i u(x) - D_i u(y)|}{|x-y|^\alpha} &\le 2\|D_i u\|_{L^\infty(\Omega)} (\Lambda^1_p)^{\alpha/2} \\ &\le c(\|u\|_{L^\infty(\mathbf{R}^n)} + \|Au\|_{L^\infty(\mathbf{R}^n)}) \end{aligned}$$

Therefore, $D(A^B_\infty) \subset C^{1,\alpha}(\overline{\Omega})$ for $\alpha \in]0,1[$.

□

From Theorems 2.3.2 and 2.3.3 we get the following result.

Corollary 2.3.4. *Let Λ_0 be as in Theorem 2.3.3. Set*

$$\begin{cases} D(A^B_C) = \{u \in \bigcap_{p \ge 1} W^{2,p}_{loc}(\Omega);\quad u, \mathcal{A}u \in C_b(\overline{\Omega}), \mathcal{B}u_{|\partial\Omega} = 0\}, \\ A^B_C u : D(A^B_C) \to C_b(\overline{\Omega}), \qquad A^B_C = \mathcal{A}u. \end{cases}$$

Then the resolvent set of A^B_C contains the half plane $\{\lambda \in \mathbf{C};\ \mathrm{Re}\,\lambda > \Lambda_0\}$, and A^B_C is sectorial.

Proof. Since $D(A^B_\infty) \subset C_b(\overline{\Omega})$, then $\rho(A^B_\infty) \subset \rho(A^B_C)$. Therefore $\rho(A^B_C)$ contains the half plane $\{\mathrm{Re}\,\lambda > \Lambda_0\}$. Estimate (2.64) and Proposition 1.2.7 prove that A^B_C is sectorial. □

2.4 Elliptic boundary value problems in some Sobolev spaces of negative order

In this section, as in the preceding one, we suppose that Ω is a domain with uniformly C^2 boundary $\partial\Omega$. Here our aim is to prove existence, uniqueness and some useful estimates for the solution of a boundary value problem for an elliptic operator $\mathcal{A}$ in suitable Sobolev spaces of negative order. Actually, we are interested in deducing L^1 norm estimates of the gradient of the resolvent of the realization of $\mathcal{A}$ in L^1 (see Theorem 2.5.3). This can be done by duality starting from the solution of the dual problem.
In this section we follow, with significant modifications, ideas from [47], [48]. Before stating the main result, let us introduce some notation.
Let $1 \le p < \infty$; we shall consider the Banach spaces $(W_0^{1,p}(\Omega))'$ and $(W^{1,p}(\Omega))'$ respectively denoted by $W^{-1,p'}(\Omega)$ and $W_*^{-1,p'}(\Omega)$ (we set $1' = \infty$). Each element $f \in W^{-1,p'}(\Omega)$ (resp. $f \in W_*^{-1,p'}(\Omega)$) admits a (not unique) $L^{p'}$ representation; that is, there exist $f_0, f_1, \dots, f_n \in L^{p'}(\Omega)$ such that

$$\langle f, v\rangle_* = \int_\Omega f_0 v\, dx + \sum_{i=1}^n \int_\Omega f_i D_i v dx \tag{2.70}$$

for every $v \in W_0^{1,p'}(\Omega)$ (resp. $v \in W^{1,p'}(\Omega)$), where $\langle\cdot,\cdot\rangle_*$ denotes the duality between $W^{-1,p}$ and $W_0^{1,p'}$ (resp. $W_*^{-1,p}$ and $W^{1,p'}$), see [1, Theorem 3.8]. In order to indicate an $L^{p'}$ representation of f we often write

$$f = f_0 - \sum_{i=1}^n D_i f_i \tag{2.71}$$

where the equality has to be intended in the distributional sense specified in (2.70). Obviously $(W^{1,p}(\Omega))'$ is continuously embedded in $(W_0^{1,p}(\Omega))'$, and there is a natural embedding of $L^{p'}(\Omega)$ in $(W^{1,p}(\Omega))'$: we can identify any $L^{p'}$ function f_0 with the functional

$$v \mapsto \int_\Omega f_0(x) v(x)\, dx.$$

We can consider these spaces as Banach spaces endowed with either the norm induced by duality or the norm defined by

$$\inf\left\{\sum_{i=0}^n \|f_i\|_{L^{p'}(\Omega)},\ f_i \text{ satisfying } (2.70)\right\}.$$

In the following lemma we prove some useful estimates that hold in these spaces.

Lemma 2.4.1. *For each $p > n$ there exist two constants c_1, c_2 such that for each $x_0 \in \overline{\Omega}$, $r > 0$ and $u \in L^p(\Omega)$ with support in $\Omega_{x_0,r}$ (given in (2.56)),*

$$\|u\|_{W_*^{-1,p}(\Omega)} \le c_1 r \|u\|_{L^p(\Omega)} \tag{2.72}$$

$$\|u\|_{W_*^{-1,\infty}(\Omega)} \le c_2 r^{1-n/p} \|u\|_{L^p(\Omega)} \tag{2.73}$$

Proof. Let $\varphi \in W^{1,p'}(\Omega)$ be such that $\|\varphi\|_{W^{1,p'}(\Omega)} \leq 1$. Then by Sobolev embedding $\varphi \in L^q(\Omega)$ with $q = (np')/(n-p')$ and $\|\varphi\|_{L^q(\Omega)} \leq c$ where c depends only on Ω. Hence

$$\|u\|_{W_*^{-1,p}(\Omega)} = \sup\left\{\int_\Omega u\varphi\,dx\,;\ \varphi \in W^{1,p'}(\Omega),\ \|\varphi\|_{W^{1,p'}(\Omega)} \leq 1\right\}$$

but the following estimate holds

$$\int_\Omega u\varphi\,dx \leq \|u\|_{L^{q'}(\Omega_{x_0,r})}\|\varphi\|_{L^q(\Omega)} \leq cr\|u\|_{L^p(\Omega)}$$

and (2.72) is proved. In a similar way one can prove (2.73). □

Here, in order to obtain a precise estimate for the L^∞ norm of the solution of an elliptic boundary value problem in $W_*^{-1,\infty}(\Omega)$, we follow a procedure similar to the one used by Stewart in [42] and in [43] starting by $W_*^{-1,p}(\Omega)$, $1 < p < \infty$.

2.4.1 Formally adjoint boundary value problems

Let $\mathcal{A}$ and $\mathcal{B}$ be the operators defined respectively in (2.1) and in (2.5) satisfying (2.4) and (2.7). Let consider the elliptic problem (2.11); we are interested in the formulation of its formally adjoint boundary value problem, hence, (at this moment) we do not take care of the smoothness properties of the coefficients and we proceed by formal computations. We define the formally adjoint differential operator $\mathcal{A}^*$ of $\mathcal{A}$ by

$$\mathcal{A}^* = \sum_{i,j=1}^n D_j(a^*_{ij}D_i) + \sum_{j=1}^n b^*_j D_j + c^* \tag{2.74}$$

with

$$a^*_{ij} = a_{ij} \quad b^*_i = -b_i \quad c^* = c - \mathrm{div} b.$$

Then by the divergence theorem

$$\int_\Omega v\mathcal{A}u\,dx = \int_\Omega u\mathcal{A}^*v\,dx + \int_{\partial\Omega}\big(\langle ADu,\nu\rangle v - \langle ADv,\nu\rangle u + \langle B,\nu\rangle uv\big)d\mathcal{H}^{n-1}$$

for all $u, v \in C^2(\Omega)\cap C^1(\bar\Omega)$. We let $\nu_A := A\nu$ and $\rho(x) := \frac{\langle\nu_A(x),\nu(x)\rangle}{\langle\beta(x),\nu(x)\rangle}$, and define a vector field by

$$\tau := \nu_A - \rho\beta.$$

We observe that $\langle\tau,\nu\rangle = 0$ and that

$$\langle D,\nu_A\rangle = \rho\langle D,\beta\rangle + \langle D,\tau\rangle. \tag{2.75}$$

Since $\rho(x) \neq 0$ for all $x \in \partial\Omega$, we can define β^* by

$$\rho\beta^* := \nu_A + \tau$$

so that
$$\langle D, \nu_A\rangle = \rho\langle D, \beta^*\rangle - \langle D, \tau\rangle. \tag{2.76}$$
We see that β^* so defined is a non-tangent vector field on $\partial\Omega$, indeed $\rho\langle\beta^*, \nu\rangle = \langle\nu_A, \nu\rangle$. From (2.75) and (2.76) we get

$$\langle ADu, \nu\rangle v - \langle ADv, \nu\rangle u = \rho(v\langle Du, \beta\rangle - u\langle Dv, \beta^*\rangle) + \langle D(uv), \tau\rangle$$

Finally we define γ^* by
$$\rho\gamma^* := \rho\gamma - \langle B, \nu\rangle + \mathrm{div}\tau$$
and the formally adjoint operator $\mathcal{B}^*$ of $\mathcal{B}$ on $\partial\Omega$ by

$$\mathcal{B}^* = \sum_{i=1}^n \beta_i^* D_i + \gamma^*. \tag{2.77}$$

Finally, applying the divergence theorem, we obtain

$$\int_\Omega v\mathcal{A}u\, dx = \int_\Omega u\mathcal{A}^*v\, dx + \int_{\partial\Omega} \rho(v\mathcal{B}u - u\mathcal{B}^*v)\ d\mathcal{H}^{n-1}$$

for all $u, v \in C^2(\Omega) \cap C^1(\bar\Omega)$.

Henceforth we focus our attention to a particular choice of the boundary operator $\mathcal{B}$. We select the conormal boundary operator

$$\mathcal{B}(x, D) = \sum_{i,j=1}^n a_{ij}(x)\nu_i(x)D_j, \tag{2.78}$$

in this way the formally adjoint operator $\mathcal{B}^*$ is defined as follows

$$\mathcal{B}^* = \langle D, \nu_A\rangle - \langle B, \nu\rangle$$

(since $\rho = 1$, $\tau = 0$, $\beta^* = \nu_A$ and $\gamma^* = -\langle B, \nu\rangle$), and $\mathcal{A}^*$ is defined in (2.74). We suppose that a_{ij}, b_i and c are real valued functions such that

$$a_{ij} = a_{ji}, \quad a_{ij},\, b_i \in W^{2,\infty}(\Omega), \quad c \in L^\infty(\Omega). \tag{2.79}$$

Assumption (2.79) guarantees that hypotheses in Section 2.1 are satisfied both for the couple of operators $(\mathcal{A}, \mathcal{B})$ and $(\mathcal{A}^*, \mathcal{B}^*)$ and Theorem 2.2.10 can be applied to each of them. We set
$$M_1 = \max_{i,j}\{\|a_{ij}\|_{W^{2,\infty}(\Omega)}, \|b_i\|_{W^{2,\infty}(\Omega)}, \|c\|_{L^\infty(\Omega)}\}. \tag{2.80}$$

Now, we consider the realization of $\mathcal{A}$ with homogeneous boundary condition given by $\mathcal{B}$ as in (2.78) in the Banach space $W_*^{-1,p}$, so defined

$$E_p : D(E_p) = W^{1,p}(\Omega) \subset W_*^{-1,p}(\Omega) \to W_*^{-1,p}(\Omega) \tag{2.81}$$

by
$$\langle E_p u, v\rangle_* = a(u, v) \qquad u \in W^{1,p}(\Omega),\, v \in W^{1,p'}(\Omega) \tag{2.82}$$

where

$$a(u,v) = -\int_\Omega \langle ADu, Dv\rangle\,dx + \int_\Omega \langle B, Du\rangle v\,dx + \int_\Omega cuv\,dx \tag{2.83}$$

in $W^{1,p}(\Omega)\times W^{1,p'}(\Omega)$. Analogously we could define the realization of $(\mathcal{A}^*,\mathcal{B}^*)$ in $W_*^{-1,p'}$ in this way:

$$E_{p'} : D(E_{p'}) = W^{1,p'}(\Omega) \subset W_*^{-1,p'}(\Omega) \to W_*^{-1,p'}(\Omega) \tag{2.84}$$

by

$$\langle E_{p'}u, v\rangle_* = a^*(u,v) \qquad u\in W^{1,p'}(\Omega),\ v\in W^{1,p}(\Omega) \tag{2.85}$$

where

$$a^*(u,v) = -\int_\Omega \langle ADu, Dv\rangle\,dx + \int_\Omega \langle B, Dv\rangle u\,dx + \int_\Omega cuv\,dx \tag{2.86}$$

in $W^{1,p'}(\Omega)\times W^{1,p}(\Omega)$.

We start with two technical results involving L^p estimates that are true for both E_p and $E_{p'}$ and that for simplicity are stated only in one case.

Theorem 2.4.2. *The operator E_p is sectorial in $W_*^{-1,p}(\Omega)$. In particular there is a constant $\omega_p\in\mathbf{R}$ depending on n,p,μ,M_1,Ω such that for each $\lambda\in\mathbf{C}$ with $\operatorname{Re}\lambda>\omega_p$ and for each $f\in W_*^{-1,p}(\Omega)$ the solution $u\in W^{1,p}(\Omega)$ of the equation $(\lambda-\mathcal{A})u=f$ satisfies*

$$|\lambda|\|u\|_{W_*^{-1,p}(\Omega)} + |\lambda|^{1/2}\|u\|_{L^p(\Omega)} + \|u\|_{W^{1,p}(\Omega)} \le K_1\|f\|_{W_*^{-1,p}(\Omega)} \tag{2.87}$$

where $K_1>0$ is a constant independent of λ and f.

Proof. Denote by A_p^B the realization of $\mathcal{A}$ in L^p with homogeneous boundary conditions $\mathcal{B}u=0$ and analogously $A^{*B^*}_{p'}$ the realization of $\mathcal{A}^*$ in $L^{p'}$ with homogeneous boundary conditions $\mathcal{B}^*u=0$. We know that $D(A_p^B)=\{u\in W^{2,p}(\Omega):\ \mathcal{B}u=0 \text{ in } \partial\Omega\}$. Then for each $u\in D(A^{*B^*}_{p'})$ and $v\in L^p(\Omega)$, we have $\langle A^{*B^*}_{p'}u,v\rangle=\langle u,(A^{*B^*}_{p'})^*v\rangle$ where $(A^{*B^*}_{p'})^*$ is the adjoint of $A^{*B^*}_{p'}$ and belongs to $\mathcal{L}(L^p(\Omega),(D(A^{*B^*}_{p'})'))$ where $(D(A^{*B^*}_{p'})'$ is the dual space of $D(A^{*B^*}_{p'})$. Note that the restriction of $(A^{*B^*}_{p'})^*$ to $D(A_p^B)$ coincides with A_p^B. Therefore, from the complex interpolation theory (see Theorem A.3.5), we have that $(A^{*B^*}_{p'})^*$ is a bounded linear operator from $[L^p(\Omega),D(A_p^B)]_{1/2}$ to $[(D(A^{*B^*}_{p'}))',L^p(\Omega)]_{1/2}$ where $[\cdot,\cdot]_{1/2}$ is the complex interpolation space of order 1/2, (see Section A.3 for the relevant definitions and results). Using [39, Theorem 4.1], which holds for domains with uniformly smooth boundary, we can characterize the complex interpolation spaces in the following way:

$$[L^p(\Omega),D(A_p^B)]_{1/2} = W^{1,p}(\Omega)$$

$$[(D(A^{*B^*}_{p'}))',L^p(\Omega)]_{1/2} = [L^{p'}(\Omega),D(A^{*B^*}_{p'})]'_{1/2} = (W^{1,p'}(\Omega))' = W_*^{-1,p}(\Omega) \tag{2.88}$$

where in the first equality in (2.88) we have used (A.16). Therefore the restriction of $(A^{*B^*}_{p'})^*$ to the space $W^{1,p}(\Omega)$ is a bounded linear operator from $W^{1,p}(\Omega)$ to $W_*^{-1,p}(\Omega)$ and coincides with E_p.
Now, we show that there exists a constant k_1 such that for each λ with $\operatorname{Re}\lambda$ large enough,

$$\|(\lambda-A_p^B)^{-1}\|_{\mathcal{L}(L^p,D(A_p^B))} \le k_1. \tag{2.89}$$

Since A_p^B is a sectorial operator, there exists $\omega_1 \in \mathbf{R}$ such that for each $\lambda \in \mathbf{C}$ with $\operatorname{Re}\lambda \geq \omega_1$ and for each $f \in L^p(\Omega)$ the equation

$$(\lambda - \mathcal{A})u = f$$

admits a solution $u \in W^{2,p}(\Omega)$ with $\mathcal{B}u = 0$ in $\partial\Omega$ satisfying (2.48). Hence

$$\begin{aligned}\|u\|_{D(A_p^B)} &= \|u\|_{L^p(\Omega)} + \|\mathcal{A}u\|_{L^p(\Omega)} \leq (1+|\lambda|)\|u\|_{L^p(\Omega)} + \|f\|_{L^p(\Omega)} \\ &\leq (\frac{1+|\lambda|}{|\lambda|} + 1)\|f\|_{L^p(\Omega)} \leq k_1\|f\|_{L^p(\Omega)}\end{aligned}$$

for $\operatorname{Re}\lambda$ large. Analogously, there exists a constant $\omega_2 \in \mathbf{R}$ and $k_2 > 0$, such that

$$\|(\lambda - A^*{}_{p'}^{B^*})^{-1}\|_{\mathcal{L}(L^{p'}, D(A^*{}_{p'}^{B^*}))} \leq k_2 \tag{2.90}$$

for $\operatorname{Re}\lambda > \omega_2$. Using (2.90) we get that

$$[(\lambda - A^*{}_{p'}^{B^*})^{-1}]^* = [(\lambda - A^*{}_{p'}^{B^*})^*]^{-1} \in \mathcal{L}((D(A^*{}_{p'}^{B^*}))', L^p)$$

hence an argument similar to the previous one yields that the operator $[(\lambda - A^*{}_{p'}^{B^*})^{-1}]^*$ belongs to $\mathcal{L}(W_*^{-1,p}(\Omega), W^{1,p}(\Omega))$ and coincides with $(\lambda - E_p)^{-1}$.
Set $K = k_1 + k_2$ and $\omega_p > \omega_1 \vee \omega_2$; then, for every λ with $\operatorname{Re}\lambda > \omega_p$ and for every $f \in W_*^{-1,p}(\Omega)$ we have that $\|u\|_{W^{1,p}(\Omega)} \leq K\|f\|_{W_*^{-1,p}(\Omega)}$ where $u = (\lambda - E_p)^{-1}f$. Then, for every $v \in W^{1,p'}(\Omega)$,

$$\langle f, v\rangle_* = \lambda\langle u, v\rangle_* - \langle E_p u, v\rangle_*$$

Thus

$$\begin{aligned}|\langle u, v\rangle_*| &\leq |\lambda|^{-1}\left(|\langle E_p u, v\rangle_*| + |\langle f, v\rangle_*|\right) \\ &\leq c|\lambda|^{-1}\left(\|u\|_{W^{1,p}(\Omega)}\|v\|_{W^{1,p'}(\Omega)} + \|f\|_{W_*^{-1,p}(\Omega)}\|v\|_{W^{1,p'}(\Omega)}\right) \\ &\leq c|\lambda|^{-1}\left(K\|f\|_{W_*^{-1,p}(\Omega)}\|v\|_{W^{1,p'}(\Omega)} + \|f\|_{W_*^{-1,p}(\Omega)}\|v\|_{W^{1,p'}(\Omega)}\right)\end{aligned}$$

Hence we have proved that

$$|\lambda|\|u\|_{W_*^{-1,p}(\Omega)} + \|u\|_{W^{1,p}(\Omega)} \leq c\|f\|_{W_*^{-1,p}(\Omega)}. \tag{2.91}$$

Therefore, (2.87) is consequence of (2.91) and of the fact that

$$(W_*^{-1,p}(\Omega), W^{1,p}(\Omega))_{1/2,p} = L^p(\Omega)$$

for $1 < p < \infty$ (see [46, Section 2.4.2, Theorem 1; Section 4.2.1, Definition 1]). □

Remark 2.4.3. We observe that if $f \in L^p(\Omega)$, then $u = (\lambda - E_p)^{-1}f \in D(A_p^B)$ and therefore $\mathcal{B}u = 0$ in $\partial\Omega$.

Lemma 2.4.4. *Let $p \geq 2$ and $f \in W_*^{-1,p}(\Omega)$ with $f = f_0 - \sum_{i=1}^n D_i f_i$; then for each $\lambda \in \mathbf{C}$ with $\operatorname{Re}\lambda > \omega_p$, for each $r < 1$ and for each $x_0 \in \overline{\Omega}$, the solution $u \in D(E_p)$ of the equation $\lambda u - \mathcal{A}u = f$ satisfies the following estimate*

$$\|u\|_{W^{1,p}(\Omega_{x_0,r})} \leq K_2\left\{\sum_{i=1}^n \|f_i\|_{L^p(\Omega_{x_0,2r})} + r\|f_0\|_{L^p(\Omega_{x_0,2r})} + r^{-1}\|u\|_{L^p(\Omega_{x_0,2r})}\right\} \tag{2.92}$$

where $\Omega_{x_0,r}$ is defined in (2.56) and K_2 is a constant independent of λ and f.

Proof. We point out that the space of functions

$$C_\nu^{-1} = \{g = g_0 - \sum_{i=1}^n D_i g_i;\ g_i \in C^1(\overline{\Omega}) \cap L^{p'}(\Omega),\ \sum_{i=1}^n g_i \nu_i = 0 \text{ on } \partial\Omega\}$$

is dense in $W_*^{-1,p}$, because every f_i in the representation of distributions in $W_*^{-1,p}$ as in (2.71) can be approximated in L^p norm. Hence, it is sufficient to prove the claim for functions in C_ν^{-1}. Then, passing to the limit in the estimate we get the claim for every $f \in W_*^{-1,p}(\Omega)$.

Suppose then that $f \in C_\nu^{-1}$; for each $x_0 \in \overline{\Omega}$ and $r < 1$, let $\theta \in C^2(\mathbf{R}^n)$ with $\theta(x) = 1$ for $|x - x_0| \leq r$, $\theta(x) = 0$ for $|x - x_0| \geq \sqrt{2}r$, $|D\theta| \leq cr^{-1}$ and $\langle AD\theta, \nu\rangle = 0$ in $\partial\Omega$. Such a function can be obtained in the following way: first we consider a cut-off function $\psi \in C^2(\mathbf{R}^n)$, $\psi(x) = 1$ in $B(x_0, r) \cap \Omega$ and $\psi = 0$ in $\Omega \cap (B(x_0, \sqrt{2}r))^c$, then we modify ψ in a neighborhood of the boundary making it constant in the direction $A\nu$ in order that $\langle D\psi, A\nu\rangle = 0$ in $\partial\Omega$. Finally we recover the regularity and preserve the homogeneous boundary condition by convolution with a family of mollifiers whose support is $B(0,\epsilon)$ with ϵ sufficiently small. In this way the function $w := \theta u$ satisfies the equation

$$\lambda w - \mathcal{A} w = E + F + G = g \tag{2.93}$$

where

$$\begin{aligned} E &= -\sum_{i,j=1}^n D_i(a_{ij} u D_j\theta) - \sum_{i=1}^n b_i u D_i\theta \\ F &= -\sum_{i,j=1}^n a_{ij} D_j u D_i\theta \\ G &= -\sum_{i=1}^n D_i(\theta f_i) + \sum_{i=1}^n f_i D_i\theta + \theta f_0 \end{aligned} \tag{2.94}$$

Thus, multiplying (2.93) by w and integrating by parts we get

$$\begin{aligned} \int_\Omega \langle AD(\theta u), D(\theta u)\rangle\,dx &= \int_\Omega \langle B, D(\theta u)\rangle \theta u\,dx - \int_\Omega (\lambda - c)(\theta u)^2\,dx \\ &+ \int_\Omega \langle AD\theta, D(\theta u)\rangle u\,dx - \int_\Omega \langle B, D\theta\rangle \theta u^2\,dx \\ &- \int_\Omega \langle AD\theta, Du\rangle \theta u\,dx + \sum_{i=1}^n \int_\Omega \theta f_i D_i(\theta u)\,dx \\ &+ \sum_{i=1}^n \int_\Omega f_i (D_i\theta)\theta u\,dx + \int_\Omega f_0 \theta^2 u\,dx \end{aligned} \tag{2.95}$$

We point out that in (2.95) all the integrals are on $\Omega \cap B(x_0, \sqrt{2}r)$. Now, using (2.4) and the properties of the function θ we get

$$
\begin{aligned}
\mu^{-1}\|Du\|^2_{L^2(\Omega\cap B(x_0,r))} \le c\big(r^{-2}\|u\|^2_{L^2(\Omega\cap B(x_0,\sqrt{2}r))} &+ \sum_{i=1}^n \|f_i\|^2_{L^2(\Omega\cap B(x_0,\sqrt{2}r))}\big)\\
&+ \int_{\Omega\cap B(x_0,\sqrt{2}r)} \langle B, Du\rangle \theta^2 u\,dx + \int_{\Omega\cap B(x_0,\sqrt{2}r)} \langle AD\theta, Du\rangle \theta u\,dx\\
&+ \sum_{i=1}^n \int_{\Omega\cap B(x_0,\sqrt{2}r)} \theta^2 f_i D_i u\,dx
\end{aligned}
$$

Finally, using the inequality $ab \le \varepsilon a^2 + \varepsilon^{-1}b^2$, we prove that there exists a constant c depending on the norm of the coefficients of $\mathcal{A}$ and on the ellipticity constant μ such that

$$
\|Du\|_{L^2(\Omega\cap B(x_0,r))} \le c\big(\sum_{i=0}^n \|f_i\|_{L^2(\Omega\cap B(x_0,\sqrt{2}r))} + r^{-1}\|u\|_{L^2(\Omega\cap B(x_0,\sqrt{2}r))}\big) \tag{2.96}
$$

which implies the statement for $p = 2$. By Theorem 2.4.2 applied to equation (2.93), we get

$$
\begin{aligned}
\|\theta u\|_{W^{1,p}(\Omega)} \le K_1\|g\|_{W^{-1,p}_*(\Omega)} \le K_1\Big(&\sum_{i=0}^n \|f_i\|_{L^p(\Omega\cap B(x_0,\sqrt{2}r))}\\
&+ r^{-1}\big(\sum_{i,j=1}^n \|a_{ij}\|_{L^\infty} + \sum_{i=1}^n \|b_i\|_{L^\infty}\big)\|u\|_{L^p(\Omega\cap B(x_0,\sqrt{2}r))}\\
&+ \sum_{i,j=1}^n \|a_{ij}D_j u D_i\theta\|_{W^{-1,p}_*(\Omega)}\Big)
\end{aligned} \tag{2.97}
$$

By the Sobolev embedding theorem, every test function $\phi \in W^{1,p'}(\Omega)$ belongs also to $L^{q'}(\Omega)$, with $q' = np/(np-n-p)$, and $\|\phi\|_{L^{q'}(\Omega)} \le k\|\phi\|_{W^{1,p'}(\Omega)}$ with $k = k(p,\Omega)$. Therefore, by (2.96) for $2 < p \le 2n/(n-2)$ if $n > 2$ (for every p if $n \le 2$), we get

$$
\begin{aligned}
\|a_{ij}D_j u D_i\theta\|_{W^{-1,p}_*(\Omega)} &\le c r^{-1}\|Du\|_{L^{np/(n+p)}(\Omega\cap B(x_0,\sqrt{2}r))}\\
&\le c r^{n(\frac{1}{p}-\frac{1}{2})}\|Du\|_{L^2(\Omega\cap B(x_0,\sqrt{2}r))}\\
&\le c r^{n(\frac{1}{p}-\frac{1}{2})}\big(\sum_{i=0}^n \|f_i\|_{L^2(\Omega\cap B(x_0,2r))} + r^{-1}\|u\|_{L^2(\Omega\cap B(x_0,2r))}\big)\\
&\le c\big(\sum_{i=0}^n \|f_i\|_{L^p(\Omega\cap B(x_0,2r))} + r^{-1}\|u\|_{L^p(\Omega\cap B(x_0,2r))}\big)
\end{aligned}
$$

where c depends on $n, \|a_{ij}\|_\infty, p, \Omega$ and it may change from a line to the other. Summing up we find

$$
\|\theta u\|_{W^{1,p}(\Omega)} \le K_2\Big(\sum_{i=0}^n \|f_i\|_{L^p(\Omega\cap B(x_0,\sqrt{2}r))} + r^{-1}\|u\|_{L^p(\Omega\cap B(x_0,2r))}\Big).
$$

Since $\theta u = u$ on $\Omega\cap B(x_0,r)$ we get the statement for every $p \in [2, 2n/(n-2)]$ when $n > 2$ and for all $p \ge 2$ if $n \le 2$. Repeating the same procedure, starting from $p = \frac{2n}{n-2}$

we can prove the statement for every $p \in [2, 2n/(n-4)]$ if $n > 4$, for every p if $n \le 4$. Thus, after $[n/2]$ steps, the proof is complete. $\square$

The following estimate is proved by using a modification of Stewart's technique. It will be useful in order to obtain the estimate of the gradient of the solution of (2.111) in $L^1(\Omega)$.

Theorem 2.4.5. *Let $p > n$, $f \in W_*^{-1,\infty}(\Omega) \cap W_*^{-1,p}(\Omega)$; then, there exists $\omega_\infty > \omega_p$ such that for each $\lambda \in \mathbf{C}$ with $\operatorname{Re}\lambda > \omega_\infty$ the solution $u \in D(E_p)$ of $\lambda u - \mathcal{A}u = f$ belongs to $W^{1,p}$ and satisfies*

$$|\lambda|^{1/2}\|u\|_{L^\infty(\Omega)} \le K_3\|f\|_{W_*^{-1,\infty}(\Omega)}, \tag{2.98}$$

where K_3 is a constant independent of λ, u and f.

Proof. Let $x_0 \in \Omega$ and $r < 1$. Let θ be a cut-off function as the one considered in proof of Lemma 2.4.4: $\theta \in C^2(\mathbf{R}^n)$, $\theta(x) = 1$ on $B(x_0, r)$ $\theta(x) = 0$ outside $B(x_0, 2r)$ and with $\|D^\alpha\theta\|_{L^\infty(\Omega)} \le cr^{-|\alpha|}$ for each $|\alpha| \le 2$. As f belongs to $W_*^{-1,\infty}(\Omega)$, it admits a distributional representation $f = f_0 - \sum_{i=1}^n D_i f_i$, where $f_i \in L^\infty(\Omega)$ for each $i = 0, 1, \ldots, n$ and $\sum_{i=0}^n \|f_i\|_{L^\infty(\Omega)} \ge \|f\|_{W^{-1,\infty}(\Omega)}$. Note that $u \in W^{1,p}(\Omega)$ for $p > n$ by Theorem 2.4.2, therefore $\theta u \in W^{1,p}(\Omega)$ and solves

$$(\lambda - \mathcal{A})(\theta u) = g \tag{2.99}$$

where g is defined in (2.94). By (2.97), (2.72) and (2.92), we get

$$\begin{aligned}
\|g\|_{W_*^{-1,p}(\Omega)} &\le K_4\left\{\|u\|_{W^{1,p}(\Omega_{x_0,2r})} + r^{-1}\|u\|_{L^p(\Omega_{x_0,2r})} + \sum_{i=0}^n \|f_i\|_{L^p(\Omega_{x_0,2r})}\right\} \\
&\le K_5\left\{\sum_{i=0}^n \|f_i\|_{L^p(\Omega_{x_0,4r})} + r^{-1}\|u\|_{L^p(\Omega_{x_0,4r})}\right\} \\
&\le K_6 r^{n/p}\left\{\sum_{i=0}^n \|f_i\|_{L^\infty(\Omega)} + r^{-1}\|u\|_{L^\infty(\Omega)}\right\},
\end{aligned} \tag{2.100}$$

where K_4, K_5 and K_6 are constants independent of r, λ, f and u. Since

$$W^{1,p}(\Omega_{x_0,2r}) \hookrightarrow C^0(\overline{\Omega}_{x_0,2r}) \hookrightarrow L^p(\Omega_{x_0,2r})$$

for $p > n$ and the first injection is compact, then for each $\varepsilon > 0$ we get

$$\|\theta u\|_{L^\infty(\Omega_{x_0,2r})} \le \varepsilon r^{1-n/p}\|\theta u\|_{W^{1,p}(\Omega_{x_0,2r})} + c(\varepsilon)r^{-n/p}\|\theta u\|_{L^p(\Omega_{x_0,2r})}, \tag{2.101}$$

where $c(\varepsilon)$ is independent of r, λ, u and f (see Lemma 5.1 of [30]).

Moreover, (2.73) and the Hölder inequality imply

$$\|\theta u\|_{W_*^{-1,\infty}(\Omega_{x_0,r})} \le c_2 r^{1-n/p}\|\theta u\|_{L^p(\Omega_{x_0,r})} \le c_2 r\|\theta u\|_{L^\infty(\Omega)}. \tag{2.102}$$

Therefore, from (2.101) and (2.102) we get

$$r^{-2}\|\theta u\|_{W_*^{-1,\infty}(\Omega)} + r^{-1}\|\theta u\|_{L^\infty(\Omega)} \le \varepsilon r^{-n/p}\|\theta u\|_{W^{1,p}(\Omega)} + c(\varepsilon)r^{-1-n/p}\|\theta u\|_{L^p(\Omega)}. \tag{2.103}$$

On the other hand, from (2.87)

$$|\lambda|\|\theta u\|_{W_*^{-1,p}(\Omega)} + |\lambda|^{1/2}\|\theta u\|_{L^p(\Omega)} + \|\theta u\|_{W^{1,p}(\Omega)} \le K_1\|g\|_{W_*^{-1,p}(\Omega)}. \tag{2.104}$$

Therefore, by (2.103), (2.104) and (2.100) we deduce

$$\begin{aligned} &r^{-2}\|\theta u\|_{W_*^{-1,\infty}(\Omega)} + r^{-1}\|\theta u\|_{L^\infty(\Omega)} \\ &\qquad \le K_1K_6\big(\varepsilon + c(\varepsilon)r^{-1}|\lambda|^{-1/2}\big)\big(r^{-1}\|u\|_{L^\infty(\Omega)} + \sum_{i=0}^n \|f_i\|_{L^\infty(\Omega)}\big). \end{aligned}$$

Set $K_7 = 4K_1K_6$ and choose $\omega_\infty \ge \omega_p$ and $\varepsilon = K_7^{-1}$, $r = K_7c(K_7^{-1})|\lambda|^{-1/2} = K_8|\lambda|^{-1/2}$. Then, if x_0 is a maximum point for the function $|u|$ we obtain

$$K_8^{-2}|\lambda|\|\theta u\|_{W_*^{-1,\infty}(\Omega)} + \frac{1}{2}K_8^{-1}|\lambda|^{1/2}\|u\|_{L^\infty(\Omega)} \le \frac{1}{2}\sum_{i=0}^n \|f_i\|_{L^\infty(\Omega)} \le \|f\|_{W_*^{-1,\infty}(\Omega)}. \tag{2.105}$$

Thus (2.98) is proved. □

2.5 Generation of analytic semigroups in $L^1(\Omega)$

In this section we prove that the realization of uniformly elliptic operators with suitable oblique boundary conditions is sectorial in $L^1(\Omega)$ where Ω is assumed to satisfy (2.2). We consider the operator $\mathcal{A}$ in divergence form with real-valued coefficients

$$\begin{aligned} \mathcal{A}(x,D) &= \sum_{i,j=1}^n D_i(a_{ij}(x)D_j) + \sum_{i=1}^n b_i(x)D_i + c(x) \\ &= \operatorname{div}(A(x)D) + B(x)\cdot D + c(x). \end{aligned} \tag{2.106}$$

We suppose that $\mathcal{A}$ is uniformly μ-elliptic, i.e.,

$$\mu^{-1}|\xi|^2 \le \sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \le \mu|\xi|^2, \quad x\in\overline{\Omega}, \xi\in\mathbf{R}^n \tag{2.107}$$

and that

$$a_{ij} = a_{ji}, \quad a_{ij},\, b_i \in W^{2,\infty}(\Omega), \quad c\in L^\infty(\Omega). \tag{2.108}$$

Actually the regularity assumption on the coefficients b_i will be weakened later. Define

$$M_1 = \max_{i,j}\{\|a_{ij}\|_{W^{2,\infty}(\Omega)}, \|b_i\|_{W^{2,\infty}(\Omega)}, \|c\|_{L^\infty(\Omega)}\}. \tag{2.109}$$

We consider the following first order differential operator acting on the boundary

$$\mathcal{B}(x,D) = \langle AD,\nu\rangle = \sum_{i=1}^n a_{ij}(x)\nu_i(x)D_j. \tag{2.110}$$

Since we would like to solve the problem in L^1 by duality from L^∞, we point out that the choice of the coefficients and the assumptions of regularity (2.108) guarantee that hypotheses in Section 2.1 hold also for $(\mathcal{A}^*, \mathcal{B}^*)$; this fact allows us to apply the results of Section 2.3 to the realization of $\mathcal{A}^*$ with homogeneous boundary conditions given by $\mathcal{B}^*$ in $L^\infty(\Omega)$.

In order to deduce a result of generation in $L^1(\Omega)$ we argue as follows. Set

$$D_{\mathcal{A}} = \{u \in L^1(\Omega) \cap C^2(\bar{\Omega});\ \mathcal{A}u \in L^1(\Omega), \mathcal{B}u = 0 \text{ in } \partial\Omega\}.$$

Lemma 2.5.1. *$\mathcal{A} : D_{\mathcal{A}} \subset L^1(\Omega) \to L^1(\Omega)$ is closable in $L^1(\Omega)$.*

Proof. Let (u_j) be a sequence in $D_{\mathcal{A}}$ such that $u_j \to 0$ and $\mathcal{A}u_j \to v$ in $L^1(\Omega)$. Then, integrating by parts,

$$\int_\Omega \varphi\, v\, dx = \lim_{j\to\infty} \int_\Omega \varphi \mathcal{A}u_j\, dx = \lim_{j\to\infty} \int_\Omega u_j \mathcal{A}^*\varphi\, dx = 0$$

for every $\varphi \in C_c^\infty(\Omega)$. Hence $v = 0$, which implies the assertion. □

By Lemma 2.5.1 we can define the realization of $\mathcal{A}$ in L^1 with boundary condition $\mathcal{B}$, (that will be denoted for simplicity by $(A_1, D(A_1))$ to be the closure of $\mathcal{A}_{|D_{\mathcal{A}}}$ in $L^1(\Omega)$, that is, the smallest closed extension of $\mathcal{A}_{|D_{\mathcal{A}}}$ in $L^1(\Omega)$. Then $D(A_1)$ is the closure of $D_{\mathcal{A}}$ with respect to the graph norm in L^1. Now we are in a position to prove the following result.

Theorem 2.5.2. *There exist $C > 0$ and $\omega_1 \in \mathbf{R}$, depending on n, μ, M_1 and Ω, such that for $\operatorname{Re}\lambda \geq \omega_1$ the problem*

$$\begin{cases} \lambda u - \mathcal{A}u = f & \text{in } \Omega \\ \mathcal{B}u = 0 & \text{in } \partial\Omega \end{cases} \tag{2.111}$$

with $f \in L^1(\Omega)$ has a unique solution $u \in L^1(\Omega)$ and

$$|\lambda| \|u\|_{L^1(\Omega)} \leq C \|f\|_{L^1(\Omega)}. \tag{2.112}$$

Proof. First of all we prove that the range of $(\lambda - A_1)$ contains the space of functions $L_c^\infty(\Omega) = \{\psi \in L^\infty(\Omega);\ \operatorname{supp}\psi \subset\subset \Omega\}$ which is dense in $L^1(\Omega)$.
Indeed, let $\pi \in C^2(\Omega)$ be such that

$$\begin{cases} \sum_{i,j=1}^n |D_{ij}\pi| + \sum_{i=1}^n |D_i\pi|^2 \leq c \\ e^{-\pi} \in L^1(\Omega) \\ \langle AD\pi, \nu\rangle = 0 \qquad \text{in } \partial\Omega \end{cases}$$

Moreover, if Ω is unbounded, we also require that $\lim_{|x|\to\infty, x\in\Omega} \pi(x) = +\infty$. Such a π exists. For instance, when $\Omega = \mathbf{R}^n$ one can choose $\pi(x) = \sqrt{1+|x|^2}$. In the general case one can adapt the previous example modifying π near the boundary in a suitable way.

We define $\Pi(x) = \exp[\pi(x)]$. Then, for every function $\psi \in L^\infty_c(\Omega)$, we get $\Pi\psi \in L^\infty_c(\Omega)$ and

$$\begin{cases} \lambda u - \mathcal{A}u = \psi \in L^\infty_c(\Omega) \\ \mathcal{B}u = 0 \quad \text{ in } \partial\Omega \end{cases}$$

if and only if

$$\begin{cases} \lambda \Pi u - \mathcal{A}_\pi(\Pi u) = \Pi\psi \in L^\infty_c(\Omega) \\ \mathcal{B}(\Pi u) = 0 \quad \text{ in } \partial\Omega \end{cases} \tag{2.113}$$

where

$$\mathcal{A}_\pi = \mathcal{A} - 2\sum_{i,j=1}^n a_{ij} D_i\pi D_j + \big(\sum_{i,j=1}^n (D_i(a_{ij}D_j\pi) - a_{ij}D_i\pi D_j\pi) + \sum_{i=1}^n b_i D_i\pi\big).$$

As it is easily seen, the operator $\mathcal{A}_\pi$ satisfies the assumptions (2.3)-(2.4); moreover, since $\mathcal{A}^0_\pi(x,\xi) = \mathcal{A}^0(x,\xi)$ then $\mathcal{A}_\pi$ satisfies also the root and the complementing conditions. Therefore, by applying Theorem 2.3.3 we get that there exists $\Pi u \in D((A_\pi)^B_\infty) \subseteq L^\infty(\Omega)$ solution of (2.113).
Hence $u \in \{v \in C^1(\bar\Omega) \cap L^1(\Omega); \mathcal{A}v \in L^1(\Omega)\}$ and ψ is therefore in the range of $(\lambda - A_1)$.
Now we prove (2.112). Let consider u solution of $\lambda u - \mathcal{A}u = f \in L^1(\Omega)$ and let

$$\mathcal{A}^* = \sum_{i,j=1}^n D_j(a_{ij}D_i) - \sum_{j=1}^n b_j D_j + (c - \operatorname{div} b)$$

Then, from Theorem 2.3.3, it follows that $(A^*)^{B^*}_\infty$ with oblique boundary conditions $\mathcal{B}^*(x,D) = \langle A(x)D, \nu(x)\rangle - \langle B(x), \nu(x)\rangle = 0$ generates an analytic semigroup in $L^\infty(\Omega)$ and so the elliptic problem

$$\begin{cases} \lambda w - \mathcal{A}^* w = \varphi \in L^\infty(\Omega) \\ \mathcal{B}^* w = 0 \quad \text{ in } \partial\Omega \end{cases} \tag{2.114}$$

has a unique solution $w \in D((A^*)^{B^*}_\infty)$ for $\operatorname{Re}\lambda$ sufficiently large. Moreover, taking $\operatorname{Re}\lambda$ sufficiently large we get

$$|\lambda| \|w\|_{L^\infty(\Omega)} + |\lambda|^{1/2}\|Dw\|_{L^\infty(\Omega)} \le \tilde{K}\|\varphi\|_{L^\infty(\Omega)}.$$

Now, we can apply the method used in Pazy (see [35]) to obtain

$$\begin{aligned} \|u\|_{L^1(\Omega)} &= \sup\Big\{\int u(x)\varphi(x)dx;\ \varphi \in L^\infty_c(\Omega),\ \|\varphi\|_{L^\infty(\Omega)} \le 1\Big\} \\ &\le \sup\Big\{\int u(x)(\lambda - \mathcal{A}^*)w_\varphi dx;\ w_\varphi \in L^\infty(\Omega)\, \text{solution of (2.114)},\ \|\varphi\|_{L^\infty(\Omega)} \le 1\Big\} \\ &\le \sup\Big\{\int w_\varphi(\lambda - \mathcal{A})u dx;\ w_\varphi \in L^\infty(\Omega)\, \text{solution of (2.114)},\ \|\varphi\|_{L^\infty(\Omega)} \le 1\Big\} \end{aligned}$$

in particular,

$$\|u\|_{L^1(\Omega)} \le \tilde{K}|\lambda|^{-1}\|f\|_{L^1(\Omega)}.$$

So, $(\lambda - A_1)$ is an injective operator with closed range in $L^1(\Omega)$ and the proof is complete. □

The following theorem establishes further properties of the resolvent operator.

Theorem 2.5.3. *Under the assumptions of Theorem 2.5.2, there exist $\omega_1' \geq \omega_1$, $K' \geq K$ and $\theta_1' \in (\pi/2, \theta_1)$ depending on n, μ, M_1 and Ω such that for every λ such that $|\arg(\lambda - \omega_1')| < \theta_1'$, the solution of* (2.111) *satisfies*

$$|\lambda|^{1/2} \|Du\|_{L^1(\Omega)} \leq K' \|f\|_{L^1(\Omega)}. \tag{2.115}$$

PROOF. Let $\phi = \mathrm{div}\psi$ with ψ any function in $L^\infty(\Omega, \mathbf{R}^n)$. By the estimate (2.98) we know that for λ with $\mathrm{Re}\,\lambda > \omega_\infty$, the solution of the following problem

$$\begin{cases} \lambda v - \mathcal{A}^* v = \mathrm{div}\psi \\ \mathcal{B}^* v = 0 \quad \text{ on } \partial\Omega \end{cases} \tag{2.116}$$

satisfies

$$|\lambda|^{1/2} \|v\|_{L^\infty(\Omega)} \leq K_3 \|\mathrm{div}\psi\|_{W_*^{-1,\infty}(\Omega)}. \tag{2.117}$$

We notice that

$$\|\mathrm{div}\psi\|_{W_*^{-1,\infty}} = \sup\{\langle \mathrm{div}\psi, \varphi\rangle : \ \varphi \in W^{1,1}(\Omega), \ \|\varphi\|_{W^{1,1}(\Omega)} \leq 1\} \leq \|\psi\|_{L^\infty}. \tag{2.118}$$

Now, if u is the solution of (2.111), we get

$$\begin{aligned}
\|Du\|_{L^1(\Omega)} &= \sup\left\{ \int_\Omega \langle Du(x), \psi(x)\rangle dx : \ \psi \in C_c^\infty(\Omega; \mathbf{R}^n), \ \|\psi\|_{L^\infty(\Omega)} \leq 1 \right\} \\
&= \sup\left\{ \int_\Omega u(x)\, \mathrm{div}\psi(x) dx : \ \psi \in C_c^\infty(\Omega; \mathbf{R}^n), \ \|\psi\|_{L^\infty(\Omega)} \leq 1 \right\} \\
&\leq \sup\left\{ \int_\Omega u(x)\, \mathrm{div}\psi(x) dx : \psi \in C_c^\infty(\Omega; \mathbf{R}^n), \ \|\mathrm{div}\psi\|_{W_*^{-1,\infty}(\Omega)} \leq 1 \right\} \\
&= \sup\left\{ \int_\Omega u\, (\lambda - \mathcal{A}^*)\, v_\psi \, dx : \ v_\psi \text{ solution of (2.116)}, \ \|\mathrm{div}\psi\|_{W_*^{-1,\infty}(\Omega)} \leq 1 \right\} \\
&= \sup\left\{ \int_\Omega [(\lambda - \mathcal{A})u]\, v_\psi \, dx : \ v_\psi \text{ solution of (2.116)}, \ \|\mathrm{div}\psi\|_{W_*^{-1,\infty}(\Omega)} \leq 1 \right\} \\
&\leq C \sup\left\{ \|f\|_{L^1(\Omega)} \|v_\psi\|_{L^\infty(\Omega)} : \ v_\psi \text{ solution of (2.116)}, \ \|\mathrm{div}\psi\|_{W_*^{-1,\infty}(\Omega)} \leq 1 \right\}.
\end{aligned} \tag{2.119}$$

Now, taking into account (2.117), we get

$$\|Du\|_{L^1(\Omega)} \leq K' |\lambda|^{-1/2} \|f\|_{L^1(\Omega)}.$$

□

As a consequence of Theorem 2.5.2 we have that A_1 is sectorial, that is there exist $K \in \mathbf{R}$ and $\theta_1 \in (\pi/2, \pi)$ such that

$$\Sigma_{\theta_1, \omega_1} = \{\lambda \in \mathbf{C};\ \lambda \neq \omega_1, |\arg(\lambda - \omega_1)| < \theta_1\} \subset \rho(A_1)$$

and

$$\|R(\lambda, A_1)\|_{\mathcal{L}(L^1(\Omega))} \leq \frac{K}{|\lambda - \omega_1|}$$

holds for each $\lambda \in \Sigma_{\theta_1, \omega_1}$.

Chapter 3

Estimates of the derivatives of solution of parabolic problems in $L^1(\Omega)$

As a consequence of Theorem 2.5.2 and Proposition 1.2.7 we have that $(A_1, D(A_1))$ is sectorial in $L^1(\Omega)$, then it generates a bounded analytic semigroup $T(t)$ and $T(t)u_0$ is the solution of

$$\begin{cases} \partial_t w - \mathcal{A}w = 0 & \text{in } (0,\infty)\times\Omega \\ w(0) = u_0 & \text{in } \Omega \\ \langle ADw, \nu\rangle = 0 & \text{in } (0,\infty)\times\partial\Omega. \end{cases}$$

for each $u_0 \in L^1(\Omega)$. Moreover there exist $c_i = c_i(\Omega, \mu, M_1)$, $i = 0, 1$ such that

$$\|T(t)\|_{\mathcal{L}(L^1(\Omega))} \leq c_0, \quad t > 0 \tag{3.1}$$

and

$$t\|A_1 T(t)\|_{\mathcal{L}(L^1(\Omega))} \leq c_1, \quad t > 0. \tag{3.2}$$

Moreover since $D(A_1)$ is dense in $L^1(\Omega)$ by construction, $T(t)$ is strongly continuous in $L^1(\Omega)$. Hence

$$\lim_{t\to 0^+} \|T(t)u_0 - u_0\|_{L^1(\Omega)} = 0 \quad \text{for all } u_0 \in L^1(\Omega) \tag{3.3}$$

Notice that for every $u \in L^1(\Omega)$ and for every $t > 0$, $T(t)u \in W^{2,1}(\Omega)$.

3.0.1 Estimates of first order derivatives

Now, using the gradient estimate (2.115) of the resolvent operator $R(\lambda, A_1)$, we estabilish the following further property of the semigroup $T(t)$.

Proposition 3.0.4. *Let Ω, $\mathcal{A}$ and $\mathcal{B}$ be as in Section 2.5 and let $T(t)$ be the semigroup generated by $(A_1, D(A_1))$. Then, there exists c_2 depending on Ω, μ, M_1 such that for $t > 0$,*

$$t^{1/2}\|DT(t)\|_{\mathcal{L}(L^1(\Omega))} \leq c_2. \tag{3.4}$$

Proof. Let θ_1' be as in Theorem 2.5.3 and suppose $\omega_1' = 0$ (otherwise we consider $A_1 - \omega_1'$). Let consider the curve

$$\Gamma = \{\lambda \in \mathbf{C}; |\arg \lambda| = \theta_1', |\lambda| \geq 1\} \cup \{\lambda \in \mathbf{C} : |\arg \lambda| \leq \theta_1', |\lambda| = 1\}$$

oriented counterclockwise. We know that for $t > 0$

$$T(t) = \frac{1}{2\pi i}\int_\Gamma e^{t\lambda} R(\lambda, A_1) d\lambda.$$

Setting $\lambda' = \lambda t$ we get

$$T(t) = \frac{1}{2\pi i}\int_\Gamma e^{\lambda'} R(\lambda'/t, A_1) t^{-1} d\lambda'$$

and

$$D_i T(t) = \frac{1}{2\pi i}\int_\Gamma e^{\lambda'} t^{-1} D_i R(\lambda'/t, A_1) d\lambda' \qquad i = 1, \ldots, n$$

therefore by (2.115)

$$\|D_i T(t)\|_{\mathcal{L}(L^1(\Omega))} \leq t^{-1/2}\int_\Gamma e^{\mathrm{Re}\,\lambda'} |\lambda'|^{-1/2} d|\lambda'| \leq ct^{-1/2} \qquad i = 1, \ldots, n$$

and the result is proved. □

Remark 3.0.5. [Neumann boundary conditions] We have stated Theorem 2.5.2 in the form we most frequently use, but the estimates hold under more general assumptions. In particular, all non tangential boundary conditions are allowed. We denote by c_ν a constant which can be used in the inequalities (3.1)–(3.4), when Neumann boundary conditions are associated with a general uniformly elliptic operator.

Remark 3.0.6. [Assumptions on the coefficients b_i] The result of generation in L^1 and estimates (3.1), (3.2) can be achieved under weaker assumptions on coefficients b_i. Assume $\mathcal{A}, \mathcal{B}$ as in (2.106), (2.110) with coefficients satisfying (2.108), (2.107). Then we know that $(A_1, D(A_1))$ generates an analytic semigroup in $L^1(\Omega)$.
We consider a first order perturbing operator $\mathcal{C} = \sum_{i=1}^n (\tilde{b}_i - b_i) D_i$ with $\tilde{b}_i \in L^\infty(\Omega)$ $b_i \neq \tilde{b}_i$. Let C_1 be the realization of $\mathcal{C}$ in $L^1(\Omega)$ with domain $D(C_1) = W^{1,1}(\Omega)$. The operator C_1 is A_1- bounded and more precisely for every $\varepsilon > 0$ there exists $c(\varepsilon) > 0$ such that

$$\|C_1 u\|_{L^1(\Omega)} \leq \varepsilon \|A_1 u\|_{L^1(\Omega)} + c(\varepsilon)\|u\|_{L^1(\Omega)}$$

holds for every $u \in D(A_1)$. Indeed let $u \in D(A_1)$, (suppose $\omega_1 = 0$, otherwise consider $A_1 - \omega_1$) then $u = R(\lambda, A_1) f$ for every $\lambda \in \mathbf{C}$ with $\mathrm{Re}\,\lambda > 0$ and $f \in L^1(\Omega)$. Moreover, by (1.7) we can write

$$u = \int_0^\infty e^{-\lambda s} T(s) f ds, \quad \mathrm{Re}\,\lambda > 0.$$

Thus, in particular for $\lambda > 0$

$$\|Du\|_{L^1(\Omega)} \le c\|f\|_{L^1(\Omega)} \int_0^\infty \frac{e^{-\lambda s}}{\sqrt{s}} ds = \frac{c}{\sqrt{\lambda}}\|f\|_{L^1(\Omega)} \le c(\sqrt{\lambda}\|u\|_{L^1(\Omega)} + \frac{1}{\sqrt{\lambda}}\|A_1 u\|_{L^1(\Omega)})$$

This implies that $D(A_1) \hookrightarrow W^{1,1}(\Omega)$; moreover, minimizing over $\lambda > 0$, we get

$$\|Du\|_{L^1(\Omega)} \le c\|u\|_{L^1(\Omega)}^{1/2}\|A_1 u\|_{L^1(\Omega)}^{1/2} \le \varepsilon\|A_1 u\|_{L^1(\Omega)} + \frac{c}{\varepsilon}\|u\|_{L^1(\Omega)} \tag{3.5}$$

and by Theorem 1.2.10 we conclude. We point out that the first inequality in (3.5) asserts that $W^{1,1}(\Omega) \in J_{1/2}(L^1(\Omega), D(A_1))$.

3.1 Estimates of second order derivatives

In order to proceed, we also need a precise L^1-estimate of the second (spatial) derivatives of $T(t)u_0$, for $u_0 \in D(A_1)$. This is proved in Proposition 3.1.3 below. The argument used here is similar to the one used in [18, Theorem 2.4], where Ω is bounded and different boundary conditions are imposed. The scheme is the following: we estimate the second order derivatives in Proposition 3.1.1, and then, using this result, we characterize the interpolation space $D_{\mathcal{A}}(\alpha, 1) = (L^1(\Omega), D(\mathcal{A}))_{\alpha,1}$ as a fractional Sobolev space and use this to improve estimate (3.6) using the $W^{1,1}$ norm of u instead of the L^1 norm. We start with the following result.

Proposition 3.1.1. *Let $\Omega, \mathcal{A}, \mathcal{B}$ be as in Section 2.5. Assume, in addition, $c \in W^{1,\infty}(\Omega)$; then, there exists c_3 depending on n, μ, Ω, M_1, $\|c\|_{W^{1,\infty}(\Omega)}$, c_0 c_1, c_2, c_ν such that for every $t \in (0,1)$ and $u \in L^1(\Omega)$ we have*

$$t\|D^2 T(t)u\|_{L^1(\Omega)} \le c_3\|u\|_{L^1(\Omega)}. \tag{3.6}$$

Proof. We set for $\sigma > 0$ $u_\sigma = T(\sigma)u$ and

$$M_2 = \max\{\|A\|_{2,\infty}, \|B\|_{2,\infty}, \|c\|_{1,\infty}\}. \tag{3.7}$$

By the regularity of the boundary $\partial\Omega$ we can consider a partition of unity $\{(\eta_h, U_h)\}_{h\in\mathbf{N}}$ such that $\operatorname{supp}\eta_h \subset U_h$, $\sum_{h=0}^\infty \eta_h(x) = 1$ for every $x \in \overline{\Omega}$ and $0 \le \eta_h \le 1$ for every $h \in \mathbf{N}$, $\overline{U}_0 \subset \Omega$, U_h for $h \ge 1$ is a ball such that $\{U_h\}_{h\ge1}$ is a covering of $\partial\Omega$ and $\{U_h\}_{h\in\mathbf{N}}$ is a covering of Ω with bounded overlapping, that is there is $\kappa > 0$ such that

$$\sum_{h\in\mathbf{N}} \chi_{U_h}(x) \le \kappa, \quad \forall x \in \overline{\Omega}. \tag{3.8}$$

Moreover we choose η_h in such a way $\langle A(x)D\eta_h(x), \nu(x)\rangle = 0$ for every $x \in \partial\Omega$ and set $\overline{M} := \sup_{h\in\mathbf{N}} \|\eta_h\|_{2,\infty}$, which is finite by the uniform C^2 regularity of $\partial\Omega$. We can also consider coordinate functions $\psi_h : V_h \to B(0,1)$ such that $\psi_h(V_h \cap \Omega) = B^+(0,1) = \{y = (y', y_n) \in B(0,1) : y_n > 0\}$, $\psi_h(V_h \cap \partial\Omega) = \{y = (y', y_n) \in B(0,1) : y_n = 0\}$,

$d(\psi_h)_x(a(x)\nu(x)) = -e_n$ for every $x \in \partial\Omega$ where $d(\psi_h)_x$ denotes the differential of ψ_h at x. Finally we suppose that there is a constant M_ψ such that

$$\sup_{h\geq 1}\left\{\|D^2\psi_h\|_{2,\infty}, \|D^2\psi_h^{-1}\|_{2,\infty}\right\} \leq M_\psi.$$

Notice also that we may assume that for all $h \geq 1$ the inclusion $U_h \subset\subset V_h$ holds, and that we can choose a C^2 domain E such that $\psi_h(U_h \cap \Omega) \subset E \subset B^+(0,1)$. Notice that $u_\sigma \in W^{1,1}(\Omega)$ and denote by $u(t) = T(t)u_\sigma$ the solution of the problem

$$\begin{cases} \partial_t w - \mathcal{A}w = 0 & \text{in } (0,\infty)\times\Omega \\ w(0) = u_\sigma & \text{in } \Omega \\ \langle ADw, \nu\rangle = 0 & \text{in } (0,\infty)\times\partial\Omega. \end{cases}$$

We want to estimate the L^1–norm of $tD^2u(t)$ by the L^1-norm of u; we shall use estimates (3.1)–(3.4). The functions $v_h(t) = u(t)\eta_h$ solve, for every $h \in \mathbf{N}$, the problem

$$\begin{cases} \partial_t w - \mathcal{A}w = \mathcal{A}_h u(t) & \text{in } (0,\infty)\times\Omega \\ w(0) = \eta_h u_\sigma & \text{in } \Omega \\ \langle ADw, \nu\rangle = 0 & \text{in } (0,\infty)\times\partial\Omega \end{cases} \tag{3.9}$$

where

$$\mathcal{A}_h u(t) = -2\langle AD\eta_h, Du(t)\rangle - u(t)\,\mathrm{div}(AD\eta_h) - u(t)\,\langle B, D\eta_h\rangle\,. \tag{3.10}$$

Notice that the derivative $D_k v_h(t)$ satisfies the equation $\partial_t(D_k v_h(t)) - \mathcal{A}(D_k v_h(t)) = \mathcal{A}_h^k u(t)$, where

$$\begin{aligned} \mathcal{A}_h^k u(t) =& \mathrm{div}\left((D_k A)D(u(t)\eta_h)\right) + \langle (D_k B), D(u(t)\eta_h)\rangle + (D_k c)u(t)\eta_h + D_k(\mathcal{A}_h u(t)) \\ =& \mathrm{div}\left((D_k A)D(u(t)\eta_h)\right) + \langle (D_k B), D(u(t)\eta_h)\rangle + (D_k c)u(t)\eta_h \\ &+ D_k[-2\langle AD\eta_h, Du(t)\rangle - u(t)\,\mathrm{div}(AD\eta_h) - u(t)\,\langle B, D\eta_h\rangle] \end{aligned} \tag{3.11}$$

For $D_k v_h(t)$ we consider the problem

$$\begin{cases} \partial_t w - \mathcal{A}w = \mathcal{A}_h^k u(t) & \text{in } (0,\infty)\times\Omega \\ w(0) = D_k(\eta_h u_\sigma) & \text{in } \Omega \\ \langle ADw, \nu\rangle = 0 & \text{in } (0,\infty)\times\partial\Omega \end{cases} \tag{3.12}$$

whose solution is $v_{hk}(t) = T(t)D_k(\eta_h u_\sigma) + \int_0^t T(t-s)\mathcal{A}_h^k u(s)ds$. Now we consider $h = 0$, i.e., we draw our attention to the inner part. Since $v_0(t) = \eta_0 u(t) = 0$ in $\Omega\setminus U_0$, it turns out that $D_k v_0(t)$ is the solution of (3.12) with $h = 0$. Then

$$D_k v_0(t) = T(t)D_k(\eta_0 u_\sigma) + \int_0^t T(t-s)\mathcal{A}_0^k u(s)ds, \tag{3.13}$$

where $\mathcal{A}_0^k$ is the operator defined in (3.11). Then, differentiating, we obtain

$$D_{lk}^2 v_0(t) = D_l[T(t)D_k(\eta_0 u_\sigma)] + \int_0^t D_l[T(t-s)\mathcal{A}_0^k v(s)]ds.$$

by which, using (3.4),

$$\begin{aligned}
\|D^2_{lk}v_0(t)\|_{L^1(\Omega)} &\leq \|D_l T(t) D_k(\eta_0 u_\sigma)\|_{L^1(\Omega)} + \int_0^t \|D_l T(t-s)\mathcal{A}_0^k u(s)\|_{L^1(\Omega)} ds \\
&\leq \frac{c_2}{\sqrt{t}}\|D_k(\eta_0 u_\sigma)\|_{L^1(\Omega)} + \int_0^t \frac{c_2}{\sqrt{t-s}}\|\mathcal{A}_0^k u(s)\|_{L^1(\Omega)} ds \\
&\leq \frac{c_2^2}{\sqrt{t}}\|\eta_0\|_{W^{1,\infty}}\|u_\sigma\|_{W^{1,1}(\Omega)} + \int_0^t \frac{c_2}{\sqrt{t-s}}\|\mathcal{A}_0^k u(s)\|_{L^1(\Omega)} ds
\end{aligned}$$

Finally, estimating $\|\mathcal{A}_0^k u(s)\|_{L^1(\Omega)}$ by (3.11) we get $\|\mathcal{A}_0^k u(s)\|_{L^1(\Omega)} \leq c\,\|u(s)\|_{W^{2,1}(\Omega)}$ where $c = c(\overline{M}, M_2)$. Summing on l and k, using (A.1) and again (3.1), we get

$$\|D^2 v_0(t)\|_{L^1(\Omega)} \leq c\Big(\frac{1}{\sqrt{t}}\|u_\sigma\|_{W^{1,1}(\Omega)} + \int_0^t \frac{1}{\sqrt{t-s}}\|D^2 u(s)\|_{L^1(\Omega)} ds\Big)$$

where $c = c(\overline{M}, M_2, c_2, n)$. We now consider $h \geq 1$, i.e., we consider a ball intersecting $\partial\Omega$.

Using the transformation $\hat{f}(y) := f(\psi_h^{-1}(y))$ for a generic f defined in $\Omega \cap V_h$, and since v_h is the solution of (3.9), we get that for every $h \geq 1$ the function $\hat{v}_h(t,y) = \eta_h(\psi_h^{-1}(y))u(t,\psi_h^{-1}(y))$ is the solution of the following initial-boundary value problem with homogeneous Neumann boundary conditions

$$\begin{cases} \partial_t w - \hat{\mathcal{A}} w = \hat{\mathcal{A}}_h \hat{v} & \text{in } (0,+\infty)\times E \\ w(0) = \hat{\eta}_h \hat{u}_\sigma & \text{in } E \\ \dfrac{\partial w}{\partial \nu} = 0 & \text{in } (0,+\infty)\times \partial E \end{cases} \tag{3.14}$$

where $\hat{\mathcal{A}}$ is the operator defined on $B(0,1)$ as follows

$$\hat{\mathcal{A}} w := \operatorname{div}(\hat{A}\,Dw) + \langle \hat{B}, Dw\rangle + \hat{c}w$$

whose coefficients (here we omit the index h to simplify the notations and by analogy with (3.9)) are given by

$$\begin{aligned}
\hat{A}(y) :=& \big(D\psi_h\big)(\psi_h^{-1}(y)) \cdot A(\psi_h^{-1}(y)) \cdot \big(D\psi_h\big)^t(\psi_h^{-1}(y)) \\
(\hat{B}(y))_l :=& \mathrm{Tr}\Big[\big(D\psi_h\big)(\psi_h^{-1}(y)) \cdot A(\psi_h^{-1}(y)) \cdot H^l(\psi_h^{-1}(y)) \cdot \big(D\psi_h^{-1}\big)^t(y)\Big] \\
&+ \mathrm{Tr}\Big[\big(D\psi_h\big)(\psi_h^{-1}(y)) \cdot G^j(y)\Big]\big(D\psi_h\big)^t_{jl}(\psi_h^{-1}(y)) - \frac{\partial}{\partial y_j}\Big[\hat{a}_{jl}(y)\Big] \\
&+ \Big[\big(D\psi_h\big)(\psi_h^{-1}(y)) \cdot B(\psi_h^{-1}(y))\Big]_l \\
\hat{c}(y) :=& c(\psi_h^{-1}(y))
\end{aligned}$$

where $H^l_{ki} = D^2_{ki}(\psi_h)_l$ and $G^j_{ki} = D_k a_{ij}(\psi_h^{-1}(y))$ and (see (3.10))

$$\hat{\mathcal{A}}_h \hat{u}(t) = -2\big\langle A(\psi_h^{-1}(y))(D\psi_h)^t D\hat{\eta}_h, (D\psi_h)^t D\hat{u}(t)\big\rangle - \hat{u}(t)\big[\operatorname{div}(\hat{A}D\hat{\eta}_h) + \langle \hat{A}, D\hat{\eta}_h\rangle\big]\,.$$

Now, as done before for $h = 0$, differentiating the equation (now $D_k = \frac{\partial}{\partial y_k}$) we obtain that $D_k\hat{v}_h$ solves $\partial_t(D_k\hat{v}_h(t)) - \hat{\mathcal{A}}(D_k\hat{v}_h(t)) = \hat{\mathcal{A}}_h^k\hat{u}(t)$, where $\hat{\mathcal{A}}_h^k\hat{v}$ can be obtained by taking the corresponding term in (3.11). Associated with this operator, we can consider the problem

$$\begin{cases} \partial_t w - \hat{\mathcal{A}} w = \hat{\mathcal{A}}_h^k \hat{u}(t) & \text{in } (0,\infty)\times E \\ w(0) = D_k\big(\hat{\eta}_h \hat{u}_\sigma\big) & \text{in } E \\ \dfrac{\partial w}{\partial \nu} = 0 & \text{in } (0,\infty)\times \partial E. \end{cases}$$

The function $D_k \hat{v}_h$ satisfies the equation and the initial condition. Notice that if $k \neq n$ also the boundary condition is satisfied since $\hat{v}_h = 0$ in a neighborhood of $\partial E \cap \{y \in \mathbf{R}^n \,|\, y_n > 0\}$, in the other part of ∂E the operator D_k is a tangential derivative and $\frac{\partial \hat{v}_h}{\partial y_n}$ is constant for $y_n = 0$. Denote by S the semigroup which gives the solution of this problem and notice that the estimates (3.1)–(3.4) hold for $S(t)$, see Remark 3.0.5. Then

$$D_k \hat{v}_h(t) = S(t) D_k \hat{v}_h(0) + \int_0^t S(t-s)\hat{\mathcal{A}}_h^k \hat{u}(s)ds. \tag{3.15}$$

Differentiating (3.15) with respect to D_j for any j, we have then proved that the following holds

$$D^2_{kj} \hat{v}_h(t) = D_j S(t) D_k \hat{v}_h(0) + \int_0^t D_j S(t-s)\hat{\mathcal{A}}_h^k \hat{u}(s)ds. \tag{3.16}$$

Thus, as for $v_0(t)$, we have for $(k,j) \neq (n,n)$

$$\begin{aligned} \|D^2_{kj}\hat{v}_h(t)\|_{L^1(E)} &\leq \frac{c_2}{\sqrt{t}}\|D_k(\hat{\eta}_h \hat{u}_\sigma)\|_{L^1(E)} + \int_0^t \frac{c_2}{\sqrt{t-s}}\|\hat{\mathcal{A}}_h^k \hat{u}(s)\|_{L^1(E)}ds \\ &\leq \frac{c}{\sqrt{t\sigma}}\|\hat{u}\|_{L^1(E)} + \int_0^t \frac{c_2}{\sqrt{t-s}}\|\hat{\mathcal{A}}_h^k \hat{u}(s)\|_{L^1(E)}ds. \end{aligned}$$

We now estimate $D^2_{nn}\hat{v}_h(t)$. Since

$$\begin{aligned} \hat{a}_{nn} D^2_{nn}\hat{v}_h(t) \;&=\; \hat{\mathcal{A}}\hat{v}_h(t) - \sum_{(i,j)\neq(n,n)} \hat{a}_{ij} D^2_{ij}\hat{v}_h(t) - \sum_{i,j=1}^n (D_i \hat{a}_{ij}) D_j \hat{v}_h(t) \\ &\quad - \sum_{i=1}^n \hat{b}_i D_i \hat{v}_h(t) - \hat{c}\hat{v}_h(t) \end{aligned}$$

and since $\hat{a}$ is uniformly elliptic with ellipticity constant proportional to μ, we can find a constant c (depending only on $n, M_2, \mu, \partial\Omega$) such that

$$\begin{aligned} \|D^2_{nn}\hat{v}_h(t)\|_{L^1(E)} &= \Big\| \frac{1}{\hat{a}_{nn}}\Big(\hat{\mathcal{A}}\hat{v}_h(t) - \sum_{(i,j)\neq(n,n)} \hat{a}_{ij} D^2_{ij}\hat{v}_h(t) + \\ &\quad - \sum_{i,j=1}^n (D_i \hat{a}_{ij}) D_j \hat{v}_h(t) - \sum_{i=1}^n \hat{b}_i D_i \hat{v}_h(t) - \hat{c}\hat{v}_h(t)\Big)\Big\|_{L^1(E)} \\ &\leq c\Big[\sum_{(i,j)\neq(n,n)} \|D^2_{ij}\hat{v}_h(t)\|_{L^1(E)} + \|\hat{\mathcal{A}}\hat{v}_h(t)\|_{L^1(E)} + \|D\hat{v}_h(t)\|_{L^1(E)} + \|\hat{v}_h(t)\|_{L^1(E)}\Big]. \end{aligned}$$

Summing up, we may argue in the same way as for $h = 0$, and get

$$\begin{aligned}\|D^2\hat{v}_h(t)\|_{L^1(E)} &\le c'\Big[\frac{1}{\sqrt{t}}\|u_\sigma \circ \psi_h^{-1}\|_{W^{1,1}(E)} + \int_0^t \frac{1}{\sqrt{t-s}}\|D^2\hat{u}(s)\|_{L^1(E)}ds \\ &\quad + \|\hat{\mathcal{A}}\hat{v}_h(t)\|_{L^1(E)}\Big] \\ &\le c'\Big[\frac{1}{\sqrt{t\sigma}}\|u \circ \psi_h^{-1}\|_{L^1(E)} + \int_0^t \frac{1}{\sqrt{t-s}}\|D^2\hat{u}(s)\|_{L^1(E)}ds + \|\hat{\mathcal{A}}\hat{v}_h(t)\|_{L^1(E)}\Big]\end{aligned}$$

where $c' = c(\overline{M}, M_2, M_\psi, n, c_2, c_\nu)$. Coming back to $\Omega \cap U_h$ we obtain

$$\begin{aligned}\|D^2 v_h(t)\|_{L^1(\Omega\cap U_h)} &\le c''\Big[\frac{1}{\sqrt{t}}\|u_\sigma\|_{W^{1,1}(\Omega\cap U_h)} + \int_0^t \frac{1}{\sqrt{t-s}}\|D^2u(s)\|_{L^1(\Omega\cap U_h)}ds \\ &\quad + \|\mathcal{A}v_h(t)\|_{L^1(\Omega\cap U_h)}\Big] \qquad (3.17)\\ &\le c''\Big[\frac{1}{\sqrt{t\sigma}}\|u\|_{L^1(\Omega\cap U_h)} + \int_0^t \frac{1}{\sqrt{t-s}}\|D^2u(s)\|_{L^1(\Omega\cap U_h)}ds + \|\mathcal{A}v_h(t)\|_{L^1(\Omega\cap U_h)}\Big]\end{aligned}$$

where c'' depends on $\overline{M}, M_2, M_\psi, n, c_2, c_\nu$. Now, using (3.1), (3.2) and (3.8), we have

$$\begin{aligned}\|D^2u(t)\|_{L^1(\Omega)} &= \|D^2\big(\sum_{h=0}^{\infty} v_h(t)\big)\|_{L^1(\Omega)} = \|\sum_{h=0}^{\infty} D^2 v_h(t)\|_{L^1(\Omega)} \qquad (3.18)\\ &\le \kappa c''\Big[\frac{1}{\sqrt{t}}\|u_\sigma\|_{W^{1,1}(\Omega)} + \int_0^t \frac{1}{\sqrt{t-s}}\|D^2u(s)\|_{L^1(\Omega)}ds + \|\mathcal{A}u(t)\|_{L^1(\Omega)}\Big] \\ &\le c'''\Big[\frac{1}{\sqrt{t\sigma}}\|u\|_{L^1(\Omega)} + \int_0^t \frac{1}{\sqrt{t-s}}\|D^2u(s)\|_{L^1(\Omega)}ds + \frac{1}{\sqrt{t\sigma}}\|u\|_{L^1(\Omega)}\Big],\end{aligned}$$

where c''' depends on κ, c'', c_0, c_1. Now using Gronwall's generalized inequality (see Lemma 1.5.7), we get

$$\|D^2u(t)\|_{L^1(\Omega)} \le \frac{c}{\sqrt{t\sigma}}\|u\|_{L^1(\Omega)}. \qquad (3.19)$$

Then, by taking $\sigma = t$, we get $\|D^2u(t)\|_{L^1(\Omega)} \le c_3 t^{-1}\|u\|_{L^1(\Omega)}$ for every $t \in (0,1)$. □

3.1.1 Characterization of interpolation spaces between $D(A_1)$ and $L^1(\Omega)$

We can use Proposition 3.1.1 to characterize some interpolation spaces between $D(A_1)$ and $L^1(\Omega)$.

Theorem 3.1.2. *Let A_1 be as in Proposition 3.1.1; then for every $\alpha \in (0,1/2)$ we have*

$$(L^1(\Omega), D(A_1))_{\alpha,1} = W^{2\alpha,1}(\Omega)$$

where $W^{2\alpha,1}$ denotes the Sobolev space of fractional order (see Section A.2.1 for details).

Proof. It is sufficient to prove that

$$(L^1(\Omega), D(A_1))_{\alpha,1} = (L^1(\Omega), W^{2,1}(\Omega) \cap W^{1,1}_{A,\nu}(\Omega))_{\alpha,1}$$

in fact using Theorem A.2.7 we complete the proof.
First of all, let us observe that $W^{2,1}(\Omega)\cap W^{1,1}_{A,\nu}(\Omega) \hookrightarrow D(A_1)$. Therefore, using Definition A.2.2, we obtain

$$(L^1(\Omega), W^{2,1}(\Omega)\cap W^{1,1}_{A,\nu}(\Omega))_{\alpha,1} \hookrightarrow (L^1(\Omega), D(A_1))_{\alpha,1}.$$

Conversely, let $u_0 \in (L^1(\Omega), D(A_1))_{\alpha,1}$ and set for $t \in [0,1]$

$$u_0 = u_0 - T(t)u_0 + T(t)u_0 = -\int_0^t A_1 T(s)u_0 ds + T(t)u_0 = v_1 + v_2.$$

We have

$$\|v_1\|_{L^1(\Omega)} \leq \int_0^t \|A_1 T(s)u_0\|_{L^1(\Omega)} ds$$

and since $v_2 \in W^{2,1}(\Omega)\cap W^{1,1}_{A,\nu}(\Omega)$, using (A.1), (3.1) and Proposition 3.1.1, we have

$$\begin{aligned}
\|v_2\|_{W^{2,1}(\Omega)} &= \|T(t)u_0\|_{L^1(\Omega)} + \sum_{i,j=1}^n \|D_{ij}[T(t)u_0 - T(1)u_0 + T(1)u_0]\|_{L^1(\Omega)} \\
&\leq c_0\|u_0\|_{L^1(\Omega)} + \sum_{i,j=1}^n \|D_{ij}\int_t^1 T(s/2)A_1T(s/2)u_0 ds\|_{L^1(\Omega)} + c_3\|u_0\|_{L^1(\Omega)} \\
&\leq c\left\{\|u_0\|_{L^1(\Omega)} + \int_t^1 s^{-1}\|A_1T(s/2)u_0\|_{L^1(\Omega)} ds\right\}
\end{aligned}$$

Therefore for $t \in [0,1]$, setting $K(t,u_0) := K(t,u_0,L^1(\Omega), W^{2,1}(\Omega)\cap W^{1,1}_{A,\nu}(\Omega))$ we obtain

$$\begin{aligned}
K(t,u_0) &= \inf_{u_0=u_0^1+u_0^2}(\|u_0^1\|_{L^1(\Omega)} + t\|u_0^2\|_{W^{2,1}(\Omega)}) \\
&\leq \|v_1\|_{L^1(\Omega)} + t\|v_2\|_{W^{2,1}(\Omega)} \\
&\leq c\Big(\int_0^t \|A_1T(s)u_0\|_{L^1(\Omega)} ds + t\|u_0\|_{L^1(\Omega)} \\
&+ t\int_t^1 s^{-1}\|A_1T(s/2)u_0\|_{L^1(\Omega)} ds\Big)
\end{aligned}$$

On the other hand, choosing $u_0^1 = u_0$ and $u_0^2 = 0$ we get

$$K(t,u_0) \leq \|u_0\|_{L^1(\Omega)}.$$

Therefore

$$\begin{aligned}
K(t,u_0) \leq c\Big(\min(1,t)\|u_0\|_{L^1(\Omega)} &+ \int_0^t \|A_1T(s)u_0\|_{L^1(\Omega)} ds \\
&+ t\int_t^1 s^{-1}\|A_1T(s/2)u_0\|_{L^1(\Omega)} ds\Big).
\end{aligned}$$

Therefore for each $\alpha \in (0,1)$ we get

$$\begin{aligned}\int_0^\infty t^{-(1+\alpha)} K(t,u_0)dt \le c\Big\{ &\|u_0\|_{L^1(\Omega)} \int_0^\infty t^{-(1+\alpha)} \min(1,t)dt \\ &+ \int_0^\infty (t^{-(1+\alpha)} \int_0^t \|A_1 T(s) u_0\|_{L^1(\Omega)} ds) dt \\ &+ \int_0^\infty (t^{-\alpha} \int_t^\infty s^{-1} \|A_1 T(s) u_0\|_{L^1(\Omega)} ds) dt \Big\}\end{aligned}$$

so that using Hardy inequalities stated in Theorem 1.5.6, we get

$$\int_0^\infty t^{-(1+\alpha)} K(t,u_0)dt \le c\Big\{ \|u_0\|_{L^1(\Omega)} + \int_0^\infty s^{-\alpha} \|A_1 T(s) u_0\|_{L^1(\Omega)} ds \Big\}$$

and hence from Theorem 1.3.2 we get

$$(L^1(\Omega), D(A_1))_{\alpha,1} \hookrightarrow (L^1(\Omega), W^{2,1}(\Omega) \cap W^{1,1}_{A,\nu}(\Omega))_{\alpha,1}$$

so, the result is proved. □

Using Theorem 3.1.2 we can improve the estimate of Proposition 3.1.1, under additional assumption on the initial datum; in fact, we have the following.

Proposition 3.1.3. *Let $\Omega, \mathcal{A}, \mathcal{B}$ be as in Section 2.5. Assume, in addition, $c \in W^{1,\infty}(\Omega)$; then, there exist $\delta \in (1/2, 1)$ and c_4 depending on n, μ, Ω, M_2, c_0, c_1, c_2, c_3 c_ν such that for every $t \in (0,1)$ and $u \in D(A_1)$ we have*

$$t^\delta \|D^2 T(t) u\|_{L^1(\Omega)} \le c_4 \|u\|_{W^{1,1}(\Omega)}. \tag{3.20}$$

Proof. We can repeat the proof of Proposition 3.1.1 until the first inequality in (3.18), with $\sigma > 0$, so that we have

$$\begin{aligned}\|D^2 u(t)\|_{L^1(\Omega)} \le &\kappa c'' \Big[\frac{1}{\sqrt{t}} \|u_\sigma\|_{W^{1,1}(\Omega)} + \int_0^t \frac{1}{\sqrt{t-s}} \|D^2 u(s)\|_{L^1(\Omega)} ds \\ &+ \|\mathcal{A} u(t)\|_{L^1(\Omega)} \Big]\end{aligned} \tag{3.21}$$

Using (1.10), we get that for any $\alpha, \beta \in (0,1)$ there is C such that

$$t^{1-\alpha+\beta} \|\mathcal{A} T(t) u\|_{D_{\mathcal{A}}(\beta,1)} \le C \|u\|_{D_{\mathcal{A}}(\alpha,1)}.$$

By definition of interpolation, $D_{\mathcal{A}}(\beta,1)$ is continuously embedded in $L^1(\Omega)$ for any $\beta \in (0,1)$. Using the fact that $D_{\mathcal{A}}(\alpha,1)$ is the fractional Sobolev space $W^{2\alpha,1}(\Omega)$ for $\alpha < 1/2$ and that $W^{1,1}(\Omega)$ embeds in $W^{2\alpha,1}(\Omega)$ for such α, we obtain, with constants C that may change from a line to the other,

$$\begin{aligned}\|\mathcal{A} T(t) u\|_{L^1(\Omega)} \le &C \|\mathcal{A} T(t) u\|_{D_{\mathcal{A}}(\beta,1)} \le \frac{C}{t^{1-\alpha+\beta}} \|u\|_{D_{\mathcal{A}}(\alpha,1)} \\ = &\frac{C}{t^{1-\alpha+\beta}} \|u\|_{W^{2\alpha,1}(\Omega)} \le \frac{C}{t^{1-\alpha+\beta}} \|u\|_{W^{1,1}(\Omega)}\end{aligned}$$

We choose then $\alpha \in (0,1/2)$ and $\beta \in (0,1)$ is such a way that $\delta = 1-\alpha+\beta \in (1/2,1)$, and (3.21) becomes

$$\|D^2u(t)\|_{L^1(\Omega)} \leq \frac{C}{t^\delta}\|u_\sigma\|_{W^{1,1}(\Omega)} + \int_0^t \frac{C'}{\sqrt{t-s}}\|D^2u(s)\|_{L^1(\Omega)}ds.$$

Therefore applying the Gronwall's lemma and passing to the limit as $\sigma \to 0$ we get (3.20). $\square$

Chapter 4

BV functions and parabolic problems: the first characterization

This chapter is entirely devoted to functions of bounded variation and sets of finite perimeter. We have collected several results related to these functions, from the classical ones present in literature to a new characterization of such functions. This chapter is organized as follows: in the first section we recall definitions, basic properties and classical results for functions of bounded variation and sets of finite perimeter.
In the second one we extend classical definitions and properties to functions with possibly weighted bounded variation on Ω and finally, in the last section we give a first characterization for such class of functions in terms of the short-time behavior of $T(t)$.

4.1 The space BV: definitions and preliminary results

First we give a brief introduction to the definition of BV functions in non-weighted Euclidean domains (complete discussions and proofs can be found in [5] and [20]). These are integrable functions whose weak first-order distributional derivatives are finite Radon measures. Throughout this chapter we denote by Ω a generic open set of $\mathbf{R}^n$. The classical integration by parts formula shows that if $f \in C^1(\Omega)$ and $\varphi \in C^1_c(\Omega, \mathbf{R}^n)$, then

$$\int_\Omega f \operatorname{div}\varphi \, dx = -\int_\Omega \varphi \cdot Df \, dx.$$

The definition of Sobolev functions is based upon a generalization of the integration by parts formula. A locally summable function $g : \Omega \mapsto \mathbf{R}^n$ is called a weak derivative of f

if for all $\varphi \in C_c^\infty(\Omega, \mathbf{R}^n)$,

$$\int_\Omega f \mathrm{div}\varphi \, dx = -\int_\Omega \varphi \cdot g \, dx.$$

If $|g|$ is integrable, then f belongs to the Sobolev space $W^{1,1}(\Omega)$.

Definition 4.1.1. *Let $f \in L^1(\Omega)$; we say that f is a* function of bounded variation in Ω *if there exists a vector-valued Radon measure $\mu_f = (\mu_f^1, \dots, \mu_f^n)$ on Ω with $|\mu_f|(\Omega)$ finite such that for all $\varphi \in C_c^\infty(\Omega, \mathbf{R}^n)$,*

$$\int_\Omega f \mathrm{div}\varphi dx = -\int_\Omega \varphi \cdot d\mu_f = -\sum_{i=1}^n \int_\Omega \varphi_i d\mu_f^i(x). \tag{4.1}$$

The vector space of all functions of bounded variation is denoted by $BV(\Omega)$.

By (4.1) it follows that a BV function f belongs to the Sobolev space $W^{1,1}(\Omega)$ if and only if μ_f is absolutely continuous with respect to the Lebesgue measure on Ω. In this case $\mu_f = \nabla f dx$ (see [20, Sec 5.1]), where ∇f denotes the density of μ_f with respect to dx provided by the Besicovitch differentiation Theorem 1.4.10 and coincides with the approximate gradient of u. According to the notation adopted in the Sobolev case we denote by Df the distributional derivative measure μ_f. The following proposition leads to the current working definition for BV functions.

Proposition 4.1.2. *Let $f \in L^1(\Omega)$. Then $f \in BV(\Omega)$ if and only if*

$$|Df|(\Omega) = \sup\{\int_\Omega f \mathrm{div}\varphi \, dx : \ \varphi \in C_c^1(\Omega, \mathbf{R}^n), \|\varphi\|_{L^\infty(\Omega)} \leq 1\} < \infty.$$

The space BV is a Banach space if endowed with the norm

$$\|f\|_{BV(\Omega)} := \|f\|_{L^1(\Omega)} + |\mu_f|(\Omega) \tag{4.2}$$

but the norm-topology is too strong for many applications. Indeed, continuously differentiable functions are not dense in $BV(\Omega)$. For example let $\Omega := \mathbf{R}$, $f := \chi_{(1,2)} \in L^1(\mathbf{R})$ and consider $\{f_k\}$ a sequence of smooth functions obtained by convolution. Then f_k does not converge to f with respect to the norm (4.2). In fact Df_k is absolutely continuous with respect the Lebesgue measure whereas Df is singular with respect the Lebesgue measure, being $Df = \delta_1 - \delta_2$ a measure concentrated on two points. Therefore

$$|Df_k - Df|(\Omega) = |Df_k|(\Omega) + |Df|(\Omega) \geq |Df|(\Omega) \geq 1.$$

This is true because $|\lambda - \mu| = |\lambda| + |\mu|$ for mutually singular measures λ, μ.

An important application of BV function theory is the study of sets of finite perimeter introduced by R. Caccioppoli in [10]; a detailed analysis of these sets was carried on by E. De Giorgi (see [16]) and H. Federer (see [21] and the references there).

4.1.1 Sets of finite perimeter

Given a subset $E \subset \mathbf{R}^n$, we denote by $|E|$ its Lebesgue measure, and by $\mathcal{H}^{n-1}(E)$ its $(n-1)$-dimensional Hausdorff measure.

Definition 4.1.3. *Let E be a measurable subset of $\mathbf{R}^n$. The perimeter of E in Ω is the variation of χ_E in Ω, i.e.*

$$\mathcal{P}(E,\Omega) = \sup\left\{\int_{\Omega\cap E} \mathrm{div}\varphi\, dx : \ \varphi \in C^1_c(\Omega, \mathbf{R}^n), \|\varphi\|_{L^\infty(\Omega)} \leq 1\right\}. \tag{4.3}$$

We say that E is a set of finite perimeter in Ω if $\mathcal{P}(E,\Omega) < \infty$.

When $\Omega = \mathbf{R}^n$, $\mathcal{P}(E,\mathbf{R}^n)$ will be simply denoted by $\mathcal{P}(E)$. The class of sets of finite perimeter in Ω contains all sets E with C^1 boundary inside Ω such that $\mathcal{H}^{n-1}(\Omega\cap\partial E) < \infty$. Indeed, by the Gauss-Green theorem, for these sets E we have

$$\int_E \mathrm{div}\varphi\, dx = -\int_{\partial E} \langle \varphi, \nu_E\rangle d\mathcal{H}^{n-1} \qquad \forall \varphi \in C^1_c(\Omega, \mathbf{R}^n) \tag{4.4}$$

where ν_E is the inner unit normal to E. Using this formula the supremum in (4.3) can be easily computed and it turns out that $\mathcal{P}(E,\Omega) = \mathcal{H}^{n-1}(\Omega\cap\partial E)$

The theory of sets of finite perimeter is closely connected to the theory of BV functions. First of all we notice that if $E \subset \mathbf{R}^n$ has finite measure in Ω, that is $\chi_E \in L^1(\Omega)$, then by Proposition 4.1.2, E has finite perimeter in Ω if and only if the characteristic function χ_E belongs to $BV(\Omega)$; in this case $\mathcal{P}(E,\Omega)$ coincides with $|D\chi_E|(\Omega)$, the total variation in Ω of the distributional derivative of χ_E.
The variational measure $D\chi_E$ can be used to define a measure theoretic boundary denoted by $\mathcal{F}E$ and called *reduced boundary* of E, defined as follows.

Definition 4.1.4. *(Reduced boundary) Let E be a measurable subset of $\mathbf{R}^n$ with finite perimeter in Ω. We define*

$$\mathcal{F}E = \left\{x \in \mathrm{supp}\,|D\chi_E| \cap \Omega : \exists \lim_{\varrho\to 0} \frac{D\chi_E(B_\varrho(x))}{|D\chi_E|(B_\varrho(x))} = \nu_E(x), \ \textit{and}\ |\nu_E(x)| = 1\right\}. \tag{4.5}$$

The function $\nu_E : \mathcal{F}E \to \mathbf{S}^{n-1}$ is called the generalized inner normal *to E,*

By the Besicovitch differentiation theorem (see Theorem 1.4.10) we know that $|D\chi_E|$ is concentrated on $\mathcal{F}E$ and $D\chi_E = \nu_E|D\chi_E|$. De Giorgi proved that $\mathcal{F}E\cap\Omega$ is a countably $(n-1)$- rectifiable set (i.e. $\mathcal{F}E = \bigcup_{h\in\mathbf{N}} K_h \cup N_0$ with $\mathcal{H}^{n-1}(N_0) = 0$ and K_h compact subsets of C^1 manifolds M_h, see Definition 1.4.14) and that

$$D\chi_E = \nu_E \mathcal{H}^{n-1} \llcorner \mathcal{F}E. \tag{4.6}$$

These results imply that the classical Gauss-Green formula can be rewritten for sets of finite perimeter in Ω in the form

$$\int_{E\cap\Omega} \mathrm{div}\varphi\, dx = -\int_{\mathcal{F}E} \langle \varphi, \nu_E\rangle d\mathcal{H}^{n-1} \qquad \forall \varphi \in C^1_c(\Omega, \mathbf{R}^n). \tag{4.7}$$

Observe that in (4.7) the inner normal and the boundary have to be thought in a measure theoretic sense and not in the topological one.
Another important result due to De Giorgi is a blow-up property for points of the reduced boundary (see [16] for the original reference).

Theorem 4.1.5. *(De Giorgi) For any $x \in \mathcal{F}E$ the following properties hold*

(i) the sets $E^x_\rho = (E - x)/\rho$ locally converge in measure in $\mathbf{R}^n$ to the half space $H_{\nu_E(x)}$ orthogonal to $\nu_E(x)$ and containing $\nu_E(x)$ as $\rho \to 0^+$

$$H_{\nu_E(x)} = \{y \in \mathbf{R}^n : \ \langle \nu_E(x), y - x\rangle \geq 0\};$$

(ii) $\mathcal{L}^n \llcorner E^x_\rho \xrightarrow{w^*_{loc}} \mathcal{L}^n \llcorner H_{\nu_E(x)}$ as $\rho \to 0^+$, i.e.

$$\lim_{\rho \to 0^+} \int_{\Omega \cap E^x_\rho} \phi(y)dy = \int_{H_{\nu_E(x)}} \phi(y)dy \quad \forall \phi \in C_c(\mathbf{R}^n).$$

Now we examine the density properties of sets of finite perimeter.

Definition 4.1.6. *Let E be a measurable subset of $\mathbf{R}^n$. For every $\alpha \in [0,1]$ we denote by E^α the set of points of $\mathbf{R}^n$ where E has density α, that is*

$$E^\alpha = \left\{ x \in \mathbf{R}^n : \exists \lim_{\varrho \to 0} \frac{|E \cap B_\varrho(x)|}{|B_\varrho(x)|} = \alpha \right\}; \tag{4.8}$$

The essential boundary is then defined as $\partial^* E = \mathbf{R}^n \setminus (E^0 \cup E^1)$, i.e., the set of points where the density of E is neither 0 nor 1.

Theorem 4.1.7. *(Federer) Let E be a set of finite perimeter in Ω. Then*

$$\mathcal{F}E \cap \Omega \subset E^{1/2} \subset \partial^* E \quad \textit{and} \quad \mathcal{H}^{n-1}(\Omega \setminus (E^0 \cup \mathcal{F}E \cup E^1)) = 0$$

In particular, $\mathcal{H}^{n-1}$- a.e. $x \in \partial^* E \cap \Omega$ belongs to $\mathcal{F}E$.

4.2 Weighted BV functions

A natural way to extend the definition of functions of bounded variation in the weighted Euclidean case on Ω is described here. Given a symmetric positive definite matrix $P = (p_{ij})^n_{i,j=1}$, and a function $f \in L^1(\Omega)$, we define the weighted total variation, by setting

$$|Df|_P(\Omega) = \sup \left\{ \int_\Omega f \mathrm{div} \psi dx : \psi \in C^1_c(\Omega, \mathbf{R}^n), \|P^{-1/2}\psi\|_\infty \leq 1 \right\} \tag{4.9}$$

and say that f *has finite total weighted variation*, if $|Df|_P(\Omega) < +\infty$. Thus, as in the classical case we denote by BV_P as the space of L^1 functions that have finite weighted total variation. Notice that if P has entries $p_{ij} \in C^1(\Omega)$, then the total variation can be equivalently defined by

$$|Du|_P(\Omega) = \sup \left\{ \int_\Omega u \mathrm{div}(P^{1/2}\phi)dx : \phi \in C^1_c(\Omega, \mathbf{R}^n), \|\phi\|_{L^\infty(\Omega)} \le 1 \right\}.$$

Of course, if P is the identity matrix then $|Df|_P$ reduces to the classical definition of total variation for an L^1 function and in this case we write $f \in BV(\Omega)$ and drop the P everywhere. The space $BV_P(\Omega)$ turns out to be a Banach space with the norm

$$\|f\|_{BV_P} = \|f\|_{L^1(\Omega)} + |Df|_P(\Omega).$$

In a similar way, a set E is said to have finite weighted perimeter if $|D\chi_E|_P(\Omega) < +\infty$. In this case, its total variation measure is the perimeter of E and it is denoted also by $\mathcal{P}_P(E, \Omega) = |D\chi_E|_P(\Omega)$.

Henceforth, we assume that P is a symmetric μ elliptic matrix i.e., there exists $\mu \ge 1$ such that $\mu^{-1}|\xi|^2 \le \langle P(x)\xi, \xi\rangle \le \mu|\xi|^2$ for all $\xi \in \mathbf{R}^n$ and all $x \in \Omega$. We also assume that the coefficients $p_{ij} \in C_b(\overline{\Omega})$, then, the seminorms $|Df|(\Omega)$ and $|Df|_P(\Omega)$ are equivalent, more precisely

$$\frac{1}{\sqrt{\mu}}|Df|(\Omega) \le |Df|_P(\Omega) \le \sqrt{\mu}|Df|(\Omega),$$

where μ is the ellipticity constant of P and this immediately implies that $BV(\Omega) = BV_P(\Omega)$ with equivalence of the norms.

We also notice that if f is regular, then the equality

$$|Df|_P(\Omega) = \int_\Omega |Df(x)|_P dx$$

holds, where $|Df(x)|_P = |P^{1/2}Df(x)| = \langle PDf(x), Df(x)\rangle^{1/2}$.

Remark 4.2.1. *(Lower semicontinuity of the total variation)* It is useful to notice that $|D \cdot|_P(\Omega)$ is lower semicontinuous with respect to the convergence in $L^1_{\mathrm{loc}}(\Omega)$. Indeed for any $\varphi \in C^1_c(\Omega, \mathbf{R}^n)$ with $\|P^{-1/2}\varphi\|_\infty \le 1$ the integral $\displaystyle\int_\Omega f \mathrm{div}\varphi\, dx$ is continuous with respect to the L^1-norm of f, hence $|Df|_P$, as the supremum of continuous functionals, is lower semicontinuous.

As in the unweighted case, the norm topology is in some respects too strong, since for instance smooth functions are not dense with respect to it. Nevertheless, a classical weaker approximation result is given by the Anzellotti-Giaquinta theorem, see e.g. [5, Theorem 3.9]. It states that for every $f \in BV(\Omega)$ there exists a sequence of functions $(f_k)_k \subset C^\infty(\Omega) \cap BV(\Omega)$ such that

$$\|f - f_k\|_{L^1(\Omega)} \to 0, \quad \int_\Omega |Df_k|dx \to |Df|(\Omega);$$

Such a sequence is said to converge *in variation* to f.
The Anzellotti-Giaquinta theorem can be adapted also to the case of weighted BV functions as follows: given a matrix Q, we define

$$C_Q(\Omega) = \left\{f \in C^\infty(\Omega) \cap C^1(\overline{\Omega}); \langle QDf, \nu\rangle = 0 \text{ on } \partial\Omega\right\}, \tag{4.10}$$

and the following approximation result holds. We point out that we shall use this proposition in order to approximate a function in $BV(\Omega)$ with functions in the domain of A_1 which verify a condition on $\partial\Omega$.

Proposition 4.2.2. *Let Ω, $P = (p_{ij})_{i,j=1}^n$ be as above, and let $Q = (q_{ij})_{i,j=1}^n$ be an elliptic matrix with $q_{ij} \in C_b^1(\overline{\Omega})$. Then, for every $f \in BV_P(\Omega)$ there exists a sequence of functions $(f_k)_k \subset C_Q(\Omega)$ such that*

$$\lim_{k\to\infty} \|f - f_k\|_{L^1(\Omega)} = 0, \quad \lim_{k\to\infty} \int_\Omega |Df_k|_P dx = |Df|_P(\Omega).$$

Proof. The proof goes as the classical one, except that we have to modify the usual approximation sequence in a neighborhood of the boundary of Ω.
Fix $\varepsilon > 0$; since $f \in BV(\Omega)$, there exist functions $\{f_k\}_k \in C^\infty(\Omega) \cap BV(\Omega)$ such that

$$f_k \to f \text{ in } L^1(\Omega)$$

$$\int_\Omega |Df_k|\, dx \to |Df|(\Omega) \text{ as } k \to \infty.$$

We can find $\delta_0 > 0$ such that for every $\delta \in (0, \delta_0)$ the set $\Omega^\delta = \{x \in \Omega;\ \mathrm{dist}(x, \partial\Omega) > \delta\}$ satisfies

$$\|f_k\|_{L^1(\Omega\setminus\Omega^\delta)} \le \varepsilon, \qquad \int_{\Omega\setminus\Omega^\delta} |\nabla f_k| dx \le \varepsilon \qquad \forall k \in \mathbf{N}. \tag{4.11}$$

The assumption on the regularity on $\partial\Omega$ is used to modify the approximating sequence to make it constant in the direction $Q\nu$. Indeed, for every $x \in \Omega\setminus\Omega^\delta$ there is the projection on $\partial\Omega^\delta$, say $P_Q(x)$, such that x may be written $x = (1-t)P_Q(x) + \delta t Q(P_Q(x))\nu(P_Q(x))$ for some $t \in [0, 1)$ ($\nu(y)$ is the outer normal to $\partial\Omega^\delta$ in y). This is possible since the map $\psi : \partial\Omega^\delta \times [0, \varepsilon) \to \Omega$, $\psi(y, t) = y + tQ(y)\nu(y)$ defines, for sufficiently small $\varepsilon > 0$, a diffeomorphism on its image, and then we can define $P_Q(x) = \pi_1(\psi^{-1}(x))$ for any $x \in \psi(\partial\Omega^\delta \times [0, \varepsilon))$, where $\pi_1 : \partial\Omega^\delta \times [0, \varepsilon) \to \partial\Omega^\delta$ is given by $\pi_1(y, t) = y$.
Let us modify the functions f_k in the following way

$$\tilde{f}_k(x) := \begin{cases} f_k(P_Q(x)) & x \in \Omega\setminus\Omega^\delta \\ f_k(x) & x \in \Omega^\delta. \end{cases}$$

Then, choosing δ sufficiently small, we have that

$$\left|\int_\Omega |\nabla\tilde{f}_k| dx - \int_\Omega |\nabla f_k| dx\right| \le \varepsilon \tag{4.12}$$

Finally, for every $\tau < \frac{\delta}{2}$ we can define the approximants as follows

$$g_k^\tau(x) := \begin{cases} f_k(P_Q(x)) & x \in \Omega\setminus\Omega^{\frac{\delta}{2}} \\ (\rho_\tau * \tilde{f}_k)(x) & x \in \Omega^{\frac{\delta}{2}}\setminus\Omega^{\frac{3}{2}\delta} \\ f_k(x) & x \in \Omega^{\frac{3}{2}\delta}. \end{cases}$$

where ρ_τ is the standard mollifier. Then $g_k^\tau \in C_c^\infty(\overline{\Omega})$, $(\nabla g_k^\tau, Q\nu) = 0$ in $\partial\Omega \quad \forall k \in \mathbf{N}$. Finally with a standard procedure of diagonalization we can find a sequence $\{g_k^{\tau(k)}\} \subset \{g_k^\tau\}$ such that

$$\lim_{k\to\infty} \|g_k^{\tau(k)} - f\|_{L^1(\Omega)} \le 3\,\varepsilon\,, \qquad \Big|\int_\Omega |\nabla g_k^{\tau(k)}|dx - |Df|(\Omega)\Big| \le 3\,\varepsilon\,.$$

Now, let $\varphi \in C_c^1(\Omega, \mathbf{R}^n)$ with $\|P^{-1/2}\varphi\|_{L^\infty(\Omega)} \le 1$. Then taking into account (4.11) and (4.12) we have

$$\begin{aligned}\int_\Omega g_k^{\tau(k)} \mathrm{div}\varphi\; dx &= \int_{\Omega\setminus\Omega^{\frac{\delta}{2}}} f_k(P_Q(x))\mathrm{div}\varphi\; dx + \int_{\Omega^{\frac{\delta}{2}}\setminus\Omega^{\frac{3}{2}\delta}} (\rho_{\tau_k} * \tilde{f}_k)\mathrm{div}\varphi\; dx\\ &\quad + \int_{\Omega^{\frac{3}{2}\delta}} f_k \mathrm{div}\varphi\; dx\\ &\le 2\varepsilon\|\varphi\|_{W^{1,\infty}} + \int_{\Omega^{\frac{3}{2}\delta}} (f_k - f)\mathrm{div}\varphi\; dx + \int_\Omega f\mathrm{div}\varphi\; dx\\ &\quad - \int_{\Omega\setminus\Omega^{\frac{3}{2}\delta}} f\mathrm{div}\varphi\; dx\\ &\le 3\varepsilon\|\varphi\|_{W^{1,\infty}} + |Df|_P(\Omega)\end{aligned}$$

and so

$$\int_\Omega |Dg_k^{\tau(k)}|_P\; dx \le |Df|_P(\Omega) + 3\varepsilon\|\varphi\|_{W^{1,\infty}}.$$

This estimate and Remark 4.2.1 complete the proof. □

Remark 4.2.3. A particular case of Proposition 4.2.2 is given when $Q = A$; in this case we have that $C_A(\Omega) \subset D(A_1)$ (it is a core for A_1, i.e. it is dense in $D(A_1)$ for the graph norm $\|\cdot\|_{L^1(\Omega)} + \|A_1\cdot\|_{L^1(\Omega)}$), and then the weighted BV functions can be approximated in variation via functions in the domain of the operator A_1.

There are several other useful properties connecting BV functions to sets of finite perimeter such as the coarea formula. Next we state a weighted version of it, a particular case of (see [13, Lemma 2.4]). We relate the weighted variation measure of f and the weighted perimeter of its level sets.
For $f : \Omega \to \mathbf{R}$ and $t \in \mathbf{R}$, define

$$E_t = \{x \in \Omega : \;\; f(x) > t\}.$$

Lemma 4.2.4. *If $f \in BV(\Omega)$, the mapping*

$$t \in \mathbf{R} \mapsto \mathcal{P}_P(E_t, \Omega)$$

is $\mathcal{L}^1$-measurable.

Proof. Since $f \in L^1(\Omega)$, the mapping $(x,t) \mapsto \chi_{E_t}(x)$ is $\mathcal{L}^n \times \mathcal{L}^1$- measurable, and thus, for each $\varphi \in C_c^1(\Omega, \mathbf{R}^n)$, the function

$$t \mapsto \int_\Omega \chi_{E_t} \mathrm{div}\varphi\; dx$$

is $\mathcal{L}^1$-measurable. Let D denote any countable dense subset of $C_c^1(\Omega, \mathbf{R}^n)$. Then

$$t \mapsto \mathcal{P}(E_t, \Omega) = \sup\left\{ \int_{E_t} \mathrm{div}\varphi \; dx; \; \varphi \in D, |\varphi| \leq 1 \right\}$$

is $\mathcal{L}^1$-measurable since it is the supremum of a countable family of measurable functions.

□

Theorem 4.2.5. *Let $f \in BV(\Omega)$. Then E_t has finite perimeter for $\mathcal{L}^1$ a.e. $t \in \mathbf{R}$ and*

$$|Df|_P(\Omega) = \int_{\mathbf{R}} \mathcal{P}_P(E_t, \Omega) dt. \tag{4.13}$$

Conversely, if $f \in L^1(\Omega)$ and

$$\int_{\mathbf{R}} \mathcal{P}_P(E_t, \Omega) dt < \infty$$

then $f \in BV(\Omega)$.

Proof. Let $\varphi \in C_c^1(\Omega, \mathbf{R}^n)$, $\|P^{-1/2}\varphi\|_{L^\infty(\Omega)} \leq 1$. Then

$$\int_\Omega f \mathrm{div}\varphi \; dx = \int_{\mathbf{R}} \Big(\int_{E_t} \mathrm{div}\varphi \; dx \Big) \; dt. \tag{4.14}$$

Indeed, suppose first $f \geq 0$, so that

$$f(x) = \int_0^\infty \chi_{E_t}(x) \; dt \qquad \text{a.e. } x \in \Omega.$$

Thus

$$\begin{aligned}
\int_\Omega f \mathrm{div}\varphi \; dx &= \int_\Omega \Big(\int_0^\infty \chi_{E_t}(x) \, dt \Big) \mathrm{div}\varphi(x) \, dx \\
&= \int_0^\infty \Big(\int_\Omega \chi_{E_t}(x) \mathrm{div}\varphi(x) \, dx \Big) \, dt \\
&= \int_0^\infty \Big(\int_{E_t} \mathrm{div}\varphi \, dx \Big) \, dt.
\end{aligned}$$

Similarly if $f \leq 0$,

$$f(x) = \int_{-\infty}^0 (\chi_{E_t}(x) - 1) \, dt,$$

whence

$$\begin{aligned}
\int_\Omega f \mathrm{div}\varphi \; dx &= \int_\Omega \Big(\int_{-\infty}^0 (\chi_{E_t}(x) - 1) \, dt \Big) \mathrm{div}\varphi(x) \, dx \\
&= \int_{-\infty}^0 \Big(\int_\Omega (\chi_{E_t}(x) - 1) \mathrm{div}\varphi(x) \, dx \Big) \, dt \\
&= \int_{-\infty}^0 \Big(\int_{E_t} \mathrm{div}\varphi \, dx \Big) \, dt.
\end{aligned}$$

For the general case, write $f = f^+ - f^-$ and (4.14) is proved. From (4.14) we see that for all φ as above,

$$\int_\Omega f \operatorname{div}\varphi \, dx \leq \int_{\mathbf{R}} \mathcal{P}_P(E_t, \Omega) dt.$$

Hence

$$|Df|_P(\Omega) \leq \int_{\mathbf{R}} \mathcal{P}_P(E_t, \Omega) \, dt. \tag{4.15}$$

Now, we prove that assertion (4.13) holds for all $f \in BV_P(\Omega) \cap C^\infty(\Omega)$.
Let

$$m(t) := \int_{\Omega \setminus E_t} |Df|_P \, dx = \int_{\{f \leq t\}} |Df|_P \, dx.$$

Then the function m is non decreasing, and thus m' exists for a.e. $t \in \mathbf{R}$, with

$$\int_{\mathbf{R}} m'(t) \, dt \leq \int_\Omega |Df|_P \, dx. \tag{4.16}$$

Now, fix $t \in \mathbf{R}$, $r > 0$, and define $\eta : \mathbf{R} \to \mathbf{R}$ this way:

$$\eta(s) = \begin{cases} 0 & \text{if } s \leq t\,, \\ \frac{s-t}{r} & \text{if } t \leq s \leq t + r\,, \\ 1 & \text{if } s \geq t + r. \end{cases}$$

Then

$$\eta'(s) = \begin{cases} \frac{1}{r} & \text{if } t < s < t + r\,, \\ 0 & \text{if } s < t \text{ or } s > t + r. \end{cases}$$

Hence, for all $\varphi \in C^1_c(\Omega, \mathbf{R}^n)$ with $\|P^{-1/2}\varphi\|_{L^\infty(\Omega)} \leq 1$

$$-\int_\Omega \eta(f(x)) \operatorname{div}\varphi \, dx = \int_\Omega \eta'(f(x)) Df \cdot \varphi \, dx = \frac{1}{r} \int_{E_t \setminus E_{t+r}} Df \cdot \varphi \, dx.$$

Now,

$$\begin{aligned} \frac{m(t+r) - m(t)}{r} &= \frac{1}{r} \left[\int_{\Omega \setminus E_{t+r}} |Df|_P \, dx - \int_{\Omega \setminus E_t} |Df|_P \, dx \right] \\ &= \frac{1}{r} \int_{E_t \setminus E_{t+r}} |Df|_P \, dx \\ &\geq \frac{1}{r} \int_{E_t \setminus E_{t+r}} Df \cdot \varphi \, dx \\ &= -\int_\Omega \eta(f(x)) \operatorname{div}\varphi \, dx \end{aligned}$$

For those t such that $m'(t)$ exists, we then let $r \to 0$:

$$m'(t) \geq -\int_{E_t} \operatorname{div}\varphi \, dx \quad \text{a.e. } t \in \mathbf{R}.$$

Taking the supremum over all φ as above:

$$\mathcal{P}_P(E_t, \Omega) \leq m'(t),$$

and recalling (4.16) we find

$$\int_{\mathbf{R}} \mathcal{P}_P(E_t,\Omega) \le \int_\Omega |Df|_P\,dx = |Df|_P(\Omega).$$

This estimate and (4.15) complete the proof for $f \in BV(\Omega)\cap C^\infty(\Omega)$. Finally, fix $f \in BV_P(\Omega)$ and choose $\{f_k\}_{k\in\mathbf{N}}$ as in Proposition 4.2.2. Then

$$f_k \to f \qquad \text{in } L^1(\Omega) \text{ as } k\to\infty.$$

Define

$$E_t^k = \{x\in\Omega:\ f_k(x) > t\}.$$

Now,

$$\int_{\mathbf{R}} |\chi_{E_t^k}(x) - \chi_{E_t}(x)|\,dt = \int_{\min\{f(x),f_k(x)\}}^{\max\{f(x),f_k(x)\}} dt = |f_k(x)-f(x)|,$$

consequently

$$\int_\Omega |f_k(x)-f(x)|\,dx = \int_{\mathbf{R}} (\int_\Omega |\chi_{E_t^k}(x)-\chi_{E_t}(x)|\,dx)\,dt.$$

Since $f_k \to f$ in $L^1(\Omega)$, there exists a subsequence which, upon reindexing by k if necessary, satisfies

$$\chi_{E_t^k} \to \chi_{E_t} \quad \text{in } L^1(\Omega), \quad \text{a.e.}\, t\in\mathbf{R}.$$

Then by the lower semicontinuity of the the total variation,

$$\mathcal{P}_P(E_t,\Omega) \le \liminf_{k\to\infty} \mathcal{P}_P(E_t^k,\Omega).$$

Thus Fatou's Lemma implies

$$\begin{aligned}\int_{\mathbf{R}} \mathcal{P}_P(E_t,\Omega)\,dt &\le \liminf_{k\to\infty} \mathcal{P}_P(E_t^k,\Omega)\\ &= \lim_{k\to\infty} |Df_k|_P(\Omega)\\ &= |Df|_P(\Omega)\end{aligned}$$

This calculation and (4.15) complete the proof. □

Remark 4.2.6. The coarea formula is true for Borel sets. If $f\in BV(\Omega)$ the set E_t has finite perimeter for $\mathcal{L}^1$-a.e. $t\in\mathbf{R}$ and

$$|Df|(B) = \int_{\mathbf{R}} |D\chi_{E_t}|(B)\,dt, \qquad Df(B) = \int_{\mathbf{R}} D\chi_{E_t}(B)\,dt$$

for any Borel set $B\subset\Omega$.

For the weighted total variation also the following continuity property under uniform convergence holds.

Proposition 4.2.7. *Let $P = (p_{ij})_{i,j=1}^n$ be a symmetric μ-elliptic matrix valued function and let $(P_{(k)})_{k\in\mathbf{N}}$ be a sequence of matrices valued functions uniformly convergent to P. Then, for every $f \in L^1(\Omega)$ the following holds:*

$$\lim_{k\to+\infty} |Df|_{P_{(k)}}(\Omega) = |Df|_P(\Omega). \tag{4.17}$$

Proof. We denote by $c_k = \|P^{-1/2} - P_{(k)}^{-1/2}\|_\infty$; by the uniform convergence, we have that $c_k \to 0$ as $k \to +\infty$; moreover, we may assume that the $P_{(k)}$ are $(\mu + 1/k)$-elliptic, that is

$$\frac{1}{\mu + 1/k}|\xi|^2 \le |P_{(k)}^{1/2}\xi|^2 \le (\mu + 1/k)|\xi|^2,$$

or, simply defining $w = P_{(k)}^{1/2}\xi$,

$$\frac{1}{\sqrt{\mu + 1/k}}|w| \le |P_{(k)}^{-1/2}w| \le \sqrt{\mu + 1/k}|w|.$$

Then, if $\psi \in C_c^1(\Omega, \mathbf{R}^n)$ with $\|P_{(k)}^{-1/2}\psi\|_\infty \le 1$, we get

$$\begin{aligned}
\|P^{-1/2}\psi\|_\infty &\le \|P_{(k)}^{-1/2}\psi\|_\infty + \|(P^{-1/2} - P_{(k)}^{-1/2})\psi\|_\infty \\
&\le \|P_{(k)}^{-1/2}\psi\|_\infty + c_k\|\psi\|_\infty \\
&\le \|P_{(k)}^{-1/2}\psi\|_\infty + c_k\sqrt{\mu + 1/k}\|P_{(k)}^{-1/2}\psi\|_\infty \\
&\le 1 + c_k\sqrt{\mu + 1/k}.
\end{aligned}$$

By definition of weighted variation, we get

$$\int_\Omega f \operatorname{div}\psi dx \le (1 + c_k\sqrt{\mu + 1/k})|Df|_P(\Omega)$$

whence

$$|Df|_{P_{(k)}}(\Omega) \le (1 + c_k\sqrt{\mu + 1/k})|Df|_P(\Omega).$$

With a similar computation, we also get

$$|Df|_P(\Omega) \le (1 + c_k\sqrt{\mu})|Df|_{P_{(k)}}(\Omega),$$

and then (4.17) follows by letting $k \to +\infty$. $\square$

4.3 A first characterization of BV functions

In this last section we show some connections between the total variation of a generic function $u_0 \in L^1$ and the short time behavior of the solution of a parabolic problem with initial datum u_0. More precisely we connect the total variation of u_0 to the L^1 norm of the gradient of such solution. This result is strictly linked with the original definition given by E. De Giorgi [15] of functions of bounded variation which is recalled in the following paragraph.

Consider the heat semigroup $(W(t))_{t\geq 0}$ in $\mathbf{R}^n$. We show how it is linked to the definition of function with bounded variation originally given by De Giorgi (see [15]). For a given function $f \in L^1(\mathbf{R}^n)$, we consider the function

$$\begin{aligned} W(t)f(x) &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbf{R}^n} f(x+\sqrt{2t}y)e^{-\frac{|y|^2}{2}}\, dy \\ &= \frac{1}{(4\pi t)^{n/2}} \int_{\mathbf{R}^n} f(y)e^{-\frac{|x-y|^2}{4t}} dy \\ &= (G_t * f)(x) \end{aligned}$$

where $G_t(x) = (4\pi t)^{-n/2}e^{-|x|^2/4t}$ denotes the Gauss-Weierstrass kernel. By using simple tools of analysis one can easily prove that $W(t)f(x) \to f(x)$ almost everywhere and also in $L^1(\mathbf{R}^n)$ as $t \to 0^+$. The operator $W(t)$ is also contractive, thus $\|W(t)f\|_{L^1(\mathbf{R}^n)} \leq \|f\|_{L^1(\mathbf{R}^n)}$ for any $f \in L^1(\mathbf{R}^n)$ and any $t > 0$. Moreover, if the function g is regular, then $DW(t)g(x) = W(t)Dg(x)$. Finally, since $W(t+s)f(x) = W(s)W(t)f(x)$, using the previous property for $g(x) = W(t)f(x)$, we get

$$\int_{\mathbf{R}^n} |DW(t+s)f(x)|\, dx = \int_{\mathbf{R}^n} |W(s)(DW(t)f)(x)|\, dx \leq \int_{\mathbf{R}^n} |DW(t)f(x)|\, dx.$$

This computation shows that the function

$$t \mapsto \int_{\mathbf{R}^n} |DW(t)f(x)|\, dx$$

is monotone decreasing for every $f \in L^1(\mathbf{R}^n)$ and then it is well defined the quantity:

$$\mathcal{I}[f] = \lim_{t\to 0} \int_{\mathbf{R}^n} |DW(t)f(x)|\, dx, \tag{4.18}$$

that a priori can be finite or not. De Giorgi called $\mathcal{I}[f]$ the total variation of f in $\mathbf{R}^n$ and he defined the space $BV(\mathbf{R}^n)$ as the space of functions such that $\mathcal{I}[f] < \infty$.

In Theorem 4.3.4 we prove that (4.18) still holds in Ω, when the left hand side reduces to (4.3) and $T(t)$ is the semigroup generated by the second order uniformly elliptic operator $(A_1, D(A_1))$. More in detail we prove that

$$|Du_0|_P(\Omega) = \lim_{t\to 0} \int_\Omega |DT(t)u_0|_P\, dx, \tag{4.19}$$

for every $u_0 \in L^1(\Omega)$, where $|D\cdot|_P(\Omega)$ is defined in (4.9).

Remark 4.3.1. Notice that, since $(T(t))_{t\geq 0}$ is a strongly continuous semigroup on $L^1(\Omega)$, then by the lower semicontinuity of the total variation with respect to the L^1 convergence we obtain

$$|Du_0|_P(\Omega) \leq \liminf_{t\to 0} \int_\Omega |DT(t)u_0|_P dx \tag{4.20}$$

for every u_0 in $L^1(\Omega)$. Therefore in order to prove (4.19) it is sufficient to prove

$$\limsup_{t\to 0}\int_\Omega |DT(t)u_0|_P\,dx \le |Du_0|_P(\Omega).$$

Now observe that, for functions in the domain of the operator A_1, (4.19) is true. Actually for these functions the result is stronger than (4.19), indeed the following equality holds

$$\lim_{t\to 0}\|DT(t)u_0 - Du_0\|_{L^1(\Omega)} = 0.$$

This can be easily seen if we take into account that, by Remark 3.0.6, $D(A_1)$ is continuously embedded in $W^{1,1}(\Omega)$, i.e., there exists $k = k(\Omega, \mu, M_1) > 0$ such that $u_0 \in D(A_1)$ implies $u_0 \in W^{1,1}(\Omega)$ and

$$\|u_0\|_{W^{1,1}(\Omega)} \le k(\|u_0\|_{L^1(\Omega)} + \|A_1 u_0\|_{L^1(\Omega)}); \tag{4.21}$$

Furthermore $T(t)A_1u_0 = A_1T(t)u_0$ and by the strong continuity of $T(t)$ in $L^1(\Omega)$ we get

$$\begin{aligned}\|DT(t)u_0 - Du_0\|_{L^1(\Omega)} &\le k\left(\|T(t)u_0 - u_0\|_{L^1(\Omega)} + \|A_1T(t)u_0 - A_1u_0\|_{L^1(\Omega)}\right)\\ &= k\left(\|T(t)u_0 - u_0\|_{L^1(\Omega)} + \|T(t)A_1u_0 - A_1u_0\|_{L^1(\Omega)}\right).\end{aligned}$$

Example 1. Another simple case in which the existence of the limit as $t \to 0$ of $\displaystyle\int_\Omega |DT(t)u_0(x)|\,dx$ is guaranteed is when Ω is convex and $A = P = I$, $B = c = 0$, i.e., $(T(t))_{t\ge 0}$ is the heat semigroup generated by the Neumann Laplacian and the total variation is the classical (non-weighted) one. In this case, it is easily seen that $F(t) = \|DT(t)u_0\|_{L^1(\Omega)}$ is decreasing (as is the case if $\Omega = \mathbf{R}^n$), *provided that* Ω *is convex.* In fact, in this case computations significantly simplify and go as follows, where we set $u(t,x) = (T(t)u_0)(x)$ and $F(t) = \displaystyle\int_\Omega |Du|\,dx$,

$$\begin{aligned}F'(t) &= \int_\Omega \partial_t|Du|\;dx = \int_\Omega \frac{1}{|Du|}\langle Du, D\partial_t u\rangle\;dx = \int_\Omega \frac{1}{|Du|}\sum_{i,k} D_iuD_iD^2_{kk}u\;dx\\ &= \int_{\partial\Omega}\frac{1}{|Du|}\sum_{i,k} D_iuD^2_{ik}u\nu_k\,d\mathcal{H}^{n-1} - \int_\Omega\sum_{i,k} D_k\frac{D_iu}{|Du|}D^2_{ik}u\;dx\\ &= -\int_{\partial\Omega}\frac{1}{|Du|}\langle D\nu Du, Du\rangle\,d\mathcal{H}^{n-1} + \int_\Omega\frac{1}{|Du|}\left[\left|D^2u\frac{Du}{|Du|}\right|^2 - \mathrm{Tr}\;(D^2u)^2\right]\;dx \le 0\end{aligned}$$

where we have taken into account the Neumann boundary conditions and the fact that if Ω is convex then all the curvatures (i.e., the eigenvalues of the matrix $D\nu$) are nonnegative. This estimate and (4.20) allow us to conclude.
The monotonicity is not true in general also when $\mathcal{A} = \Delta$; if Ω is not convex F may not be non-increasing. In [22, Theorem 2.16] there is an example with Ω non convex and $F'(0) > 0$.

Before stating the main result, we recall an useful boundary trace theorem whose proof can be found in [1, Theorem 5.3.6].

Theorem 4.3.2. *Let Ω be an open subset of $\mathbf{R}^n$ with uniformly C^2 boundary; then the trace operator is continuous from $W^{1,1}(\Omega)$ onto $L^1(\partial\Omega, \mathcal{H}^{n-1})$, that is, there exists $c_\Omega > 0$ such that for every $u \in W^{1,1}(\Omega)$ the trace $v = u_{|\partial\Omega}$ of u on $\partial\Omega$ is well defined and*

$$\|v\|_{L^1(\partial\Omega,\mathcal{H}^{n-1})} \leq c_\Omega \|u\|_{W^{1,1}(\Omega)}. \tag{4.22}$$

The following result is a monotonicity estimate for $F(t) = \displaystyle\int_\Omega |DT(t)u_0|dx$ and gives a localized version of (4.19). Here we assume stronger regularity conditions on the coefficient c and recall that

$$M_2 = \max\{\|A\|_{2,\infty}, \|B\|_{2,\infty}, \|c\|_{1,\infty}\}.$$

Without loss of generality, in what follows we take for simplicity the same ellipticity constant μ both for the matrix of the coefficients A of $\mathcal{A}$ and P.

Proposition 4.3.3. *Let $v \in D(A_1)$, where $\mathcal{A}$ is as in* (2.106)-(2.108)*, with coefficients $c \in W^{1,\infty}(\Omega)$. Let $P = (p_{ij})_{i,j=1}^n$ be a non-negative μ-elliptic matrix with $p_{ij} \in W^{1,\infty}(\Omega)$ and $p_{ij} = a_{ij}$ on $\partial\Omega$. Then for every $\eta \in C_b^1(\overline{\Omega})$, η non-negative, there exists a constant*

$$c_5 = c_5(n, \Omega, M_2, \|P\|_{1,\infty}, \|\eta\|_{W^{1,\infty}}, \mu)$$

such that

$$\int_\Omega \eta |DT(t)v|_P \, dx \leq \int_\Omega \eta |Dv|_P \, dx + c_5 t^{1-\delta} \|v\|_{W^{1,1}(\Omega)} \tag{4.23}$$

holds for every $t \in (0,1)$, where $\delta \in (1/2, 1)$ is the parameter in (3.20).

Proof. For $v \in D(A_1)$ and $\eta \in C_b^1(\overline{\Omega})$, $\eta \geq 0$, we define the function $F_\eta : (0,1) \to \mathbf{R}$ by

$$F_\eta(t) = \int_\Omega \eta |DT(t)v|_P \, dx.$$

This function is differentiable since $T(t)v$ is regular for every $t > 0$ and the equality

$$\partial_t |DT(t)v|_P = \frac{1}{|DT(t)v|_P} \langle PDT(t)v, D\mathcal{A}T(t)v\rangle$$

holds for a.e. $x \in \Omega$. Moreover, $T(t)v \in D(A_1)$ for every $t > 0$ and then

$$A_1 T(t)v = T\Big(\frac{t}{2}\Big) A_1 T\Big(\frac{t}{2}\Big) v;$$

this implies also that $A_1 T(t)v \in D(A_1)$. Then, thanks to (4.21) and from the fact that

$$\frac{|\langle PDT(t)v, DA_1T(t)v\rangle|}{|DT(t)v|_P} \leq |DA_1T(t)v|_P,$$

we can differentiate under the integral sign. Denoting by $u(t,x)$ the solution $(T(t)v)(x)$, we obtain

$$
\begin{array}{lcl}
F'_\eta(t) & = & \displaystyle \frac{d}{dt}\int_\Omega \eta |Du|_P\,dx = \int_\Omega \frac{\eta}{|Du|_P}\langle PDu, D\mathcal{A}u\rangle\,dx \\
& = & \displaystyle \sum_{i,j,h,k=1}^n \int_\Omega \eta \frac{p_{ij}D_juD_i(D_h(a_{hk}D_ku))}{|Du|_P}\,dx \\
& & \displaystyle + \sum_{i,j,h=1}^n \int_\Omega \eta \frac{p_{ij}D_juD_i(b_hD_hu)}{|Du|_P}\,dx + \sum_{i,j=1}^n \int_\Omega \eta \frac{p_{ij}D_juD_i(cu)}{|Du|_P}\,dx \\
(I_1) & = & \displaystyle \sum_{i,j,h,k=1}^n \int_\Omega \eta \frac{p_{ij}D_ju(D^2_{ih}a_{hk}D_ku + D_ha_{hk}D^2_{ik}u + D_ia_{hk}D^2_{hk}u)}{|Du|_P}\,dx \\
(I_2) & & \displaystyle + \sum_{i,j,h,k=1}^n \int_\Omega \eta \frac{1}{|Du|_P} p_{ij}D_ju\, a_{hk}D^3_{ihk}u\ dx \\
(I_3) & & \displaystyle + \sum_{i,j,h,k=1}^n \int_\Omega \eta \frac{1}{|Du|_P} p_{ij}D_ju\Big(D_ib_hD_hu + b_hD^2_{ih}u\Big)\,dx \\
(I_4) & & \displaystyle + \sum_{i,j,h,k=1}^n \int_\Omega \eta \frac{1}{|Du|_P} p_{ij}D_ju\Big(D_ic\,u + cD_iu\Big)\,dx\,.
\end{array}
$$

Notice that there is a constant $k = k(n, M_2, \|\eta\|_{L^\infty}, \|P\|_\infty)$ such that

$$
|I_1| + |I_3| + |I_4| \le k\|u\|_{W^{2,1}(\Omega)}\,.
$$

It remains to estimate I_2; integrating by parts with respect to x_k, we have that

$$
\begin{array}{lcl}
& & \displaystyle \sum_{i,j,h,k=1}^n \int_\Omega \frac{\eta}{|Du|_P} p_{ij}D_ju\, a_{hk}D^3_{ihk}u\,dx \\
(II_1) & = & \displaystyle \frac{1}{2}\sum_{i,j,h,k,l,m=1}^n \int_\Omega \frac{\eta}{|Du|^3_P} p_{ij}D_ju\, a_{hk}D^2_{ih}uD_kp_{lm}D_muD_lu\,dx \\
(II_2) & & \displaystyle + \sum_{i,j,h,k,l,m=1}^n \int_\Omega \frac{\eta}{|Du|^3_P} p_{ij}D_ju\, a_{hk}D^2_{ih}u\, p_{lm}D_muD^2_{kl}u\,dx \\
(II_3) & & \displaystyle - \sum_{i,j,h,k=1}^n \int_\Omega \frac{\eta}{|Du|_P}\Big(D_kp_{ij}D_ju\, a_{hk} + p_{ij}D_juD_ka_{hk}\Big)D^2_{ih}u\,dx \\
(II_4) & & \displaystyle - \sum_{i,j,h,k=1}^n \int_\Omega \frac{\eta}{|Du|_P} p_{ij}D^2_{kj}u\, a_{hk}D^2_{ih}u\,dx \\
(II_5) & & \displaystyle - \sum_{i,j,h,k=1}^n \int_\Omega \frac{1}{|Du|_P} p_{ij}D_ju\, a_{hk}D^2_{ih}u\, D_k\eta\,dx \\
(II_6) & & \displaystyle + \sum_{i,j,h,k=1}^n \int_{\partial\Omega} \frac{\eta}{|Du|_P} p_{ij}D_ju\, a_{hk}D^2_{ih}u\,\nu_k\,d\mathcal{H}^{n-1}.
\end{array}
$$

This implies the existence of a constant $k = k(M_1, \|P\|_{1,\infty}, \|\eta\|_{1,\infty})$, such that

$$|II_1| + |II_3| + |II_5| \le k \int_\Omega |D^2 u|\, dx\,.$$

where M_1 was so defined

$$M_1 = \max_{i,j}\{\|a_{ij}\|_{W^{2,\infty}(\Omega)}, \|b_i\|_{W^{2,\infty}(\Omega)}, \|c\|_{L^\infty(\Omega)}\}.$$

Notice that for II_2 we have

$$\begin{aligned}\sum_{i,j,k,l,m=1}^n p_{ij} D_j u a_{hk} D^2_{ih} u\, p_{lm} D_m u D^2_{kl} u &= \langle D^2 u\, A\, D^2 u\, P Du, PDu\rangle\\ &= \Big\langle P^{1/2} D^2 u\, A\, D^2 u\, P^{1/2}(P^{1/2} Du), P^{1/2} Du\Big\rangle,\end{aligned}$$

and for II_4 we can write

$$\begin{aligned}\sum_{i,j,h,k=1}^n p_{ij} D^2_{kj} u\, a_{hk} D^2_{ih} u &= \sum_{i,j,h,k,m=1}^n p^{\frac12}_{im} p^{\frac12}_{mj} D^2_{kj} u\, a_{hk} D^2_{ih} u\\ &= \mathrm{Tr}\,\Big(P^{1/2} D^2 u\, A\, D^2 u\, P^{1/2}\Big),\end{aligned}$$

where Tr denotes the trace of a matrix. Then

$$\begin{aligned}II_2 + II_4 &= \int_\Omega \frac{1}{|Du|_P}\Big(\Big\langle P^{1/2} D^2 u\, A\, D^2 u\, P^{1/2} \frac{P^{1/2} Du}{|Du|_P}, \frac{P^{1/2} Du}{|Du|_P}\Big\rangle\\ &\quad - \mathrm{Tr}\,\Big(P^{1/2} D^2 u\, A\, D^2 u\, P^{1/2}\Big)\Big)\,\eta\, dx \le 0\end{aligned} \tag{4.24}$$

since $P^{1/2} D^2 u\, A\, D^2 u\, P^{1/2}$ is positive definite because

$$\Big\langle (P^{1/2} D^2 u\, A\, D^2 u\, P^{1/2})\xi, \xi\Big\rangle = \Big\langle A^{1/2} D^2 u\, P^{1/2}\xi, A^{1/2} D^2 u\, P^{1/2}\xi\Big\rangle.$$

Finally, for the term II_6, we notice that

$$\begin{aligned}\sum_{i,j,h,k=1}^n p_{ij} D_j u\, a_{hk} D^2_{ih} u\, \nu_k &= \sum_{i=1}^n \Big(\sum_{h,k=1}^n a_{hk} D^2_{ih} u\, \nu_k \sum_{j=1}^n p_{ij} D_j u\Big)\\ &= \sum_{i=1}^n \sum_{h,k=1}^n \Big(D_i\Big(a_{hk} D_h u\, \nu_k\Big) - D_h u D_i(a_{hk}\nu_k)\Big) \sum_{j=1}^n p_{ij} D_j u\\ &= \langle D\langle ADu, \nu\rangle, PDu\rangle - \langle D(A\nu)Du, PDu\rangle = -\langle D(A\nu)Du, PDu\rangle\end{aligned} \tag{4.25}$$

since $P \equiv A$ on $\partial\Omega$. Observe that the regularity of the boundary and the ellipticity of a_{ij} imply that there exists a constant $\tilde c$ depending on $\|A\|_{1,\infty}$ and L (see Definition 1.5.1) such that $|D(A\nu)| \le \tilde c$. As a consequence, we obtain that

$$\begin{aligned}&\Big|\sum_{i,j,h,k=1}^n \int_{\partial\Omega} \frac{1}{|Du|_P}\eta\, p_{ij} D_j u\, a_{hk} D^2_{ih} u\, \nu_k\, d\mathcal{H}^{n-1}\Big|\\ =\;& \Big|\int_{\partial\Omega} \frac{1}{|Du|_P}\eta\, \langle D(A\nu)Du, PDu\rangle\, d\mathcal{H}^{n-1}\Big| \le k \int_{\partial\Omega} \eta |Du|_P d\mathcal{H}^{n-1}\\ \le\;& k\|\eta\|_\infty \sqrt{\mu} \int_{\partial\Omega} |Du| d\mathcal{H}^{n-1} \le k \int_\Omega \big[|Du| + |D^2 u|\big]\, dx\,,\end{aligned}$$

where $k = k(M_2, L, \mu, \|\eta\|_{L^\infty}, c_\Omega)$, and c_Ω is introduced in (4.22).

Taking now into account that $u(t,x)$ satisfies (3.4) and (3.20), we have proved there is a constant c_5 such that for every $t \in (0,1)$ the inequality

$$F'_\eta(t) = \frac{d}{dt}\int_\Omega \eta |Du|_P \, dx \le c_5 t^{-\delta} \|v\|_{W^{1,1}(\Omega)}.$$

holds. Then, by integration (4.23) follows. □

In the following theorem we show the announced characterization of the space $BV(\Omega)$ in terms of the short-time behavior of $\|DT(t)u_0\|_{L^1(\Omega)}$, analogous to (4.18). Here we may relax the regularity assumption on the coefficients b_i according to Remark 3.0.6.

Theorem 4.3.4. *Assume $\Omega \subset \mathbf{R}^n$ has uniformly C^2 boundary. Let $\mathcal{A}$ be as in Section 2.5 with*

$$a_{ij} \in W^{2,\infty}(\Omega), \qquad b_i, c \in L^\infty(\Omega)$$

and P be a non negative μ-elliptic matrix with $p_{ij} \in C_b(\overline{\Omega})$. If $(T(t))_{t\ge 0}$ is the semigroup generated by $(A_1, D(A_1))$ in $L^1(\Omega)$, then, for every $u_0 \in L^1(\Omega)$, the equality

$$\lim_{t\to 0}\int_\Omega |DT(t)u_0(x)|_P \, dx = |Du_0|_P(\Omega)$$

holds. In particular, u_0 belongs to $BV(\Omega)$ if and only if the above limit is finite.

Proof. We start first assuming that $p_{ij} \in C_b^2(\overline{\Omega})$ and considering the operator $\hat{\mathcal{A}} = \operatorname{div}(ADu)$, i.e., $b_i = c = 0$, $i = 1, \dots n$. We denote by $(\hat{A}_1, D(\hat{A}_1))$ its realization in L^1 (as specified in Section 2.5) and by $\hat{T}$ the generated semigroup. Thanks to (4.20), we have only to prove that

$$\limsup_{t\to 0}\int_\Omega |D\hat{T}(t)u_0(x)|_P \, dx \le |Du_0|_P(\Omega), \tag{4.26}$$

which is trivially satisfied if $u_0 \in L^1(\Omega) \setminus BV(\Omega)$. We then consider $u_0 \in BV(\Omega)$. Fix $\varepsilon > 0$ and consider two open neighborhoods $U \subset V$ of $\partial\Omega$ with disjoint boundaries such that, if we take $S' = \Omega \cap U$ and $S = \Omega \cap \overline{V}$, we get

$$|Du_0|_P(S) < \varepsilon. \tag{4.27}$$

Let then $\eta \in C^2(\Omega)$ be a function such that

$$0 \le \eta \le 1, \quad \eta \equiv 1 \text{ on } S', \quad \eta \equiv 0 \text{ on } \Omega \setminus S$$

and define the matrix

$$P_A = \eta^2 A + (1 - \eta^2)P.$$

By Proposition 4.2.2 there exists a sequence

$$\begin{aligned}(u_k)_k &\subset \left\{v \in C^\infty(\Omega) \cap C^1(\overline{\Omega}) : \langle ADv, \nu\rangle = 0 \text{ on } \partial\Omega\right\} \\ &= \left\{v \in C^\infty(\Omega) \cap C^1(\overline{\Omega}) : \langle P_A Dv, \nu\rangle = 0 \text{ on } \partial\Omega\right\} \subset D(A_1)\end{aligned}$$

such that $u_k \to u_0$ in $L^1(\Omega)$ and

$$\lim_{k\to+\infty} \int_\Omega |Du_k|_P \, dx = |Du_0|_P(\Omega).$$

Notice that since P is μ-elliptic we get

$$\int_\Omega |Du_k| dx \le \sqrt{\mu} \int_\Omega |Du_k|_P dx$$

and then there exists $M > 0$ such that

$$\|u_k\|_{W^{1,1}(\Omega)} \le M. \tag{4.28}$$

Since $\Omega \setminus S$ is an open set, by lower semicontinuity we have

$$|Du_0|_P(\Omega \setminus S) \le \liminf_{k\to+\infty} \int_{\Omega\setminus S} |Du_k|_P dx$$

and also

$$\int_S |Du_k|_P dx = \int_\Omega |Du_k|_P dx - \int_{\Omega\setminus S} |Du_k|_P dx$$

whence

$$\begin{aligned}\limsup_{k\to+\infty} \int_S |Du_k|_P dx &\le \lim_{k\to+\infty} \int_\Omega |Du_k|_P dx - \liminf_{k\to+\infty} \int_{\Omega\setminus S} |Du_k|_P dx \\ &\le |Du_0|_P(\Omega) - |Du_0|_P(\Omega \setminus S) = |Du_0|_P(S).\end{aligned}$$

This proves that

$$\limsup_{k\to+\infty} \int_S |Du_k|_P dx \le |Du_0|_P(S); \tag{4.29}$$

by the μ-ellipticity of A and P, we get that $|\xi|_A \le \sqrt{\mu}|\xi|_P$ therefore the following holds:

$$\limsup_{k\to+\infty} \int_S |Du_k|_A dx = \limsup_{k\to+\infty} \int_S \langle ADu_k, Du_k\rangle^{1/2} dx \le \mu \limsup_{k\to+\infty} \int_S |Du_k|_P dx,$$

whence by (4.29) and (4.27)

$$\limsup_{k\to+\infty} \int_S |Du_k|_A dx \le \mu\varepsilon. \tag{4.30}$$

We also notice that

$$\begin{aligned}|\xi|_P^2 &= \langle P\xi, \xi\rangle = \langle P_A\xi, \xi\rangle + \langle (P - P_A)\xi, \xi\rangle \\ &= \langle P_A\xi, \xi\rangle + \eta^2 \langle (P - A)\xi, \xi\rangle = |\xi|_{P_A}^2 + \eta^2 \langle (P - A)\xi, \xi\rangle\end{aligned}$$

and, since P and A are μ-elliptic,

$$|\langle (P - A)\xi, \xi\rangle| \le 2\mu|\xi|^2 \le 2\mu^2|\xi|_A^2, \quad \forall \xi \in \mathbf{R}^n.$$

We have then obtained that $|\xi|_P \le |\xi|_{P_A} + \mu\sqrt{2}\eta|\xi|_A$ and as a consequence

$$\int_\Omega |D\hat{T}(t)u_k|_P dx \le \int_\Omega |D\hat{T}(t)u_k|_{P_A} dx + \mu\sqrt{2}\int_\Omega \eta|D\hat{T}(t)u_k|_A dx.$$

We can apply Proposition 4.3.3 to both terms in the right hand side in order to obtain, using (4.28), that

$$\int_\Omega |D\hat{T}(t)u_k|_P dx \le \int_\Omega |Du_k|_{P_A} dx + \mu\sqrt{2}\int_\Omega \eta|Du_k|_A dx + (1+\mu\sqrt{2})c_5 M t^{1-\delta}.$$

By definition of P_A, we have that

$$|\xi|^2_{P_A} = \eta^2|\xi|^2_A + (1-\eta^2)|\xi|^2_P, \quad \forall \xi \in \mathbf{R}^n,$$

and then

$$\int_\Omega |Du_k|_{P_A} dx \le \int_\Omega \eta|Du_k|_A dx + \int_\Omega \sqrt{1-\eta^2}|Du_k|_P dx \le \int_S |Du_k|_A dx + \int_\Omega |Du_k|_P dx.$$

We have then obtained the following estimate

$$\int_\Omega |D\hat{T}(t)u_k|_P dx \le \int_\Omega |Du_k|_P dx + (1+\mu\sqrt{2})\int_S |Du_k|_A dx + (1+\mu\sqrt{2})c_5 M t^{1-\delta}. \quad (4.31)$$

Using (4.30), (4.31) and the fact that $\hat{T}(t)u_k \to \hat{T}(t)u_0$ in $L^1(\Omega)$ as $n \to +\infty$, we get

$$\begin{aligned}\int_\Omega |D\hat{T}(t)u_0|_P dx &\le \liminf_{k\to+\infty}\int_\Omega |D\hat{T}(t)u_k|_P dx \le \limsup_{k\to+\infty}\int_\Omega |D\hat{T}(t)u_k|_P dx \\ &\le |Du_0|_P(\Omega) + \mu(1+\mu\sqrt{2})\varepsilon + (1+\mu\sqrt{2})c_5 M t^{1-\delta}\end{aligned}$$

and the result for P regular then follows by letting $t \to 0$, since ε is arbitrary. The case with $p_{ij} \in C_b(\overline{\Omega})$ is a consequence of the approximation result given in Proposition 4.2.7.

Finally, we consider non zero coefficients b_i and c and $\mathcal{A}u = \mathrm{div}(ADu) + \langle B, Du\rangle + cu$ with $b_i, c \in L^\infty(\Omega)$, $i = 1, \dots n$. Notice that the boundary operators associated with A_1 and $\hat{A}_1$ as in (2.110) coincide, and then the set $C_A(\Omega)$ defined in (4.10) is a core both for $(A_1, D(A_1))$ and $(\hat{A}_1, D(\hat{A}_1))$. We denote by $(T(t))_{t\ge 0}$ the semigroup generated by $(A_1, D(A_1))$. Notice that if we define $\hat{u}(t) := \hat{T}(t)u_0$ and $u = T(t)u_0$, with $u_0 \in C_A(\Omega)$, the function $w := \hat{u} - u$ is the solution of the problem

$$\begin{cases} \partial_t w - \mathcal{A}w = \mathcal{E}\hat{u} := -\langle B, D\hat{u}\rangle - c\hat{u} & \text{in } (0,\infty)\times\Omega \\ w(0) = 0 & \text{in } \Omega \\ \langle ADw, \nu\rangle = 0 & \text{in } (0,\infty)\times\partial\Omega\,. \end{cases}$$

Thus, since $w(t) = \int_0^t T(t-s)\mathcal{E}\hat{u}(s)ds$, we get

$$Dw(t) = D(\hat{u}-u)(t) = \int_0^t DT(t-s)\mathcal{E}\hat{u}(s)ds$$

and then using (3.4)

$$
\begin{aligned}
\|D\hat{T}(t)u_0 - DT(t)u_0\|_{L^1(\Omega)} &\le c_2\|\mathcal{E}\hat{T}(t)u_0\|_{L^1(\Omega)}\int_0^t \frac{1}{\sqrt{t-s}}ds \\
&\le 2c_2\sqrt{t}\left(\|B\|_\infty\|D\hat{T}(t)u_0\|_{L^1(\Omega)} + \|c\|_\infty\|\hat{T}(t)u_0\|_{L^1(\Omega)}\right)
\end{aligned}
\tag{4.32}
$$

Since $\|\hat{T}(t)u_0\|_{L^1(\Omega)} \to \|u_0\|_{L^1(\Omega)}$ and $\limsup_{t\to 0}\|D\hat{T}(t)u_0\|_{L^1(\Omega)}$ is bounded we can conclude that $\lim_{t\to 0}\|D\hat{T}(t)u_0 - DT(t)u_0\|_{L^1(\Omega)} = 0$ and consequently, for $v \in C_A(\Omega)$, it follows

$$
\begin{aligned}
\limsup_{t\to 0}\int_\Omega |DT(t)v|_P\,dx &\le \limsup_{t\to 0}\int_\Omega |D\hat{T}(t)v|_P\,dx \\
&+ \lim_{t\to 0}\int_\Omega |D\hat{T}(t)v - DT(t)v|_P\,dx = \int_\Omega |Dv|_P\,dx.
\end{aligned}
$$

The thesis then follows from the density of $C_A(\Omega)$ in $BV_P(\Omega)$ (see Proposition 4.2.2); given $u_0 \in BV_P(\Omega)$, we take a sequence $(u_k) \subset C_A(\Omega)$ approximating u_0 in P-variation. Then, using (4.32) with u_k in place of u_0 and (4.31), we get

$$
\begin{aligned}
&\int_\Omega |DT(t)u_k|_P dx \le \int_\Omega |D\hat{T}(t)u_k|_P dx + \int_\Omega |DT(t)u_k - D\hat{T}(t)u_k|_P dx \\
\le&(1 + 2c_2\mu\sqrt{t}\|B\|_\infty)\int_\Omega |Du_k|_P dx \\
&+ (1+\mu\sqrt{2})(1 + 2c_2\mu\sqrt{t}\|B\|_\infty)\int_S |Du_k|_A dx \\
&+ (1+\mu\sqrt{2})(1 + 2c_2\mu\sqrt{t}\|B\|_\infty)c_5Mt^{1-\delta} + 2c_2\sqrt{\mu t}\,\|c\|_{L^\infty}\int_\Omega |\hat{T}(t)u_k|dx
\end{aligned}
$$

and consequently it follows

$$
\begin{aligned}
&|Du_0|_P(\Omega) \le \liminf_{t\to 0}\int_\Omega |DT(t)u_0|_P dx \le \limsup_{t\to 0}\limsup_{k\to+\infty}\int_\Omega |DT(t)u_k|_P dx \\
\le&\limsup_{t\to 0}\big\{(1 + 2c_2\mu\sqrt{t}\|B\|_\infty)|Du_0|_P(\Omega) + (1+\mu\sqrt{2})(1 + 2c_2\mu\sqrt{t}\|B\|_\infty)\varepsilon \\
&+ (1+\mu\sqrt{2})(1 + 2c_2\mu\sqrt{t}\|B\|_\infty)c_5Mt^{1-\delta} + c_2\sqrt{\mu t}\,\|c\|_{L^\infty}\|u_0\|_{L^1(\Omega)}\big\} \\
=&|Du_0|_P(\Omega) + (1+\mu\sqrt{2})\varepsilon
\end{aligned}
$$

The result then follows since ε is arbitrary. □

Chapter 5

BV functions and parabolic problems: the second characterization

In this chapter we present a second characterization of BV functions obtained using in a different way the semigroup $T(t)$ generated by the L^1 realization of

$$\mathcal{A} = \sum_{i,j=1}^{n} D_i(a_{ij}(x)D_j) + \sum_{i=1}^{n} b_i(x)D_i + c(x) \tag{5.1}$$

with coefficients

$$a_{ij} \in W^{2,\infty}(\Omega) \quad b_i,\, c \in L^{\infty}(\Omega) \tag{5.2}$$

satisfying (2.107) and with homogeneous boundary condition given by $\mathcal{B}$ in (2.5); in that case, it is possible to associate a positive function $p(t,x,y) \in C_b^1((0,\infty) \times \Omega \times \Omega)$ to the semigroup $T(t)$ (see [45, Sections 5.3, 5.4] for more details) generated by $(A_1, D(A_1))$ and the following representation holds

$$(T(t)u_0)(x) = \int_{\Omega} p(t,x,y)u_0(y)\, dy. \tag{5.3}$$

This function $p(t,x,y)$ is called the kernel of $T(t)$ and this formula is a keystone for proving some interesting relations between BV functions and solutions of parabolic initial boundary value problems; more precisely, in the spirit of [33], we give a complete characterization of sets of finite perimeter and then, using it in connection with the coarea formula, we prove that

$$|Du|_A(\Omega) = \lim_{t\to 0} \frac{\sqrt{\pi}}{2\sqrt{t}} \int_{\Omega}\int_{\Omega} p(t,x,y)|u(x)-u(y)|\, dydx, \tag{5.4}$$

where $|Du|_A$ denotes the A-weighted total variation of u. This characterization is analogous to some results in [8], [14] and [27], [33], where general kernels depending on $|x-y|$ are considered.

5.1 The heat kernel in $\mathbf{R}^n$

In [27], Ledoux investigated in a different perspective some connections between the heat semigroup $(W(t))_{t\geq 0}$ on $L^2(\mathbf{R}^n)$ and the isoperimetric inequality.
We recall that the classical isoperimetric inequality in $\mathbf{R}^n$ states that *among all subset $E \subset \mathbf{R}^n$ with fixed volume and smooth boundary, Euclidean balls minimize the surface measure of the boundary.* In [27] Ledoux observed that the L^2- inequality for the Gauss-Weiesrstrass semigroup in $\mathbf{R}^n$

$$\|W(t)\chi_E\|_{L^2(\mathbf{R}^n)} \leq \|W(t)\chi_B\|_{L^2(\mathbf{R}^n)} \qquad t \geq 0 \tag{5.5}$$

for sets E with smooth boundary and with $|E| = |B|$ can be used to prove the isoperimetric inequality. In order to reach this, he provided an estimate for the L^2 norm of $W(t)\chi_E$ in terms of the perimeter of E in $\mathbf{R}^n$. We refer to [27, Proposition 1.1] for the proof.

Proposition 5.1.1 (Ledoux)**.** *For every subset E of finite measure in $\mathbf{R}^n$ and smooth boundary ∂E and for every $t \geq 0$, the inequality*

$$\int_{E^c} W(t)\chi_E(x)\,dx \leq \sqrt{\frac{t}{\pi}}\mathcal{P}(E) \tag{5.6}$$

holds.

Moreover, if B is an Euclidean ball, he checked that

$$\lim_{t\to 0}\sqrt{\frac{\pi}{t}}\int_{B^c} W(t)\chi_B(x)\,dx = \mathcal{P}(B). \tag{5.7}$$

Finally if $|E| = |B|$, then the L^2- inequality (5.5) is equivalent to the following

$$\int_{E^c} W(t)\chi_E(x)dx \geq \int_{B^c} W(t)\chi_B(x)dx. \tag{5.8}$$

This is easy to see; in fact,

$$\begin{aligned}
\int_{E^c} W(t)\chi_E(x)\,dx &= \int_{\mathbf{R}^n} W(t)\chi_E(x)\chi_{E^c}(x)\,dx \\
&= \int_{\mathbf{R}^n} W(t)\chi_E(x)(1-\chi_E(x))\,dx \\
&= \int_{\mathbf{R}^n} W(t)\chi_E(x)\,dx - \int_{\mathbf{R}^n} W(t)\chi_E(x)\chi_E(x)\,dx \\
&= \|W(t)\chi_E\|_{L^1(\mathbf{R}^n)} - \int_{\mathbf{R}^n} W(t/2)\chi_E(x)W(t/2)\chi_E(x)\,dx \\
&= \|\chi_E\|_{L^1(\mathbf{R}^n)} - \|W(t/2)\chi_E\|_{L^2(\mathbf{R}^n)} \\
&\geq \|\chi_B\|_{L^1(\mathbf{R}^n)} - \|W(t/2)\chi_B\|_{L^2(\mathbf{R}^n)} \\
&= \int_{B^c} W(t)\chi_B(x)\,dx,
\end{aligned}$$

whence

$$\int_{E^c} W(t)\chi_E(x)\,dx \geq \int_{B^c} W(t)\chi_B(x)\,dx.$$

Putting all these results together it is easy to prove that (5.5) implies the isoperimetric inequality. Indeed, under properties (5.6)-(5.8), for every $t > 0$

$$\mathcal{P}(E) \geq \sqrt{\frac{\pi}{t}} \int_{E^c} W(t)\chi_E(x)\,dx \geq \sqrt{\frac{\pi}{t}} \int_{B^c} W(t)\chi_B(x)\,dx$$

and as $t \to 0$, $\mathcal{P}(E) \geq \mathcal{P}(B)$.
Notice that the reverse of the Ledoux result is due to the following Riesz-Sobolev inequality (see [28, Theorem 3.7]):

$$\int_{\mathbf{R}^n \times \mathbf{R}^n} f(x)g(x-y)h(y)dxdy \leq \int_{\mathbf{R}^n \times \mathbf{R}^n} f^*(x)g^*(x-y)h^*(y)dxdy. \tag{5.9}$$

where f^*, g^*, h^* denote respectively the spherical symmetrization of f, g, h. Now, taking $f = h = \chi_E$ and $g = g^* = G_t(\cdot)$ (where $G_t(z)$ denotes the heat kernel in $\mathbf{R}^n$) in (5.9), so that $f^* = h^* = \chi_B$, the inequality (5.5) follows immediately:

$$\begin{aligned}
\|W(t)\chi_E\|^2_{L^2(\mathbf{R}^n)} &= \int_{\mathbf{R}^n} W(2t)\chi_E(x)\chi_E(x)dx \\
&= \int_{\mathbf{R}^n \times \mathbf{R}^n} G_{2t}(x-y)\chi_E(x)\chi_E(y)dxdy \\
&\leq \int_{\mathbf{R}^n \times \mathbf{R}^n} G_{2t}(x-y)\chi_B(x)\chi_B(y)dxdy \\
&= \int_{\mathbf{R}^n} W(2t)\chi_B(x)\chi_B(x)dx = \|W(t)\chi_B\|^2_{L^2(\mathbf{R}^n)}
\end{aligned}$$

Thus we can state the following equivalence.

Theorem 5.1.2. *Let E, B be subset of $\mathbf{R}^n$ with $|E| = |B|$, B an Euclidean ball. Then*

$$\mathcal{P}(E) \geq \mathcal{P}(B) \Longleftrightarrow \|W(t)\chi_E\|_{L^2(\mathbf{R}^n)} \leq \|W(t)\chi_B\|_{L^2(\mathbf{R}^n)} \quad \textit{for all } t \geq 0. \tag{5.10}$$

An immediate interpretation of (5.10) can be deduced by taking into account that in our assumption, (5.5) is equivalent to (5.8) and that $\int_{E^c} W(t)\chi_E(x)\,dx$ measures the amount of heat that is outside the set E at time $t \geq 0$. Therefore (5.10) tells that *among all regular sets of the same volume and at the same initial temperature, the Euclidean ball (having minimum perimeter) is that which minimizes the heat outflow*.
In [33], formula (5.7) has been generalized to all sets of finite perimeter. The proof of such result is based upon the measure-theoretic properties of the reduced boundary. Moreover, in [33] it is also proved that the finiteness of the limit on the left hand side characterizes sets of finite perimeter.
Let us point out that the same characterization of finite perimeter sets is also proved, following a different approach based on the study of behavior of the difference quotient of u, in the papers [8], [14], [36], where convolution kernels more general than the Gauss-Weierstrass one are considered. In [33] the following theorem is proved.

Theorem 5.1.3. *Let $E, F \subset \mathbf{R}^n$ be sets of finite perimeter. Then the following equality holds:*

$$\lim_{t\to 0}\sqrt{\frac{\pi}{t}}\int_F (\chi_E(x) - W(t)\chi_E(x))\,dx = \int_{\mathcal{F}E\cap\mathcal{F}F}\nu_E(x)\cdot\nu_F(x)d\mathcal{H}^{n-1}(x). \tag{5.11}$$

Proof. Since

$$W(t)\chi_E - \chi_E = \int_0^t \Delta W(s)\chi_E ds,$$

we have

$$\int_F (W(t)\chi_E - \chi_E)\,dx = \int_0^t\int_F (\Delta W(s)\chi_E)\,dx\,ds.$$

Moreover, by (4.6), integrating by parts we obtain

$$\begin{aligned}\int_F (\Delta W(s)\chi_E)\,dx &= \int_{\mathbf{R}^n}\Delta W(s)\chi_E(x)\chi_F(x)dx = -\int_{\mathbf{R}^n} D_x W(s)\chi_E(x)\cdot dD\chi_F(x)\\ &= -\int_{\mathcal{F}F} D_x W(s)\chi_E(x)\cdot\nu_F(x)d\mathcal{H}^{n-1}(x).\end{aligned}$$

Notice that, if we define for every $x \in \mathcal{F}E$ and $s > 0$ the measures

$$d\mu_{s,x} = \mathcal{L}^n \llcorner \left(\frac{E-x}{\sqrt{s}}\right),$$

and set $z = \dfrac{y-x}{\sqrt{s}}$, we have

$$\begin{aligned}D_x W(s)\chi_E(x) &= \int_E D_x\left(\frac{e^{-\frac{|x-y|^2}{4s}}}{(4\pi s)^{n/2}}\right)dy = -\int_E \frac{(x-y)}{2s}\frac{e^{-\frac{|x-y|^2}{4s}}}{(4\pi s)^{n/2}}dy\\ &= \frac{1}{2\sqrt{s}}\int_{\frac{E-x}{\sqrt{s}}}\frac{e^{-|z|^2/4}}{(4\pi)^{n/2}}zdz\\ &= \frac{1}{2\sqrt{s}}\int_{\mathbf{R}^n}\frac{e^{-|z|^2/4}}{(4\pi)^{n/2}}zd\mu_{s,x}(z).\end{aligned}$$

Moreover, setting, for every $x \in \mathcal{F}E$,

$$H_{\nu_E(x)} = \{z \in \mathbf{R}^n : z\cdot\nu_E(x) \geq 0\},$$

the existence of the approximate tangent plane for $x \in \mathcal{F}E$, see (4.1.5), implies that the measures $\mu_{s,x}$ are locally weakly* convergent as $s \to 0$ to the measure

$$\mu_x = \begin{cases} 0 & \text{if } x \in E^0,\\ \mathcal{L}^n & \text{if } x \in E^1,\\ \mathcal{L}^n \llcorner H_{\nu_E(x)} & \text{if } x \in \mathcal{F}E.\end{cases}$$

Moreover for every $\varepsilon > 0$ we can find a compact set $K \subset \mathbf{R}^n$ such that

$$\int_{\mathbf{R}^n\setminus K} z\cdot\nu_F(x)\frac{e^{-|z|^2/4}}{(4\pi)^{n/2}}d\mu_{s,x}(z) < \varepsilon, \quad \int_{\mathbf{R}^n\setminus K} z\cdot\nu_F(x)\frac{e^{-|z|^2/4}}{(4\pi)^{n/2}}d\mu_x(z) < \varepsilon$$

hence, since $\mu_{s,x}$ are locally weakly* convergent as $s \to 0$ to μ_x

$$\lim_{s\to 0} \int_{\mathbf{R}^n} z \cdot \nu_F(x) \frac{e^{-|z|^2/4}}{(4\pi)^{n/2}} d\mu_{s,x}(z) = \int_{\mathbf{R}^n} z \cdot \nu_F(x) \frac{e^{-|z|^2/4}}{(4\pi)^{n/2}} d\mu_x. \tag{5.12}$$

Summing up, we can write

$$\sqrt{\frac{\pi}{t}} \int_F (\chi_E - W(t)\chi_E)\, dx = \frac{\sqrt{\pi}}{(4\pi)^{n/2}} \int_{\mathcal{F}F} \frac{1}{\sqrt{t}} \int_0^t \frac{1}{2\sqrt{s}} g(x,s)\ ds\ d\mathcal{H}^{n-1}(x)\,, \tag{5.13}$$

where $g : \mathcal{F}F \times (0,t)$ is given by

$$g(x,s) = \int_{\mathbf{R}^n} e^{-|z|^2/4} z \cdot \nu_F(x) d\mu_{s,x}(z),$$

and by (5.12) we have

$$\lim_{s\to 0^+} g(x,s) = \begin{cases} \displaystyle\int_{H_{\nu_E(x)}} z \cdot \nu_F(x) e^{-|z|^2/4} dz & \text{for } x \in \mathcal{F}E \cap \mathcal{F}F \\ 0 & \text{for } x \in \left(E^0 \cup E^1\right) \cap \mathcal{F}F, \end{cases}$$

where E^0, E^1 are defined according to (4.8). This implies that for all $\varepsilon > 0$ there exists $t_0 > 0$ such that if $t < t_0$ and $x \in \left(E^0 \cup E^1\right) \cap \mathcal{F}F$, then

$$\left| \frac{1}{\sqrt{t}} \int_0^t \frac{1}{2\sqrt{s}} g(x,s)\ ds \right| \leq \frac{1}{\sqrt{t}} \int_0^t \frac{\varepsilon}{\sqrt{s}}\, ds = 2\varepsilon.$$

Now, by Theorem 4.1.7, we have that $\mathcal{H}^{n-1}(\partial^* E \setminus \mathcal{F}E) = 0$, then the right hand side of (5.13) reduces to the integral on $\mathcal{F}E \cap \mathcal{F}F$ and we obtain that there exists

$$\begin{aligned} \lim_{t\to 0} \sqrt{\frac{\pi}{t}} \int_F (\chi_E - W(t)\chi_E)\, dx &= \frac{\sqrt{\pi}}{(4\pi)^{n/2}} \int_{\mathcal{F}E\cap\mathcal{F}F} \int_{H_{\nu_E(x)}} z \cdot \nu_F(x) e^{-|z|^2/4} dz d\mathcal{H}^{n-1}(x) \\ &= \frac{\sqrt{\pi}}{(4\pi)^{n/2}} \int_{\mathcal{F}E\cap\mathcal{F}F} \int_{H_{\nu_E(x)}} (\nu_E(x)\cdot\nu_F(x))(z\cdot\nu_E(x)) e^{-|z|^2/4} dz d\mathcal{H}^{n-1}(x) \\ &= \int_{\mathcal{F}E\cap\mathcal{F}F} \nu_E(x)\cdot\nu_F(x) d\mathcal{H}^{n-1}(x), \end{aligned}$$

because $\nu_F(x) = (\nu_E(x)\cdot\nu_F(x))\nu_E(x)$ for $\mathcal{H}^{n-1}$-a.e. $x \in \mathcal{F}E \cap \mathcal{F}F$ and

$$\int_{H_{\nu_E(x)}} z \cdot \nu_E(x) e^{-|z|^2/4} dz = 2(4\pi)^{(n-1)/2} \qquad \forall\ x \in \mathcal{F}E.$$

□

Remark 5.1.4. Notice that if $|F \setminus E| = 0$ in the preceding statement, then $\nu_E(x) = \nu_F(x)$ for $\mathcal{H}^{n-1}$-a.e. $x \in \mathcal{F}E \cap \mathcal{F}F$, hence the equality

$$\lim_{t\to 0} \sqrt{\frac{\pi}{t}} \int_F (\chi_E - W(t)\chi_E)\, dx = \mathcal{H}^{n-1}(\mathcal{F}E \cap \mathcal{F}F) \tag{5.14}$$

holds.

As a special case, we may take $E = F$ in the above theorem, and obtain the following result, which generalizes formula (5.7).

Theorem 5.1.5. *Let $E \subset \mathbf{R}^n$ be a set of finite perimeter; then the following equality holds*

$$\lim_{t\to 0} \sqrt{\frac{\pi}{t}} \int_{E^c} W(t)\chi_E\, dx = \mathcal{P}(E). \tag{5.15}$$

PROOF. Since $\|W(t)\chi_E\|_{L^1(\mathbf{R}^n)} = |E|$ for all $t \geq 0$, we obtain

$$\int_E (\chi_E - W(t)\chi_E)\, dx = \int_{\mathbf{R}^n} (\chi_E - W(t)\chi_E)(1-\chi_{E^c})\, dx = \int_{E^c} W(t)\chi_E\, dx$$

and the assertion follows inserting $F = E$ in (5.14). □

A sort of reverse implication is also stated.

Theorem 5.1.6. *Let $E \subset \mathbf{R}^n$ be a set such that either E or E^c has finite measure, and*

$$\liminf_{t\to 0^+} \frac{1}{\sqrt{t}} \int_{E^c} W(t)\chi_E\, dx < +\infty.$$

Then E has finite perimeter.

PROOF. Assume that $|E| < +\infty$. We can write

$$\begin{aligned}
\frac{1}{\sqrt{t}}\langle W(t)\chi_E, \chi_{E^c}\rangle &= \frac{1}{(4\pi)^{n/2}\sqrt{t}} \int_{\mathbf{R}^n}\int_{\mathbf{R}^n} \chi_{E^c}(x)\chi_E(x+\sqrt{t}y)e^{-|y|^2/4}dydx \\
&= \frac{1}{(4\pi)^{n/2}\sqrt{t}} \int_{\mathbf{R}^n} e^{-|y|^2/4}\int_{\mathbf{R}^n} \Big(\chi_{E-\sqrt{t}y}(x) - \chi_E(x)\chi_{E-\sqrt{t}y}(x)\Big)\, dxdy \\
&= \frac{1}{(4\pi)^{n/2}\sqrt{t}} \int_{\mathbf{R}^n} e^{-|y|^2/4}\Big(|E| - |E\cap(E-\sqrt{t}y)|\Big)\, dy \\
&= \frac{1}{2(4\pi)^{n/2}\sqrt{t}} \int_{\mathbf{R}^n} e^{-|y|^2/4}|E\triangle(E-\sqrt{t}y)|dy \\
&= \frac{1}{2(4\pi)^{n/2}} \int_{\mathbf{R}^n} |y|e^{-|y|^2/4}\frac{|E\triangle(E-\sqrt{t}y)|}{\sqrt{t}|y|}dy,
\end{aligned}$$

where $E\triangle F = (E\cup F)\setminus(E\cap F)$. Then, if we define

$$|D_\nu\chi_E| = \liminf_{t\to 0^+} \frac{|E\triangle(E-t\nu)|}{t},$$

from the previous estimate we get that

$$\int_{\mathbf{R}^n} |y|e^{-|y|^2/4}|D_{y/|y|}\chi_E|dy \leq \liminf_{t\to 0^+} \int_{\mathbf{R}^n} |y|e^{-|y|^2/4}\frac{|E\triangle(E-\sqrt{t}y)|}{\sqrt{t}|y|}dy < +\infty.$$

Noticing that

$$\int_{\mathbf{R}^n} |y|e^{-|y|^2/4}|D_{y/|y|}\chi_E|dy = C_n \int_{\mathbf{S}^{n-1}} |D_\nu\chi_E|d\nu,$$

we have proved that

$$\int_{\mathbf{S}^{n-1}} |D_\nu \chi_E| d\nu < +\infty.$$

This implies that the function $\nu \mapsto |D_\nu \chi_E|$ is finite for a.e. $\nu \in \mathbf{S}^{n-1}$; in particular, there exist $M > 0$ and an orthonormal system of coordinates $\nu_1, \ldots, \nu_n$ of Lebesgue points of $|D_\nu \chi_E|$ such that

$$|D_{\nu_i} \chi_E| \leq M, \quad \forall i = 1, \ldots, n.$$

Without loss of generality, we can assume that $\nu_i = e_i$; now, if $\phi \in C_c^1(\mathbf{R}^n)$, the function

$$\phi_t(x) = \frac{\phi(x + te_i) - \phi(x)}{t}$$

is uniformly convergent to $\partial_i \phi(x)$. This implies that

$$\int_{\mathbf{R}^n} \chi_E(x) \partial_i \phi(x) dx = \lim_{t \to 0^+} \int_{\mathbf{R}^n} \chi_E(x) \phi_t(x) dx.$$

But

$$\int_{\mathbf{R}^n} \chi_E(x) \phi_t(x) dx = \int_{\mathbf{R}^n} \frac{\chi_E(x - te_i) - \chi_E(x)}{t} \phi(x) dx,$$

hence

$$\left| \int_{\mathbf{R}^n} \chi_E(x) \phi_t(x) dx \right| \leq \|\phi\|_\infty \frac{|E \triangle (E + te_i)|}{t}.$$

From this it follows that

$$\begin{aligned} \left| \int_{\mathbf{R}^n} \chi_E(x) \partial_i \phi(x) dx \right| &\leq \|\phi\|_\infty \liminf_{t \to 0^+} \frac{|E \triangle (E + te_i)|}{t} \\ &= \|\phi\|_\infty |D_i \chi_E| \leq M \|\phi\|_\infty. \end{aligned}$$

In the end, we have proved that

$$\int_{\mathbf{R}^n} \chi_E(x) \operatorname{div} \phi(x) dx \leq nM \|\phi\|_\infty, \quad \forall \phi \in C_c^1(\mathbf{R}^n),$$

and then $\chi_E \in BV(\mathbf{R}^n)$. □

In connection with these results, it seems to be interesting to pursue the investigation of the relationships between the perimeter of a set in a domain and the short-time behavior of the semigroup $T(t)$ generated by a more general operator like $(A_1, D(A_1))$.

Remark 5.1.7. In what follows Gaussian upper and lower bounds of the fundamental solution associated with the operator $\partial_t - \mathcal{A}$ are of relevant importance. They can be found in Appendix B and are used in a form neglecting $e^{\omega t}$. This is not important for our computations since we are interested in the behavior of $T(t)$ for small t, see Remark B.2.1.

5.2 Preliminary results for problems in a domain

For every $s>0$ and $x_0 \in \Omega$, we set

$$\Omega^{s,x_0} = \frac{\Omega - x_0}{\sqrt{s}} = \left\{ y \in \mathbf{R}^n : x_0 + \sqrt{s}y \in \Omega \right\}$$

and, given $f : \Omega \to \mathbf{R}$, $f^{s,x_0}(y) = f(x_0 + \sqrt{s}y)$. With this notation, we define the operator $\mathcal{A}^{s,x_0}$ on Ω^{s,x_0} by

$$\begin{aligned}
\mathcal{A}^{s,x_0}(y)v(y) &= \operatorname{div}(A^{s,x_0}(y)Dv(y)) + \sqrt{s}\langle B^{s,x_0}(y), Dv(y)\rangle + sc^{s,x_0}(y)v(y) \\
&= \sum_{h,k=1}^{n} a_{hk}(x_0+\sqrt{s}y)\frac{\partial^2 v}{\partial y^h \partial y^k}(y) + \sqrt{s}\sum_{k=1}^{n}\left(\sum_{h=1}^{n} D_h a_{hk}(x_0+\sqrt{s}y)\right)\frac{\partial v}{\partial y^k}(y) \\
&\quad + \sqrt{s}\sum_{h=1}^{n} b_h(x_0+\sqrt{s}y)\frac{\partial v}{\partial y^h}(y) + sc(x_0+\sqrt{s}y)v(y),
\end{aligned}$$

and the operator $\mathcal{A}^x$ on $\mathbf{R}^n$ by

$$\mathcal{A}^x v(y) = \sum_{h,k=1}^{n} a_{hk}(x)\frac{\partial^2 v}{\partial y^h \partial y^k}(y).$$

By setting $x = x_0 + \sqrt{s}y$, it is easily seen that $\mathcal{A}^{s,x_0}(y) = s\mathcal{A}(x)$. We have the following lemma.

Lemma 5.2.1. *Setting $u(t,x) = T(t)u_0(x)$, we can define the function $v : (0,+\infty) \times \Omega^{s,x_0} \to \mathbf{R}$ by $v(t,y) = u(ts, x_0 + \sqrt{s}y)$; then v is the solution of the problem*

$$\begin{cases} \partial_t w = \mathcal{A}^{s,x_0}(y)w & \text{in } (0,+\infty)\times\Omega^{s,x_0} \\ w(0,y) = u_0^{s,x_0}(y) & \text{in } \Omega^{s,x_0} \\ \langle A^{s,x_0}Dw, \nu\rangle = 0 & \text{in } (0,+\infty)\times\partial\Omega^{s,x_0}. \end{cases} \tag{5.16}$$

Proof. By definition, we have $v(0,y) = u(0, x_0+\sqrt{s}y) = u_0(x_0+\sqrt{s}y) = u_0^{s,x_0}(y)$. Moreover, if we set $x = x_0 + \sqrt{s}y$, we have that $\partial/\partial y^h = \sqrt{s}\partial/\partial x^h$ and also that the unit outward normal to $\partial\Omega^{s,x_0}$ at y coincides with the unit outward normal to $\partial\Omega$ at x; therefore,

$$\langle A^{s,x_0}(y)D_y v(t,y), \nu(y)\rangle = \sqrt{s}\langle A(x)D_x u(ts,x), \nu(x)\rangle = 0.$$

In the same way, we have

$$\partial_t v(t,y) = su'(ts, x_0+\sqrt{s}y) = su'(ts,x) = s\mathcal{A}(x)u(ts,x) = \mathcal{A}^{s,x_0}(y)v(t,y),$$

where u' denotes the derivative of u with respect to its first variable, and this concludes the proof. □In order to follow the computations in Section 5.1, based on the Gauss-Weierstrass kernel G we recall that the semigroup generated by $\mathcal{A}$, $\mathcal{A}^{s,x}$, $\mathcal{A}^x$, are represented through an integral kernel that will be introduced with a coherent notation (see e.g. [45]). We also denote by $(T^{s,x_0}(t))_{t\geq 0}$ the semigroup associated with problem (5.16) and by $p^{s,x_0}(t,y,z)$ its kernel. We also denote by $(T^{x_0}(t))_{t\geq 0}$ the semigroup associated with the problem

$$\begin{cases} \partial_t w(t,y) = \mathcal{A}^{x_0}(y)w(t,y) & \text{in } (0,+\infty)\times \mathbf{R}^n \\ w(0,y) = w_0(y) & \text{in } \mathbf{R}^n \end{cases}$$

and by $p^{x_0}(t,y,z)$ its kernel.

Lemma 5.2.2. *For the kernels the following holds*

$$p(t,x,y) = s^{-n/2} p^{s,x_0}\Big(\frac{t}{s}, \frac{x-x_0}{\sqrt{s}}, \frac{y-x_0}{\sqrt{s}}\Big). \tag{5.17}$$

Proof. The proof of Lemma 5.2.1 gives that $v(t,y) = T^{s,x_0}(t)u_0^{s,x_0}(y) = T(ts)u_0(x_0+\sqrt{s}y)$; using the kernels, we get that

$$\begin{aligned} \int_\Omega p(t,x,y)u_0(y)dy &= T(t)u_0(x) = T^{s,x_0}\left(\frac{t}{s}\right)u_0^{s,x_0}\left(\frac{x-x_0}{\sqrt{s}}\right) \\ &= \int_{\Omega^{s,x_0}} p^{s,x_0}\left(\frac{t}{s}, \frac{x-x_0}{\sqrt{s}}, z\right)u_0(x_0+\sqrt{s}z)dz \\ &= s^{-n/2}\int_\Omega p^{s,x_0}\left(\frac{t}{s}, \frac{x-x_0}{\sqrt{s}}, \frac{y-x_0}{\sqrt{s}}\right)u_0(y)dy. \end{aligned}$$

The arbitrarity of u_0 gives the thesis. $\square$

We have the following result.

Proposition 5.2.3. *For every $f\in L^1(\mathbf{R}^n)$, let $u^{s,x}(t,\xi)$ be the solution of the problem*

$$\begin{cases} \partial_t w(t,\xi) = \mathcal{A}^{s,x}(\xi)w(t,\xi) & \text{in } (0,+\infty)\times\Omega^{s,x} \\ \langle A(x+\sqrt{s}\xi)Dw(t,\xi), \nu_{\Omega^{s,x}}(\xi)\rangle = 0 & \text{in } (0,+\infty)\times\partial\Omega^{s,x} \\ w(0,\xi) = f(\xi) & \text{in } \Omega^{s,x} \end{cases}$$

and let $u^x(t,\xi)$ be the solution of the problem

$$\begin{cases} \partial_t w(t,\xi) = \mathcal{A}^{x}(\xi)w(t,\xi) & \text{in } (0,+\infty)\times\mathbf{R}^n \\ w(0,\xi) = f(\xi) & \text{in } \mathbf{R}^n \end{cases}.$$

Then for every $t>0$ we have that $u^{s,x}(t,\cdot)$ converges to $u^x(t,\cdot)$ in $L^1_{\mathrm{loc}}(\mathbf{R}^n)$ as $s\to 0$.

Proof. We start by taking $f\in C_c(\mathbf{R}^n)$ and denoting by $u^{s,x}(t,\xi)$ the solution of the problem

$$\begin{cases} \partial_t w(t,\xi) = \mathcal{A}^{x}(\xi)w(t,\xi) & \text{in } (0,+\infty)\times\Omega^{s,x} \\ \langle A^{s,x}(\xi)D_\xi w(t,\xi), \nu(\xi)\rangle = 0 & \text{in } (0,+\infty)\times\partial\Omega^{s,x} \\ w(0,\xi) = f(\xi) & \text{in } \Omega^{s,x}. \end{cases} \tag{5.18}$$

Since $u^{s,x}$ is a classical solution, for every regular function $\varphi : [0,t_0]\times\mathbf{R}^n\to\mathbf{R}$ with $\varphi(t_0,\cdot)=0$, the following holds:

$$\begin{aligned} -\int_{\Omega^{s,x}} f(\xi)\varphi(0,\xi)d\xi \;\; = \;\; & \int_0^{t_0}\int_{\Omega^{s,x}} \Big\{u^{s,x}(t,\xi)\left(\partial_t\varphi(t,\xi) + sc^{s,x}(\xi)\right) \\ & + \frac{\partial u^{s,x}(t,\xi)}{\partial\xi^k}\Big[-a^{s,x}_{hk}(\xi)\frac{\partial\varphi(t,\xi)}{\partial\xi^h} + \sqrt{s}\varphi(t,\xi)b^{s,x}_k(\xi)\Big]\Big\}d\xi dt. \end{aligned} \tag{5.19}$$

Moreover, notice that $sc^{s,x} \to 0$, $a^{s,x}_{hk} \to a_{hk}(x)$, $\sqrt{s}b^{s,x}_k \to 0$ uniformly on compact sets as $s \to 0$.

As an auxiliary tool, let us use the L^2 theory, see e.g. [45, Section 5.4], recalling that there is $M > 0$ independent of $s \in [0,1]$, such that

$$\|u^{s,x}(t)\|_{L^2(\Omega^{s,x})} \le M\|f\|_{L^2(\Omega^{s,x})} \le M\|f\|_{L^2(\mathbf{R}^n)}, \tag{5.20}$$

$$\|Du^{s,x}(t)\|_{L^2(\Omega^{s,x})} \le \frac{M}{\sqrt{t}}\|f\|_{L^2(\Omega^{s,x})} \le \frac{M}{\sqrt{t}}\|f\|_{L^2(\mathbf{R}^n)}, \tag{5.21}$$

and

$$\|D^2u^{s,x}(t)\|_{L^2(\Omega^{s,x})} \le \frac{M}{t}\|f\|_{L^2(\Omega^{s,x})} \le \frac{M}{t}\|f\|_{L^2(\mathbf{R}^n)}. \tag{5.22}$$

These conditions imply that for every bounded open set $A \subset \mathbf{R}^n$, $t > 0$ fixed and s_0 small enough, the family $(u^{s,x}(t,\cdot))_{0<s<s_0}$ is bounded in $W^{2,2}(A)$, and then, up to subsequences, it is strongly convergent in $W^{1,2}(A)$ and also in $W^{1,1}(A)$.

We can now fix a countable dense set $D \subset [0,t_0]$ in such a way that $u^{s_h,x}(t,\cdot)$ converges to some $g(t,\cdot)$ in $W^{1,1}(A)$ for every $t \in D$ and some sequence $s_h \to 0$. By (3.2) we get that

$$\begin{aligned}
&\|u^{s,x}(t_2,\cdot) - u^{s,x}(t_1,\cdot)\|_{L^1(\Omega^{s,x})} = \Big\| \int_{t_1}^{t_2} \partial_t u^{s,x}(t,\cdot)dt \Big\|_{L^1(\Omega^{s,x})} \\
&\le \int_{t_1}^{t_2} \|A^{s,x}u^{s,x}(t,\cdot)\|_{L^1(\Omega^{s,x})}dt \le c_1\|f\|_{L^1(\Omega^{s,x})} \int_{t_1}^{t_2} \frac{1}{t}dt \le c_1\|f\|_{L^1(\mathbf{R}^n)} \log\frac{t_2}{t_1},
\end{aligned}$$

that is, the function $t \mapsto u^{s,x}(t,\cdot)$ is continuous from $(0,t_0)$ to $L^1(\Omega^{s,x})$; in particular, if we consider $t_1, t_2 \in D$, then the inequality

$$\begin{aligned}
\|g(t_2,\cdot) - g(t_1,\cdot)\|_{L^1(A)} &\le \|g(t_2,\cdot) - u^{s_h,x}(t_2,\cdot)\|_{L^1(A)} \\
&\quad + \|u^{s_h,x}(t_2,\cdot) - u^{s_h,x}(t_1,\cdot)\|_{L^1(A)} + \|u^{s_h,x}(t_1,\cdot) - g(t_1,\cdot)\|_{L^1(A)}
\end{aligned}$$

holds and the convergence of $u^{s,x}$ on D shows that we can extend g to a continuous map from $(0,t_0)$ to $L^1_{\text{loc}}(\mathbf{R}^n)$; we also notice that by (3.4) we deduce also that $g(t,\cdot) \in W^{1,1}(A)$ for every $t \in (0,t_0)$. By continuity, and by the convergence of $u^{s_h,x}(t,\cdot)$ on D we deduce that $u^{s_h,x}(t,\cdot) \to g(t,\cdot)$ in $L^1_{\text{loc}}(\mathbf{R}^n)$ for every $t \in (0,t_0)$. In addition, conditions (3.1) allow us to apply the dominated convergence theorem, and then, taking the limit in (5.19), we get

$$-\int_A f(\xi)\varphi(0,\xi)d\xi = \int_0^{t_0}\int_A \Big(g(t,\xi)\partial_t\varphi(t,\xi) - \langle A(x)D_\xi\varphi(t,\xi), D_\xi g(t,\xi)\rangle\Big)d\xi dt$$

for all φ as above, and then (see e.g. [40, Prop. 2.1, Ch. III]) $g(t,\cdot)$ is the solution of the problem

$$\begin{cases} \partial_t w(t,\xi) = a_{hk}(x)\frac{\partial^2 w}{\partial\xi^h\partial\xi^k}(t,\xi) & \text{in } (0,t_0)\times\mathbf{R}^n \\ w(0,\xi) = f(\xi) & \text{in } \mathbf{R}^n \end{cases}$$

for every $f \in C_c(\mathbf{R}^n)$. Then, it follows that

$$g(t,\xi) = u^x(t,\xi) = \int_{\mathbf{R}^n} p^x(t,\xi,z)f(z)dz,$$

where using the Fourier transform the kernel p^x is given by

$$p^x(t,\xi,z) = \frac{1}{(4\pi t)^{n/2}|\det A^{1/2}(x)|}\exp\left(-\frac{\langle A^{-1}(x)(\xi-z),(\xi-z)\rangle}{4t}\right). \tag{5.23}$$

By the density of C_c in L^1 we conclude. $\square$

The following statement is an immediate consequence of Proposition 5.2.3.

Corollary 5.2.4. *For every $t>0$ and a.e. $\xi\in\mathbf{R}^n$, the family of measures $d\mu^{s,x} = p^{s,x}(t,\xi,\cdot)d\mathcal{L}^n \llcorner \Omega^{s,x}$ is weakly* convergent to the measure $d\mu^x = p^x(t,\xi,\cdot)d\mathcal{L}^n$ as $s\to 0$, that is, for every $\varphi\in C_c(\mathbf{R}^n)$ the following equality holds*

$$\lim_{s\to 0}\int_{\Omega^{s,x}}\varphi(z)p^{s,x}(t,\xi,z)dz = \int_{\mathbf{R}^n}\varphi(z)p^x(t,\xi,z)dz.$$

Henceforth, given the function $p(t,\xi,z)$, we shall denote by $D_1p(t,\xi,z)$ the gradient with respect to the first spatial variables ξ and by $D_2p(t,\xi,z)$ the gradient with respect to the second spatial variables z.

Proposition 5.2.5. *For every $t>0$ and a.e. $\xi\in\mathbf{R}^n$, the equality*

$$\lim_{s\to 0}\int_{\Omega^{s,x}}\langle D_2p^{s,x}(t,\xi,z),\varphi(z)\rangle dz = \int_{\mathbf{R}^n}\langle D_2p^x(t,\xi,z),\varphi(z)\rangle dz \tag{5.24}$$

holds for every $\varphi\in L^\infty(\mathbf{R}^n,\mathbf{R}^n)$.

Proof. We start by considering $\varphi\in C_c^1(\mathbf{R}^n,\mathbf{R}^n)$; we choose $s_0>0$ in such a way that $\operatorname{supp}\varphi\subset\Omega^{s,x}$ for all $s\le s_0$; then

$$\int_{\Omega^{s,x}}\langle D_2p^{s,x}(t,\xi,z),\varphi(z)\rangle dz = -\int_{\Omega^{s,x}}p^{s,x}(t,\xi,z)\mathrm{div}\varphi(z)dz$$

and then, by Corollary 5.2.4

$$\begin{aligned}\lim_{s\to 0}\int_{\Omega^{s,x}}\langle D_2p^{s,x}(,\xi,z),\varphi(z)\rangle dz &= \lim_{s\to 0}-\int_{\Omega^{s,x}}p^{s,x}(t,\xi,z)\mathrm{div}\varphi(z)dz\\ &= -\int_{\mathbf{R}^n}p^x(t,\xi,z)\mathrm{div}\varphi(z)dz = \int_{\mathbf{R}^n}\langle D_2p^x(t,\xi,z),\varphi(z)\rangle dz.\end{aligned}$$

For an arbitrary $\varphi\in L^\infty(\mathbf{R}^n,\mathbf{R}^n)$ we use an approximation procedure.
First of all recall that for every $\varepsilon>0$ we can find $R>0$ and $s_0>0$ such that

$$\int_{\Omega^{s,x}\setminus B_R(0)}|D_2p^{s,x}(t,\xi,z)|\,dz\le\varepsilon,\qquad \int_{\mathbf{R}^n\setminus B_R(0)}|D_2p^x(t,\xi,z)|\,dz\le\varepsilon.$$

for all $s\le s_0$. Now, let $\eta\in C_c^\infty(\mathbf{R}^n)$, $0\le\eta\le 1$, $\eta=1$ in $B_R(0)$ and $\eta=0$ in $\mathbf{R}^n\setminus B_{2R}(0)$, and select $\varepsilon<R/2$. Then $\varphi_\varepsilon=\rho_\varepsilon*(\eta\varphi)\in C_c^1(\mathbf{R}^n,\mathbf{R}^n)$ such that $\|\varphi-\varphi_\varepsilon\|_{L^p(B_R(0))}\le\varepsilon$ for all $1\le p<\infty$ and then

$$\begin{aligned}\int_{\Omega^{s,x}}\langle D_2p^{s,x}(t,\xi,z),\varphi(z)\rangle dz &= \int_{\Omega^{s,x}}\langle D_2p^{s,x}(t,\xi,z),\varphi_\varepsilon(z)\rangle dz\\ &\quad+\int_{\Omega^{s,x}\cap B_R(0)}\langle D_2p^{s,x}(t,\xi,z),(\varphi(z)-\varphi_\varepsilon(z))\rangle dz\\ &\quad+\int_{\Omega^{s,x}\setminus B_R(0)}\langle D_2p^{s,x}(t,\xi,z),(\varphi(z)-\varphi_\varepsilon(z))\rangle dz.\end{aligned}$$

Taking into account that $p^{s,x}(t,\xi,z)=s^{n/2}p(ts,x+\sqrt{s}\xi,x+\sqrt{s}z)$ and also that

$$\begin{aligned}D_2p^{s,x}(t,\xi,z) &= D_z s^{n/2}p(ts,x+\sqrt{s}\xi,x+\sqrt{s}z)\\ &= s^{(n+1)/2}D_2p(ts,x+\sqrt{s}\xi,x+\sqrt{s}z),\end{aligned}$$

by (B.2) we obtain

$$\begin{aligned}&\Big|\int_{\Omega^{s,x}\cap B_R(0)}\langle D_2p^{s,x}(t,\xi,z),(\varphi(z)-\varphi_\varepsilon(z))\rangle dz\Big|\\ &\qquad\leq s^{(n+1)/2}\|\varphi-\varphi_\varepsilon\|_{L^2(B_R)}\Big(\int_{\Omega^{s,x}}|D_2p(ts,x+\sqrt{s}\xi,x+\sqrt{s}z)|^2dz\Big)^{1/2}\leq C\varepsilon\end{aligned}$$

with C independent of s. Of course, the inequality

$$\Big|\int_{B_R(0)}\langle D_2p^x(t,\xi,z),(\varphi(z)-\varphi_\varepsilon(z))\rangle dz\Big|\leq C\varepsilon$$

holds as well, and then

$$\begin{aligned}&\lim_{s\to 0}\Big|\int_{\Omega^{s,x}}\langle D_2p^{s,x}(t,\xi,z),\varphi(z)\rangle dz-\int_{\mathbf{R}^n}\langle D_2p^x(t,\xi,z),\varphi(z)\rangle dz\Big|\\ &\leq\lim_{s\to 0}\Big|\int_{\Omega^{s,x}\cap B_R(0)}\langle D_2p^{s,x}(t,\xi,z),(\varphi(z)-\varphi_\varepsilon(z))\rangle dz\Big|\\ &\quad+\lim_{s\to 0}\Big|\int_{\Omega^{s,x}\setminus B_R(0)}\langle D_2p^{s,x}(t,\xi,z),(\varphi(z)-\varphi_\varepsilon(z))\rangle dz\Big|\\ &\quad+\lim_{s\to 0}\Big|\int_{\Omega^{s,x}}\langle D_2p^{s,x}(t,\xi,z),\varphi_\varepsilon(z)\rangle dz-\int_{\mathbf{R}^n}\langle D_2p^x(t,\xi,z),\varphi_\varepsilon(z)dz\rangle\Big|\\ &\quad+\lim_{s\to 0}\Big|\int_{B_R(0)}\langle D_2p^x(t,\xi,z),(\varphi(z)-\varphi_\varepsilon(z))\rangle dz\Big|\\ &\quad+\lim_{s\to 0}\Big|\int_{\mathbf{R}^n\setminus B_R(0)}\langle D_2p^x(t,\xi,z),(\varphi(z)-\varphi_\varepsilon(z))\rangle dz\Big|\leq C\varepsilon\end{aligned}$$

and the thesis follows from the arbitrariness of ε. □

5.3 A second characterization of BV functions

The main step in the proof of (5.4) is the following result, where an asymptotic formula relating two sets of finite perimeter is shown. In the statement, we assume that E has finite measure in order to give a meaning to the left hand side in (5.27) below. But, notice that, since E has finite perimeter in Ω, then by the relative isoperimetric inequality in the regular set Ω

$$\min\{|E\cap\Omega|,|\Omega\setminus E|\}\leq k\mathcal{P}(E,\Omega)^{n/n-1},$$

either $|E\cap\Omega|$ or $|\Omega\setminus E|$ is finite. Therefore, if $|E\cap\Omega|$ is infinite, then $|\Omega\setminus E|$ is finite and (5.27) applies with $\Omega\setminus E$ in place of E.

Proposition 5.3.1. *Assume that Ω be as in (2.2). Let $\mathcal{B}$ be as in (2.5), and consider $\mathcal{A}_0 = \mathrm{div}(AD)$, with $A = (a_{ij})_{ij}$ satisfying (2.107)–(2.108); let $(T_0(t))_{t\geq 0}$ be the semigroup generated by the realization of $\mathcal{A}_0$ in $L^1(\Omega)$ with homogeneous boundary condition $\mathcal{B}u = 0$; then, if $E, F \subset \mathbf{R}^n$ are sets of finite perimeter in Ω, the following holds*

$$\lim_{t\to 0}\sqrt{\frac{\pi}{t}}\int_{\Omega\cap F}(\chi_E(x) - T_0(t)\chi_E(x))dx = \int_{\Omega\cap\mathcal{F}F\cap\mathcal{F}E}\langle A(x)\nu_E(x), \nu_F(x)\rangle d\mathcal{H}^{n-1}(x).$$

Proof. We have

$$\begin{aligned}\int_{\Omega\cap F}(T_0(t)\chi_E(x) - \chi_E(x))dx &= \int_{\Omega\cap F}\int_0^t \frac{d}{ds}T_0(s)\chi_E(x)dsdx\\ &= \int_0^t\int_{\Omega\cap F}\mathcal{A}_0T_0(s)\chi_E(x)dxds\\ &= \int_0^t\int_{\Omega\cap F}\mathrm{div}_x(A(x)D_xT_0(s)\chi_E(x))dxds\end{aligned}$$

We introduce now the kernel $p_{0,*}(t,x,y)$ of the semigroup generated by the adjoint operator $\mathcal{A}_0^*$ of $\mathcal{A}_0$; by the symmetry of the matrix A, the operator $\mathcal{A}_0^* = \mathcal{A}_0$. In this way we have that $p_0(t,x,y) = p_{0,*}(t,y,x)$ (see for instance [45, Theorem 5.6]) and since

$$\begin{aligned}D_{x_i}p_0(t,x,y) = \lim_{h\to 0}\frac{p_0(t,x+he_i,y) - p_0(t,x,y)}{h} &= \lim_{h\to 0}\frac{p_{0,*}(t,y,x+he_i) - p_{0,*}(t,y,x)}{h}\\ &= s^{-n/2}\lim_{h\to 0}\frac{p_{0,*}^{s,x_0}(\frac{t}{s},\frac{y-x_0}{\sqrt{s}},\frac{x-x_0}{\sqrt{s}}+\frac{he_i}{\sqrt{s}}) - p_{0,*}^{s,x_0}(\frac{t}{s},\frac{y-x_0}{\sqrt{s}},\frac{x-x_0}{\sqrt{s}})}{h}\\ &= s^{-(n+1)/2}D_2^i p_{0,*}^{s,x_0}\Big(\frac{t}{s},\frac{y-x_0}{\sqrt{s}},\frac{x-x_0}{\sqrt{s}}\Big)\end{aligned}$$

where D_2^i denotes the i-th component of the gradient with respect to the second variables. Then for $t = s$ and $x = x_0$, $D_xp_0(t,x,y) = t^{-(n+1)/2}D_2p_{0,*}^{t,x}(1,\frac{y-x}{\sqrt{t}},0)$; hence integrating by parts we get

$$\begin{aligned}\int_{\Omega\cap F}\mathrm{div}(AD_xT_0(s)\chi_E(x))dx &= \int_{\Omega\cap\mathcal{F}F}\langle D_xT_0(s)\chi_E(x), A(x)\nu_F(x)\rangle d\mathcal{H}^{n-1}(x)\\ &= \int_{\Omega\cap\mathcal{F}F}\int_{\Omega\cap E}\langle D_xp_0(s,x,y), A(x)\nu_F(x)\rangle\, dy\, d\mathcal{H}^{n-1}(x)\\ &= \int_{\Omega\cap\mathcal{F}F}\int_{\Omega\cap E}s^{-(n+1)/2}\Big\langle D_2p_{0,*}^{s,x}\Big(1,\frac{y-x}{\sqrt{s}},0\Big), A(x)\nu_F(x)\Big\rangle dyd\mathcal{H}^{n-1}(x)\\ &= -\frac{1}{\sqrt{s}}\int_{\Omega\cap\mathcal{F}F}\int_{\Omega^{s,x}\cap E^{s,x}}\langle D_2p_{0,*}^{s,x}(1,z,0), A(x)\nu_F(x)\rangle dzd\mathcal{H}^{n-1}(x)\\ &= -\frac{1}{\sqrt{s}}\int_{\Omega\cap\mathcal{F}F}\int_{\mathbf{R}^n}\langle D_2p_{0,*}^{s,x}(1,z,0), A(x)\nu_F(x)\rangle d\mu^{s,x}(z)d\mathcal{H}^{n-1}(x), \end{aligned}\tag{5.25}$$

where we have denoted by $\mu^{s,x}$ the measure

$$\mu^{s,x} = \mathcal{L}^n \llcorner (\Omega^{s,x}\cap E^{s,x})\,. \tag{5.26}$$

These measures verify the following properties:

1. $\mu^{s,x} \xrightarrow{w^*_{\text{loc}}} 0$ if $x \in E^0$:

2. $\mu^{s,x} \xrightarrow{w^*_{\text{loc}}} \mathcal{L}^n$ if $x \in E^1$;

3. $\mu^{s,x} \xrightarrow{w^*_{\text{loc}}} \mathcal{L}^n \llcorner H_{\nu_E(x)}$ for $x \in \mathcal{F}E$, where $H_{\nu_E(x)} = \{z \in \mathbf{R}^n : \langle z, \nu_E(x) \rangle \le 0\}$.

These facts imply that, for $x \in E^0$, $\displaystyle\int_{\mathbf{R}^n} \big\langle D_2 p^{s,x}_{0,*}(1,z,0), A(x)\nu_F(x)\big\rangle d\mu^{s,x}(z) \to 0$; indeed

$$\begin{aligned} \Big| \int_{\mathbf{R}^n} &\langle D_2 p^{s,x}_{0,*}(1,z,0), A(x)\nu_F(x)\rangle \, d\mu^{s,x}(z) \Big| \\ &= s^{(n+1)/2} \Big| \int_{\mathbf{R}^n} \langle D_x p_0(s,x,z\sqrt{s}+x), A(x)\nu_F(x)\rangle \, d\mu^{s,x}(z) \Big| \\ &\le c_1 \|A\|_\infty \int_{\mathbf{R}^n} e^{-b|z|^2} \, d\mu^{s,x}(z). \end{aligned}$$

Now, let $\varepsilon > 0$ be given, we consider $\eta \in C^\infty_c(\mathbf{R}^n)$ such that $\displaystyle\int_{\mathbf{R}^n} (1-\eta)\, e^{-b|z|^2} \, d\mu^{s,x}(z) \le \varepsilon$, then there exists $s_0 > 0$ such that if $|s| < s_0$

$$\int_{\mathbf{R}^n} e^{-b|z|^2} \, d\mu^{s,x}(z) = \int_{\mathbf{R}^n} \eta\, e^{-b|z|^2} \, d\mu^{s,x}(z) + \int_{\mathbf{R}^n} (1-\eta)\, e^{-b|z|^2} \, d\mu^{s,x}(z) < \varepsilon.$$

Moreover, for $x \in E^1$

$$\begin{aligned} \int_{\Omega^{s,x}} \chi_{E^{s,x}}(z) \big\langle D_2 p^{s,x}_{0,*}(1,z,0), A(x)\nu_F(x)\big\rangle \, dz &= \int_{\Omega^{s,x}} \big\langle D_2 p^{s,x}_{0,*}(1,z,0), A(x)\nu_F(x)\big\rangle \, dz \\ &+ \int_{\Omega^{s,x}\setminus E^{s,x}} \big\langle D_2 p^{s,x}_{0,*}(1,z,0), A(x)\nu_F(x)\big\rangle \, dz \end{aligned}$$

Now,

$$\int_{\Omega^{s,x}} \big\langle D_2 p^{s,x}_{0,*}(1,z,0), A(x)\nu_F(x)\big\rangle \, dz \to A(x)\nu_F(x) \cdot \int_{\mathbf{R}^n} D_2 p^x_{0,*}(1,z,0) dz = 0$$

and for every $\varepsilon > 0$ there exists t_0 small enough, such that for $|s| < t_0$, by (B.2)

$$\int_{\Omega^{s,x}\setminus E^{s,x}} |D_2 p^{s,x}_{0,*}(1,z,0)| \, dz < \varepsilon$$

therefore if $x \in E^1$ and s is sufficiently small,

$$\frac{1}{\sqrt{t}} \int_0^t \frac{1}{\sqrt{s}} \int_{\Omega^{s,x}\cap E^{s,x}} \big\langle D_2 p^{s,x}_{0,*}(1,z,0), A(x)\nu_F(x)\big\rangle \, dz < 2\varepsilon.$$

Now, taking into account that $\mathcal{H}^{n-1}(\partial^* E \setminus \mathcal{F}E) = 0$, we can consider only points $x \in \mathcal{F}F \cap \mathcal{F}E$; in this case we obtain that

$$\int_{\mathbf{R}^n} \big\langle D_2 p^{s,x}_{0,*}(1,z,0), A(x)\nu_F(x)\big\rangle d\mu^{s,x}(z) \longrightarrow \int_{H_{\nu_E(x)}} \big\langle D_2 p^x_{0,*}(1,z,0), A(x)\nu_F(x)\big\rangle dz.$$

Taking into account (5.23) and the symmetry of A, we get that

$$D_2 p^x_{0,*}(1,z,0) = -\frac{1}{2(4\pi)^{n/2}|\det A^{1/2}(x)|}\exp\left(-\left\langle A^{-1}(x)z,z\right\rangle/4\right)A^{-1}(x)z,$$

and then, since for $x \in \mathcal{F}F \cap \mathcal{F}E$ we have $\nu_F(x) = \langle \nu_E(x), \nu_F(x)\rangle \nu_E(x)$

$$\int_{H_{\nu_E(x)}} \left\langle D_2 p^x_{0,*}(1,z,0), A(x)\nu_F(x)\right\rangle dz =$$

$$= -\frac{\langle \nu_E(x), \nu_F(x)\rangle}{2(4\pi)^{n/2}|\det A^{1/2}(x)|}\int_{H_{\nu_E(x)}} \exp\left(-\left\langle A^{-1}(x)z,z\right\rangle/4\right)\langle z, \nu_E(x)\rangle dz$$

$$= -\frac{\langle \nu_E(x), \nu_F(x)\rangle}{\pi^{n/2}}\int_{H_{A^{1/2}(x)\nu_E(x)}} e^{-|z|^2}\left\langle z, A^{1/2}(x)\nu_E(x)\right\rangle dz.$$

For the computation of this last integral, we consider an orthonormal basis $\{e_1, \ldots, e_n\}$ of $\mathbf{R}^n$ with

$$e_n = \frac{A^{1/2}(x)\nu_E(x)}{|A^{1/2}(x)\nu_E(x)|};$$

we then obtain

$$\int_{H_{A^{1/2}(x)\nu_E(x)}} \left\langle z, A^{1/2}(x)\nu_E(x)\right\rangle e^{-|z|^2} dz = |A^{1/2}(x)\nu_E(x)|\int_{H_{A^{1/2}(x)\nu_E(x)}} z_n e^{-|z|^2} dz$$

$$= \pi^{(n-1)/2}|A^{1/2}(x)\nu_E(x)|\int_{-\infty}^0 z_n e^{-z_n^2} dz_n = -\frac{\pi^{(n-1)/2}}{2}|A^{1/2}(x)\nu_E(x)|.$$

At the end, we have obtained that

$$\lim_{t\to 0}\sqrt{\frac{\pi}{t}}\int_{\Omega\cap F}(T_0(t)\chi_E - \chi_E)dx = -\int_{\Omega\cap\mathcal{F}F\cap\mathcal{F}E}\langle \nu_E, \nu_F\rangle |A^{1/2}\nu_E| d\mathcal{H}^{n-1}.$$

□

With a perturbation argument we establish the result stated in Proposition 5.3.1 for the semigroup $T(t)$ generated by the complete operator $(A_1, D(A_1))$ in $L^1(\Omega)$.

Theorem 5.3.2. *Assume Ω, $\mathcal{B}$ be as in Proposition 5.3.1 and let $\mathcal{A}$ be as in* (5.1) *with coefficients satisfying* (5.2). *Denote by $T(t)$ the semigroup generated by $(A_1, D(A_1))$ in $L^1(\Omega)$, then, if $E, F \subset \mathbf{R}^n$ are sets of finite perimeter in Ω, the following holds*

$$\lim_{t\to 0}\sqrt{\frac{\pi}{t}}\int_{\Omega\cap F}(\chi_E(x) - T(t)\chi_E(x))dx = \int_{\Omega\cap\mathcal{F}F\cap\mathcal{F}E}\langle A(x)\nu_E(x), \nu_F(x)\rangle d\mathcal{H}^{n-1}(x). \qquad (5.27)$$

Proof. By using Proposition 5.3.1 and the notations fixed there, it suffices to prove that

$$\lim_{t\to 0}\frac{1}{\sqrt{t}}\int_{\Omega\cap F}(T_0(t)\chi_E(x) - T(t)\chi_E(x))dx = 0. \qquad (5.28)$$

In order to get the claim, we set $u(t,x) = (T(t)\chi_E)(x)$ and $v(t,x) = (T_0(t)\chi_E)(x)$, so that the function $z = u - v$ solves the problem

$$\begin{cases} \partial_t z - \mathcal{A}z = \langle B, Dv\rangle + cv & t > 0,\, x \in \Omega \\ z(0) = 0 & x \in \Omega \\ \langle ADz, \nu\rangle = 0 & t > 0,\, x \in \partial\Omega \end{cases}$$

and can be written as follows

$$u - v = \int_0^t T(t-s)\big(\langle B, Dv(s)\rangle + cv(s)\big)\, ds. \tag{5.29}$$

Using (3.1) we have

$$\|u - v\|_{L^1(\Omega)} \le c_0 \int_0^t \Big(\|\langle B, Dv(s)\rangle\|_{L^1(\Omega)} + \|cv(s)\|_{L^1(\Omega)}\Big) ds. \tag{5.30}$$

If we prove that

$$\|u - v\|_{L^1(\Omega)} = o(\sqrt{t}) \quad \text{as } t \to 0 \tag{5.31}$$

we conclude. For the last term in (5.30) we have that

$$\int_\Omega |c(x) T_0(s)\chi_E(x)|\, dx \le c_0 \|c\|_\infty |\Omega \cap E|$$

and then

$$\lim_{t\to 0} \frac{1}{\sqrt{t}} \int_0^t \int_\Omega |c(x) T_0(s)\chi_E(x)|\, dx ds = 0.$$

For the first term in (5.30), we notice that

$$\left| \int_\Omega \int_{\Omega\cap E} \langle B(x), D_x p_0(s,x,y)\rangle dy dx \right| \le \|B\|_\infty |\Omega \cap E| \int_\Omega |D_x p_0(s,x,y)| dx$$

and using Gaussian estimates (B.2) we get

$$\int_\Omega |D_x p_0(s,x,y)| dx \le \frac{C}{\sqrt{s}}$$

for some constant C depending only on the operator $\mathcal{A}$ and the dimension n. However we can write

$$\begin{aligned}
\int_\Omega \langle B, D_x T_0(s)\chi_E\rangle dx &= \int_\Omega dx \int_{\Omega\cap E} \langle B(x), D_x p_0(s,x,y)\rangle dy \\
&= s^{-(n+1)/2} \int_\Omega dx \int_{\Omega\cap E} \left\langle B(x), D_2 p_{0,*}^{s,x}\left(1, \frac{y-x}{\sqrt{s}}, 0\right)\right\rangle dy \\
&= \frac{1}{\sqrt{s}} \int_\Omega dx \int_{\Omega^{s,x}\cap E^{s,x}} \big\langle B(x), D_2 p_{0,*}^{s,x}(1,z,0)\big\rangle dz \\
&= \frac{1}{\sqrt{s}} \int_\Omega dx \int_{\mathbf{R}^n} \big\langle B(x), D_2 p_{0,*}^{s,x}(1,z,0)\big\rangle d\mu^{s,x}(z).
\end{aligned}$$

where $\mu^{s,x}$ is defined in (5.26) and satisfies 1., 2. and 3. of Proposition 5.3.1. With the same argument previously used, we can deduce that for $x \in E^0 \cup E^1$, the limit of the above integral as $t \to 0$ vanishes; then, taking into account that $|\Omega \setminus (E^0 \cup E^1)| = 0$, we have then obtained that

$$\lim_{t\to 0} \frac{1}{\sqrt{t}} \int_0^t \int_\Omega \int_{\Omega\cap E} |\langle B(x), D_x p_0(s,x,y)\rangle|\, dy dx ds = 0$$

for $\mathcal{H}^{n-1}$-a.a. $x \in E^0 \cup E^1$. Therefore (5.31) is proved and the proof is complete $\square$

Specializing the above result for $F = E^c$ we get the following

Corollary 5.3.3. *Under assumption of Theorem 5.3.1, let $(T(t))_{t\geq 0}$ be the semigroup generated by $(A_1, D(A_1))$ in $L^1(\Omega)$; then, if $E \subset \mathbf{R}^n$ is a set with finite perimeter in Ω, the following equality holds:*

$$\lim_{t\to 0}\sqrt{\frac{\pi}{t}}\int_{\Omega\cap E^c} T(t)\chi_E dx = \int_{\Omega\cap\mathcal{F}E} |A^{1/2}(x)\nu_E(x)| d\mathcal{H}^{n-1}(x). \tag{5.32}$$

Using an argument similar to the one used in [33, Theorem 3.4] and the lower bound for the kernel $p(t,x,y)$, it is possible to prove the converse of the statement in Corollary 5.3.3.

Proposition 5.3.4. *Let $E \subset \mathbf{R}^n$ be a set such that either E or E^c has finite measure in Ω, and such that*

$$\liminf_{t\to 0}\frac{1}{\sqrt{t}}\int_{E^c\cap\Omega} T(t)\chi_E(x)dx < +\infty,$$

then E has finite perimeter in Ω, that is $\chi_E \in BV(\Omega)$.

Proof. Define $E_\Omega := E\cap\Omega$ and assume $|E_\Omega| < \infty$. From (B.17) we have

$$\begin{aligned}
\frac{1}{\sqrt{t}}\int_{E^c} T(t)\chi_E(x)\,dx &= \int_\Omega\int_\Omega p(t,x,y)\chi_E(y)\chi_{E^c}(x)\,dy\,dx \\
&\geq \frac{C_1}{t^{(n+1)/2}}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} e^{-c_1\frac{|x-y|^2}{t}}\chi_{E_\Omega}(y)\big(\chi_\Omega(x)-\chi_E(x)\big)\,dy\,dx \\
&= \frac{C_1}{\sqrt{t}}\int_{\mathbf{R}^n} e^{-c_1|z|^2}\int_{\mathbf{R}^n}\chi_\Omega(x)\chi_{E_\Omega}(z\sqrt{t}+x)(1-\chi_E(x)\big)\,dx\,dz \\
&= \frac{C_1}{\sqrt{t}}\int_{\mathbf{R}^n} e^{-c_1|z|^2}\int_{\mathbf{R}^n}\chi_\Omega(x)\big(\chi_{E_\Omega-z\sqrt{t}}(x)-\chi_{E_\Omega-z\sqrt{t}}(x)\chi_E(x)\big)\,dx\,dz \\
&= C_1\int_{\mathbf{R}^n} e^{-c_1|z|^2}|z|\frac{|(E_\Omega\Delta(E_\Omega-z\sqrt{t}))\cap\Omega|}{\sqrt{t}|z|}\,dz
\end{aligned}$$

In fact, denoting by

$$|D_\nu\chi_E|(\Omega) = \liminf_{t\to 0}\frac{|(E\Delta(E-t\nu))\cap\Omega|}{t},$$

by assumption we get that

$$\begin{aligned}
&\int_{\mathbf{R}^n}|z|e^{-c_1|z|^2}|D_{\frac{z}{|z|}}\chi_{E_\Omega}|(\Omega)\,dz \\
&\qquad\leq \liminf_{t\to 0}\frac{1}{\sqrt{t}}\int_{\Omega\times\Omega}\chi_E(y)\chi_{E^c}(x)p(t,x,y)dxdy < +\infty.
\end{aligned}$$

This implies, using an argument similar to that used in Theorem 5.1.6, that there exist $M > 0$ such that $|D_{e_i}\chi_{E_\Omega}|(\Omega) \leq M$ for $i = 1,\dots,n$. Finally, let $\varphi \in C^1_c(\Omega,\mathbf{R}^n)$; then

$$\int_\Omega \chi_E(x)D_i\varphi(x)\,dx = \lim_{t\to 0^+}\int_\Omega \chi_E(x)\frac{\varphi(x+te_i)-\varphi(x)}{t}\,dx$$

But

$$\Big|\int_\Omega \chi_E(x)\frac{\varphi(x+te_i)-\varphi(x)}{t}\Big| = \Big|\int_\Omega \frac{\chi_{E_\Omega+te_i}(x)-\chi_{E_\Omega}(x)}{t}\varphi(x)\,dx\Big|$$
$$\leq \|\varphi\|_{L^\infty(\Omega)}\frac{|(E_\Omega\Delta(E_\Omega+te_i))\cap\Omega|}{t}$$

Thus

$$\Big|\int_\Omega \chi_E(x)D_i\varphi(x)\,dx\Big| \leq \|\varphi\|_{L^\infty(\Omega)}\liminf_{t\to 0^+}\frac{|(E_\Omega\Delta(E_\Omega+te_i))\cap\Omega|}{t}$$
$$= \|\varphi\|_{L^\infty(\Omega)}|D_{e_i}\chi_{E_\Omega}|(\Omega) \leq M\|\varphi\|_{L^\infty(\Omega)}$$

and

$$\int_\Omega \chi_E(x)\mathrm{div}\varphi(x)\,dx \leq nM\|\varphi\|_{L^\infty(\Omega)}$$

that is $|D\chi_E|(\Omega) < +\infty$. □

We are now in a position to prove the main result of this section, namely, the announced characterization of BV functions (5.4). The strategy is the same as for $\mathbf{R}^n$ and is based on (4.13).

Theorem 5.3.5. *Let Ω, $\mathcal{A}$, $\mathcal{B}$ be as in Theorem 5.3.2, let $(T(t))_{t\geq 0}$ be the semigroup generated by $(A_1, D(A_1))$ in $L^1(\Omega)$ and let $u\in L^1(\Omega)$; then $u\in BV(\Omega)$ if and only if*

$$\liminf_{t\to 0}\frac{1}{\sqrt{t}}\int_{\Omega\times\Omega}|u(x)-u(y)|p(t,x,y)dxdy < +\infty;$$

moreover, in this case the following equality holds

$$|Du|_A(\Omega) = \lim_{t\to 0}\frac{\sqrt{\pi}}{2\sqrt{t}}\int_{\Omega\times\Omega}|u(x)-u(y)|p(t,x,y)dxdy. \tag{5.33}$$

Proof. The "if" part. We start by considering $u\in L^1(\Omega)$; for $\tau\in\mathbf{R}$ we denote by $E_\tau = \{u>\tau\}$ and, since the semigroup is positive and contractive, we obtain that

$$\begin{aligned}
0 &\leq \int_{\mathbf{R}}\liminf_{t\to 0}\frac{1}{\sqrt{t}}\int_{E_\tau^c\cap\Omega}T(t)\chi_{E_\tau}dxd\tau \leq \liminf_{t\to 0}\frac{1}{\sqrt{t}}\int_{\mathbf{R}}\int_{E_\tau^c\cap\Omega}T(t)\chi_{E_\tau}dxd\tau\\
&\leq \liminf_{t\to 0}\frac{1}{\sqrt{t}}\int_{\Omega\times\Omega}\int_{\mathbf{R}}|\chi_{E_\tau}(x)-\chi_{E_\tau}(y)|p(t,x,y)dxdyd\tau\\
&= \liminf_{t\to 0}\frac{1}{\sqrt{t}}\int_{\Omega\times\Omega}|u(x)-u(y)|p(t,x,y)dxdy < +\infty
\end{aligned}$$

and then, thanks to Proposition 5.3.4, almost every level E_τ has finite perimeter and equation (5.32) holds. Then, using coarea formula (4.13), we get

$$\begin{aligned}
|Du|_A(\Omega) &= \int_{\mathbf{R}}\mathcal{P}_A(E_\tau,\Omega)d\tau = \int_{\mathbf{R}}\lim_{t\to 0}\sqrt{\frac{\pi}{t}}\int_{E_\tau^c\cap\Omega}T(t)\chi_{E_\tau}dxd\tau\\
&\leq \liminf_{t\to 0}\sqrt{\frac{\pi}{t}}\int_{\Omega\times\Omega}|u(x)-u(y)|p(t,x,y)dxdy < +\infty
\end{aligned}$$

that is $u \in BV_A(\Omega)$.
The other implication follows from (5.33). To prove (5.33), we define the function

$$g_t(\tau) = \sqrt{\frac{\pi}{t}} \int_{E_\tau^c \cap \Omega} T(t)\chi_{E_\tau}(x)dx.$$

For this function we have the following estimate

$$\begin{aligned}
|g_t(\tau)| =& \sqrt{\frac{\pi}{t}} \Big| \int_0^t \int_{E_\tau^c\cap\Omega} \mathcal{A}T(s)\chi_{E_\tau} dx ds\Big| = \sqrt{\frac{\pi}{t}} \Big| \int_0^t \Big(\int_{\mathcal{F}E_\tau \cap \Omega} \langle ADT(s)\chi_{E_\tau}, \nu_{E_\tau}\rangle d\mathcal{H}^{n-1} \\
&+ \int_{E_\tau^c\cap\Omega} \langle B, DT(s)\chi_{E_\tau}\rangle dx + \int_{E_\tau^c\cap\Omega} cT(s)\chi_{E_\tau})dx\Big) ds\Big| \\
\leq& \sqrt{\frac{\pi}{t}} \int_0^t \Big(\|A\|_\infty \int_{\mathcal{F}E_\tau} |DT(s)\chi_{E_\tau}| d\mathcal{H}^{n-1} \\
&+ \|B\|_\infty \int_{E_\tau^c\cap\Omega} \int_{E_\tau\cap\Omega} |D_x p(s,x,y)| dxdy + \|c\|_\infty \int_{E_\tau^c\cap\Omega} \int_{E_\tau\cap\Omega} |p(s,x,y)| dxdy \Big) ds \\
\leq& cM_0(\mathcal{P}(E_\tau, \Omega) + \min\{|E_\tau \cap \Omega|, |E_\tau^c \cap \Omega|\}) = h(\tau)
\end{aligned}$$

where the last inequality follows from the estimates (B.2) on the kernel $p(s, x, y)$. We have that $h \in L^1(\mathbf{R})$ since

$$\int_{\mathbf{R}} \mathcal{P}(E_\tau, \Omega) d\tau = |Du|(\Omega)$$

and, denoted by $u^+ = \max\{u, 0\}$ and $u^- = \max\{-u, 0\}$,

$$\begin{aligned}
\int_{\mathbf{R}} \min\{|E_\tau \cap \Omega|, |E_\tau^c \cap \Omega|\} d\tau \leq& \int_0^\infty |E_\tau \cap \Omega| d\tau + \int_{-\infty}^0 |E_\tau^c \cap \Omega| d\tau \\
=& \int_0^\infty \int_\Omega \chi_{E_\tau} dx d\tau + \int_{-\infty}^0 \int_\Omega \chi_{E_\tau^c} dx d\tau \\
=& \int_\Omega \int_0^\infty \chi_{\{u>\tau\}} d\tau dx + \int_\Omega \int_0^\infty \chi_{\{-u\geq\tau\}} d\tau dx \\
=& \int_\Omega u^+ dx + \int_\Omega u^- dx = \int_\Omega |u| dx.
\end{aligned}$$

Then we can apply Corollary 5.3.3 and Lebesgue dominated convergence theorem to the functions g_t in order to obtain

$$\begin{aligned}
|Du|_A(\Omega) &= \int_{\mathbf{R}} \mathcal{P}_A(E_\tau, \Omega) d\tau = \int_{\mathbf{R}} \lim_{t\to 0} \sqrt{\frac{\pi}{t}} \int_{E_\tau^c\cap\Omega} T(t)\chi_{E_\tau} dx \\
&= \lim_{t\to 0} \sqrt{\frac{\pi}{t}} \int_{\mathbf{R}} \int_{\Omega\times\Omega} (\chi_{E_\tau}(y) - \chi_{E_\tau}(y)\chi_{E_\tau}(x)) p(t,x,y) dxdyd\tau \\
&= \lim_{t\to 0} \sqrt{\frac{\pi}{t}} \int_{\Omega\times\Omega} (u(y) - \min\{u(y), u(x)\}) p(t,x,y) dxdy
\end{aligned}$$

since $\chi_{E_\tau}(y)\chi_{E_\tau}(x) \neq 0$ if and only if $\tau < \min\{u(x), u(y)\}$; finally, the assertion follows by noticing that $\min\{u(y), u(x)\} = \frac{1}{2}(u(x) + u(y) - |u(x) - u(y)|)$. $\square$

Appendix A

A brief introduction to interpolation theory

A.1 Interpolation spaces

This appendix is devoted to present an elementary treatment of the interpolation theory. This theory has a wide range of applications to partial differential operators and partial differential equations. We have used interpolation techniques in Chapter 3. In particular, Theorem 3.1.2 relies on Theorem A.2.7 and both have been proved in [6]. The most known and useful families of interpolation spaces are the real and the complex interpolation spaces.
Let X, Y be two real or complex Banach spaces. By $X = Y$ we mean that X and Y have the same elements with equivalence of the norms. By $Y \hookrightarrow X$ we mean that Y is continuously embedded in X.
Suppose that $Y \hookrightarrow X$; we say that D is an intermediate space between X and Y if

$$Y \hookrightarrow D \hookrightarrow X.$$

An interpolation space between X and Y is any intermediate space such that for every $T \in \mathcal{L}(X)$, whose restriction to Y belongs to $\mathcal{L}(Y)$, the restriction to D belongs to $\mathcal{L}(D)$. Another important class of intermediate spaces are the space of class J_α.

Definition A.1.1. *An intermediate space D between X and Y is said to be of class J_α if there exists a constant $C > 0$ such that*

$$\|y\|_D \leq C\|y\|_Y^\alpha \|y\|_X^{1-\alpha}, \quad y \in Y.$$

In this case we write $D \in J_\alpha(X, Y)$.

A.1.1 Some interpolation estimates

In the next section some important examples of interpolatory inclusion are shown. First we prove a useful interpolation estimate which allows us to estimate the L^p norm of the gradient of a function with respect to the L^p norm of the function and of its second derivatives. For a more general statement see [1, Theorem 4.17].

Proposition A.1.2. *Let $1 \leq p < \infty$, then $W^{1,p}(\mathbf{R}^n)$ is of class $J_{1/2}$ between $L^p(\mathbf{R}^n)$ and $W^{2,p}(\mathbf{R}^n)$. In other words*

$$\|Du\|_{L^p(\mathbf{R}^n)} \leq c\|D^2u\|^{1/2}_{L^p(\mathbf{R}^n)}\|u\|^{1/2}_{L^p(\mathbf{R}^n)} \tag{A.1}$$

for $u \in W^{2,p}(\mathbf{R}^n)$ and some constant $c > 0$.

Proof. We first consider the one-dimensional case. Let $u \in C_c^\infty(\mathbf{R})$ and $x \in \mathbf{R}$; then for $h > 0$

$$u(x+h) = u(x) + hu'(x) + \int_0^h (h-s)u''(s+x)ds$$

hence

$$u'(x) = \frac{u(x+h)-u(x)}{h} - \frac{1}{h}\int_0^h (h-s)u''(s+x)ds.$$

Taking the L^p norm we get

$$\begin{aligned}\|u'\|_{L^p(\mathbf{R})} &\leq \frac{2}{h}\|u\|_{L^p(\mathbf{R})} + \frac{1}{h}\int_0^h (h-s)\|u''(s+\cdot)\|_{L^p(\mathbf{R})}ds \\ &= \frac{2}{h}\|u\|_{L^p(\mathbf{R})} + \frac{h}{2}\|u''\|_{L^p(\mathbf{R})}\end{aligned}$$

Let $\varepsilon = \frac{2}{h}$ then

$$\|u'\|_{L^p(\mathbf{R})} \leq \varepsilon\|u\|_{L^p(\mathbf{R})} + \frac{1}{\varepsilon}\|u''\|_{L^p(\mathbf{R})}. \tag{A.2}$$

Now, let $u \in C_c^\infty(\mathbf{R}^n)$, then by (A.2) we get

$$\int_{\mathbf{R}} |D_iu|^p dx_i \leq 2^{p-1}\left(\varepsilon^p \int_{\mathbf{R}} |D_{ii}u|^p dx_i + \frac{1}{\varepsilon^p}\int_{\mathbf{R}} |u|^p dx_i\right),$$

and by Fubini's Theorem

$$\int_{\mathbf{R}^n} |D_iu|^p dx \leq 2^{p-1}\left(\varepsilon^p \int_{\mathbf{R}^n} |D_{ii}u|^p dx + \frac{1}{\varepsilon^p}\int_{\mathbf{R}^n} |u|^p dx\right)$$

therefore

$$\|D_iu\|_{L^p(\mathbf{R}^n)} \leq c\left(\varepsilon\|D^2u\|_{L^p(\mathbf{R}^n)} + \frac{1}{\varepsilon}\|u\|_{L^p(\mathbf{R}^n)}\right) \tag{A.3}$$

holds for every $\varepsilon > 0$ and some constant c depending only on p. Minimizing (A.3) on ε, we get

$$\|Du\|_{L^p(\mathbf{R}^n)} \leq 2c\|D^2u\|^{1/2}_{L^p(\mathbf{R}^n)}\|u\|^{1/2}_{L^p(\mathbf{R}^n)}$$

for every $u \in C_c^\infty(\mathbf{R}^n)$. Finally the estimate can be extended by density to $W^{2,p}(\mathbf{R}^n)$. □

A.2 Real interpolation spaces

Let X, Y be Banach spaces, with $Y \hookrightarrow X$ (in general this is not required; however, this simplifying assumption is satisfied in the case we are investigating). We describe briefly the K-method used to construct a family of intermediate spaces between X and Y, called real interpolation spaces and denoted by $(X,Y)_{\theta,p}$, where $0 < \theta \le 1$ and $1 \le p \le \infty$. Let I be any interval contained in $(0,+\infty)$, $1 \le p < \infty$. We denote by $L^p_*(I)$ the Lebesgue space L^p with respect to the measure $\frac{dt}{t}$ in I. If $p = \infty$, $L^\infty_*(I) = L^\infty(I)$. We set $1/\infty = 0$.

Definition A.2.1. *For every $x \in X$ and $t > 0$, set*

$$K(t,x,X,Y) = \inf_{x=a+b, a\in X, b\in Y} (\|a\|_X + t\|b\|_Y).$$

Now we define a family of intermediate spaces by means of the function K.

Definition A.2.2. *Let $0 < \theta \le 1$, $1 \le p \le \infty$, set*

$$(X,Y)_{\theta,p} = \{x \in X : \, t \mapsto t^{-\theta} K(t,x,X,Y) \in L^p_*(0,+\infty)\}$$

with

$$\|x\|_{\theta,p} = \|t^{-\theta} K(t,x,X,Y)\|_{L^p_*((0,+\infty))}$$

and

$$(X,Y)_\theta = \{x \in X : \lim_{t\to 0^+} t^{-\theta} K(t,x,X,Y) = 0\}$$

Definition (A.2.2) concerns only the behavior of $t^{-\theta}K(t,x,X,Y)$ as $t \to 0$, since $K(\cdot,x,X,Y)$ is bounded. Moreover since $K(t,x,X,Y) \ge \min\{1,t\}K(1,x,X,Y)$, for $\theta = 1$ we deduce that

$$(X,Y)_{1,p} = \{0\}, \quad p < \infty.$$

Therefore, henceforth we consider the cases $(\theta,p) \in (0,1)\times[1,+\infty]$ and $(\theta,p) = (1,\infty)$. Such spaces are called *real interpolation spaces*. One can prove that $\|x\|_{(X,Y)_{\theta,p}}$ is a norm in $(X,Y)_{\theta,p}$ and that the following results hold (see [31] for their proof).

Proposition A.2.3. *For all $(\theta,p) \in (0,1)\times[1,+\infty]$ and $(\theta,p) = (1,\infty)$, $(X,Y)_{\theta,p}$ is a Banach space. For all $\theta \in (0,1)$, $(X,Y)_\theta$ is a Banach space, endowed with the norm of $(X,Y)_{\theta,\infty}$.*

The spaces $(X,Y)_{\theta,p}$ and $(X,Y)_\theta$ are of class $J_\theta(X,Y)$ for every $p \in [1,\infty]$. They are actually interpolation spaces, as they enjoy the following property.

Theorem A.2.4. *Let X_i, Y_i be Banach spaces such that $Y_i \hookrightarrow X_i$ for $i = 1,2$. Let $T \in \mathcal{L}(X_1,X_2) \cap \mathcal{L}(Y_1,Y_2)$. Then for every $\theta \in (0,1)$ and $p \in [1,\infty]$, we have*

$$T \in \mathcal{L}((X_1,Y_1)_{\theta,p},(X_2,Y_2)_{\theta,p}) \cap \mathcal{L}((X_1,Y_1)_\theta,(X_2,Y_2)_\theta)$$

and

$$\|T\|_{\mathcal{L}((X_1,Y_1)_{\theta,p},(X_2,Y_2)_{\theta,p})} \le (\|T\|_{\mathcal{L}(X_1,X_2)_{\theta,p}})^{1-\theta}(\|T\|_{\mathcal{L}(Y_1,Y_2)_{\theta,p}})^\theta$$

Finally we state without proof the duality theorem for the real method. A proof of it can be found in [46, Section 1.11.2].

Theorem A.2.5. *(Dual space) Let Y dense in X. If $0 < \theta < 1$ then for $1 \leq p < \infty$*

$$(X,Y)'_{\theta,p} = (Y',X')_{1-\theta,p'}, \quad \frac{1}{p} + \frac{1}{p'} = 1,$$

and for $p = \infty$

$$(X,Y)'_{\theta,\infty} = (Y',X')_{1-\theta,1}. \tag{A.4}$$

A.2.1 Examples

We close this section with concrete examples of some interpolation spaces. For $\theta \in (0,1)$, $p \in [1,\infty)$, $W^{\theta,p}(\mathbf{R}^n)$ is the space of all $f \in L^p(\mathbf{R}^n)$ such that

$$[f]_{W^{\theta,p}} = \Big(\int_{\mathbf{R}^n\times\mathbf{R}^n} \frac{|f(x)-f(y)|^p}{|x-y|^{\theta p+n}}dxdy\Big)^{1/p} < \infty.$$

It is endowed with the norm $\|\cdot\|_{L^p} + [\cdot]_{W^{\theta,p}}$. When $\theta > 1$ is not integer, let $[\theta]$ and $\{\theta\}$ be the integral and fractional parts of θ. Then $W^{\theta,p}(\mathbf{R}^n)$ consists of the functions $f \in W^{[\theta],p}(\mathbf{R}^n)$ such that

$$\sum_{|\alpha|=[\theta]} [D^\alpha f]_{W^{\{\theta\},p}}$$

is finite. Analogously in this case we consider the space $W^{\theta,p}$ normed by

$$\|\cdot\|_{W^{[\theta],p}} + \sum_{|\alpha|=[\theta]} [D^\alpha \cdot]_{W^{\{\theta\},p}}.$$

Example 2. *For $0 < \theta < 1$, $1 \leq p < \infty$ we have*

$$(C_b(\mathbf{R}^n), C_b^1(\mathbf{R}^n))_{\theta,\infty} = C_b^\theta(\mathbf{R}^n)$$

$$(L^p(\mathbf{R}^n), W^{1,p}(\mathbf{R}^n))_{\theta,p} = W^{\theta,p}(\mathbf{R}^n),$$

with equivalence of the respective norms.

Example 3. *Let $0 \leq \theta_1 < \theta_2 \leq 1$, $0 < \theta < 1$, $1 \leq p < \infty$. Then*

$$(W^{\theta_1,p}(\mathbf{R}^n), W^{\theta_2,p}(\mathbf{R}^n))_{\theta,p} = W^{(1-\theta)\theta_1+\theta\theta_2,p}(\mathbf{R}^n).$$

If Ω is an open set in $\mathbf{R}^n$ with uniformly C^1 boundary, then

$$(W^{\theta_1,p}(\Omega), W^{\theta_2,p}(\Omega))_{\theta,\infty} = W^{(1-\theta)\theta_1+\theta\theta_2,p}(\Omega). \tag{A.5}$$

Example 4. *For $0 < \theta < 1$, $1 \leq p,q < \infty$, $m \in \mathbf{N}$,*

$$(L^p(\mathbf{R}^n), W^{m,p}(\mathbf{R}^n))_{\theta,q} = B^{m\theta}_{p,q}(\mathbf{R}^n).$$

Here $B^s_{p,q}(\mathbf{R}^n)$ *is the Besov space defined as follows: if* s *is not an integer, let* $[s]$ *and* $\{s\}$ *be the integer and the fractional parts of* s*, respectively. Then* $B^s_{p,q}(\mathbf{R}^n)$ *consists of the functions* $f \in W^{[s],p}(\mathbf{R}^n)$ *such that*

$$[f]_{B^s_{p,q}} = \sum_{|\alpha|=[s]} \Big(\int_{\mathbf{R}^n} \frac{dh}{|h|^{n+\{s\}q}} \Big(\int_{\mathbf{R}^n} |D^\alpha f(x+h) - D^\alpha f(x)|^p dx \Big)^{q/p} \Big)^{1/q}$$

is finite. In particular, for $p = q$ *we have* $B^s_{p,p}(\mathbf{R}^n) = W^{s,p}(\mathbf{R}^n)$*. If* $s = k \in \mathbf{N}$*, then* $B^k_{p,q}(\mathbf{R}^n)$ *consists of the functions* $f \in W^{k-1,p}(\mathbf{R}^n)$ *such that*

$$[f]_{B^k_{p,q}} = \sum_{|\alpha|=k-1} \Big(\int_{\mathbf{R}^n} \frac{dh}{|h|^{n+q}} \Big(\int_{\mathbf{R}^n} |D^\alpha f(x+2h) - 2D^\alpha f(x+h) + D^\alpha f(x)|^p dx \Big)^{q/p}$$

is finite.

For a complete proof of Examples above see [46, Sections 2.3, 2.4].

Corollary A.2.6. *For* $0 < \theta < 1/2$*,* $1 \le p < \infty$*, we have*

$$(L^p(\mathbf{R}^n), W^{2,p}(\mathbf{R}^n))_{\theta,p} = W^{2\theta,p}(\mathbf{R}^n)$$

with equivalence of the respective norms.

In the following result we characterize the interpolation space between $L^1(\Omega)$ and a subspace of $W^{1,1}(\Omega)$ which takes into account in a suitable way the boundary conditions that are to be imposed in the parabolic of our interest.

Theorem A.2.7. *Let* Ω *be a subset of* $\mathbf{R}^n$ *with uniformly* C^2 *boundary; then for every* $\theta \in (0, 1/2)$ *we have*

$$(L^1(\Omega), W^{2,1}(\Omega) \cap W^{1,1}_{A,\nu}(\Omega))_{\theta,1} = W^{2\theta,1}(\Omega) \tag{A.6}$$

where $\nu(x)$ *denotes the external normal to* $\partial\Omega$ *at* x*,* A *is the matrix in* (2.106) *and* $W^{1,1}_{A,\nu}(\Omega)$ *is the closure of* $\{u \in C^1(\overline{\Omega}) \,|\, \langle A(x) \cdot \nabla u, \nu(x) \rangle = 0 \text{ for } x \in \partial\Omega\}$ *with respect to the topology of* $W^{1,1}(\Omega)$*.*

Proof. We define for an open and regular set $\omega \subset \mathbf{R}^n$ the space

$$X^A_\theta(\omega) = (L^1(\omega), W^{2,1}(\omega) \cap W^{1,1}_{A,\nu}(\omega))_{\theta,1}$$

endowed with the norm

$$\|u\|_{X^A_\theta(\omega)} := \int_0^{+\infty} \frac{K(t,u)}{t^{1+\theta}} dt, \quad K(t,u) := \inf_{\substack{a+b=u \\ a \in L^1(\omega) \\ b \in W^{2,1}(\omega) \cap W^{1,1}_{A,\nu}(\omega)}} \big(\|a\|_{L^1(\omega)} + t\|b\|_{W^{2,1}(\omega)} \big) \tag{A.7}$$

We want to prove that $X^A_\theta(\Omega) = W^{2\theta,1}(\Omega)$ for $\theta \in (0, 1/2)$. For the result in the case when $\omega = \mathbf{R}^n$ we refer to [9, Theorem 4.3.6].

We divide this proof in two steps, in the first one we prove that $X_\theta^I(\mathbf{R}^n_+) = W^{2\theta,1}(\mathbf{R}^n_+)$, where I is the identity matrix. In the second one, we use a local change of coordinates and the regularity of the domain Ω to conclude.

First step

We want to prove that

$$(L^1(\mathbf{R}^n_+), W^{2,1}(\mathbf{R}^n_+) \cap W^{1,1}_N(\mathbf{R}^n_+))_{\theta,1} = W^{2\theta,1}(\mathbf{R}^n_+) \tag{A.8}$$

where $W^{1,1}_N(\mathbf{R}^n_+)$ denote the space $W^{1,1}_{I,-e_n}(\mathbf{R}^n_+)$.

Fix $\theta \in (0,1/2)$ and consider T the operator that to any function $u : \mathbf{R}^n_+ \to \mathbf{R}$ associates

$$Tu = \tilde{u}(x) := \begin{cases} u(x_1, \dots, x_n) & \text{if } x_n \geq 0 \\ u(x_1, \dots, -x_n) & \text{if } x_n < 0. \end{cases} \tag{A.9}$$

As it is easily seen $T \in \mathcal{L}(L^1(\mathbf{R}^n_+), L^1(\mathbf{R}^n)) \cap \mathcal{L}(W^{1,1}_N(\mathbf{R}^n_+) \cap W^{2,1}(\mathbf{R}^n_+), W^{2,1}(\mathbf{R}^n))$; therefore applying Theorem A.2.4 we get

$$T \in \mathcal{L}\big((L^1(\mathbf{R}^n_+), W^{1,1}_N(\mathbf{R}^n_+) \cap W^{2,1}(\mathbf{R}^n_+))_{\theta,1}, (L^1(\mathbf{R}^n), W^{2,1}(\mathbf{R}^n))_{\theta,1}\big).$$

As a consequence we deduce that if $u \in (L^1(\mathbf{R}^n_+), W^{1,1}_N(\mathbf{R}^n_+) \cap W^{2,1}(\mathbf{R}^n_+))_{\theta,1}$ then $Tu \in W^{2\theta,1}(\mathbf{R}^n)$ con $2\|u\|_{W^{2\theta,1}(\mathbf{R}^n_+)} = \|\tilde{u}\|_{W^{2\theta,1}(\mathbf{R}^n_+)} \leq \|u\|_{X^I_\theta(\mathbf{R}^n_+)}$, hence $u \in W^{2\theta,1}(\mathbf{R}^n_+)$.

Conversely let $u \in W^{2\theta,1}(\mathbf{R}^n_+)$; then the function $\tilde{u}$ defined in the same way of (A.9) belongs to $W^{2\theta,1}(\mathbf{R}^n)$; indeed

$$\begin{aligned} [\tilde{u}]_{W^{2\theta,1}(\mathbf{R}^n)} &= \int_{\mathbf{R}^n} dx \int_{\mathbf{R}^n} \frac{|\tilde{u}(x) - \tilde{u}(y)|}{|x-y|^{n+2\theta}} dy \\ &= 2[u]_{W^{2\theta,1}(\mathbf{R}^n_+)} + 2\int_{\mathbf{R}^{n-1}\times\mathbf{R}_+} dx \int_{\mathbf{R}^{n-1}\times\mathbf{R}_-} \frac{|\tilde{u}(x) - \tilde{u}(y)|}{|x-y|^{n+2\theta}} dy \\ &\leq 4[u]_{W^{2\theta,1}(\mathbf{R}^n_+)}. \end{aligned}$$

Thus, since $X_\theta(\mathbf{R}^n) = W^{2\theta,1}(\mathbf{R}^n)$ for $\theta \in (0,1/2)$, there exist $v_1 \in L^1(\mathbf{R}^n)$ and $v_2 \in W^{2,1}(\mathbf{R}^n)$ such that $\tilde{u} = v_1 + v_2$ and $t^{-\theta}K(t,\tilde{u}) \in L^1_*(0,+\infty)$. Now, let $g \in C^\infty_c(\mathbf{R}^n)$ with $D_n g = 0$ in $x_n = 0$, then $\tilde{u}$ can be represented as the sum $(v_1 + v_2 - g) + g =: w + g$ with $w \in L^1(\mathbf{R}^n)$, $g \in W^{2,1}(\mathbf{R}^n)$. If we consider the restriction of w and g in $\mathbf{R}^n_+$ we get that $u = \tilde{u}_{|\mathbf{R}^n_+} = w_{|\mathbf{R}^n_+} + g_{|\mathbf{R}^n_+}$ with $w_{|\mathbf{R}^n_+} \in L^1(\mathbf{R}^n_+)$, $g_{|\mathbf{R}^n_+} \in W^{2,1}(\mathbf{R}^n_+) \cap W^{1,1}_N(\mathbf{R}^n_+)$ and $t^{-\theta}K(t,u) \in L^1_*(0,+\infty)$ since $K(t,u) \leq K(t,\tilde{u})$ for all $t \in (0,\infty)$. Thus (A.8) is proved.

Second step

Now we consider the same partition of unity $\{\eta_h\}_h$ associated with the covering $\{U_h\}_h$ of Ω considered in the proof of Proposition 3.1.1. Then, for a given function u defined in Ω, writing u as $\sum_{h=0}^{+\infty} u\eta_h$, we can prove that $u\eta_0 \in X_\theta(\Omega)$ if and only if $u\eta_0 \in W^{2\theta,1}(\Omega)$. For every $h \geq 1$ we can find $\psi_h : B_+(0) \to U_h \cap \Omega$ such that $d(\psi_h)_x(a(x)\nu(x)) = -e_n$, and prove that $v_h := u\eta_h \circ \psi_h$ belongs to $X_\theta(\mathbf{R}^n_+)$ if and only if belongs to $W^{2\theta,1}(\mathbf{R}^n_+)$, by which $u\eta_h \in X_\theta(\Omega)$ if and only if $u\eta_h \in W^{2\theta,1}(\Omega)$. Now in order to conclude we have to show that $u \in X_\theta(\Omega)$ if and only if $u \in W^{2\theta,1}(\Omega)$. Notice that the result is immediate if Ω is bounded, since in that case the covering $\{U_h\}_h$ is finite.

Suppose first that $u \in X_\theta(\Omega)$. Since $X_\theta(\Omega)$ continuously embeds in $L^1(\Omega)$, it is sufficient to estimate the seminorm $[u]_{W^{2\theta,1}(\Omega)}$. Moreover, since $u \in X_\theta(\Omega)$ we also have that $u\eta_h \in X_\theta(\Omega)$ for each $h \in \mathbf{N}$. Notice that, for fixed $x \in U_h$, $y \in U_k$ there exists $I_{hk} \subset \mathbf{N}$ such that

$$u(x) - u(y) = \sum_{i \in I_{hk}} u(x)\eta_i(x) - u(y)\eta_i(y)$$

where either $\operatorname{supp}(\eta_i) \cap U_h \neq \emptyset$ or $\operatorname{supp}(\eta_i) \cap U_k \neq \emptyset$. Since $\{U_h\}_h$ has a bounded overlapping κ, then $\#(I_{hk}) \leq 2\kappa$. Then

$$\begin{aligned}\int_\Omega \int_\Omega \frac{|u(x)-u(y)|}{|x-y|^{n+2\theta}}\,dxdy &\leq \sum_{h,k=1}^{\infty} \int_{U_h} dx \int_{U_k} \frac{|\sum_{i\in I_{hk}}(u(x)\eta_i(x) - u(y)\eta_i(y))|}{|x-y|^{n+2\theta}}dy \\ &\leq \sum_{h,k=1}^{\infty} \sum_{i\in I_{hk}} \int_{U_h} dx \int_{U_k} \frac{|u(x)\eta_h(x) - u(y)\eta_h(y)|}{|x-y|^{n+2\theta}}dy\,. \end{aligned} \tag{A.10}$$

Now, we define $V_h = \bigcup_{\{j:U_j\cap U_h\neq\emptyset\}} U_j$, then there is a constant $c_\kappa > 0$ depending only on κ, the overlapping of the U_i, such that

$$\begin{aligned} &\textstyle\sum_{i\in I_{hk}} \|u\eta_i\|_{L^1(U_i)} \leq c_\kappa \|u\|_{L^1(V_h\cup V_k)} \\ &\textstyle\sum_{i\in I_{hk}} \|u\eta_i\|_{W^{2,1}(U_i)} \leq c_\kappa \overline{M} \|u\|_{W^{2,1}(V_h\cup V_k)} \end{aligned} \tag{A.11}$$

where $\overline{M} := \sup_{h\in\mathbf{N}} \|\eta_h\|_{2,\infty}$. Moreover we can write $\Omega = \bigcup_{i=1}^{\kappa} \Omega_i$ where $\Omega_i = \{x \in \Omega : \#\{j : x \in U_j\} = i\}$ and $\Omega_i \cap \Omega_k = \emptyset$ if $i \neq k$. Then

$$\begin{aligned}\sum_{h,k} \int_{V_h\cup V_k} |u|\,dx &= \sum_{h,k} \sum_{i=1}^{\kappa} \int_{(V_h\cup V_k)\cap\Omega_i} |u|\,dx \\ &= \sum_{i=1}^{\kappa} \sum_{h,k} \int_{(V_h\cup V_k)\cap\Omega_i} |u|\,dx \\ &= \sum_{i=1}^{\kappa} i \int_{\Omega_i} |u|\,dx \leq \kappa \|u\|_{L^1(\Omega)}. \end{aligned} \tag{A.12}$$

Analogously,

$$\sum_{h,k} \|u\|_{W^{2,1}(V_h\cup V_k)} \leq \kappa \|u\|_{W^{2,1}(\Omega)}. \tag{A.13}$$

Since the functions $v_h := u\eta_h \circ \psi_h$ belong both to $(L^1(\mathbf{R}^n_+), W^{2,1}(\mathbf{R}^n_+) \cap W^{1,1}_{A,\nu}(\mathbf{R}^n_+))_{\theta,1}$ and $W^{2\theta,1}(\mathbf{R}^n_+)$, and in $\mathbf{R}^N_+$ the norms of $W^{2\theta,1}(\mathbf{R}^n_+)$ and $X_\theta(\mathbf{R}^n_+)$ are equivalent, we get a constant κ_0, depending only on the norm of the embedding of $X_\theta(\mathbf{R}^n_+)$ in $W^{2\theta,1}(\mathbf{R}^n_+)$ and ψ_h, such that

$$\int_{U_h} dx \int_{U_k} \frac{|u(x)\eta_i(x) - u(y)\eta_i(y)|}{|x-y|^{n+2\theta}}dy \leq \kappa_0 \int_0^{+\infty} \frac{1}{t^{1+\theta}} K(t, u\eta_i)dt \tag{A.14}$$

where K is defined in (A.7). By definition of $K(t,\cdot)$ and by (A.11) we get

$$\begin{aligned}
\sum_{i\in I_{hk}} K(t,u\eta_i) &= \sum_{i\in I_{hk}} \inf_{\substack{\tilde a+\tilde b=u\eta_i\\ \tilde a\in L^1(\Omega),\tilde b\in W^{2,1}(\Omega)}} \big(\|\tilde a\|_{L^1(\Omega)}+t\|\tilde b\|_{W^{2,1}(\Omega)}\big)\\
&\le \sum_{i\in I_{hk}} \inf_{\substack{a+b=u\\ a\in L^1(\Omega),b\in W^{2,1}(\Omega)}} \big(\|a\eta_i\|_{L^1(\Omega)}+t\|b\eta_i\|_{W^{2,1}(\Omega)}\big)\\
&\le \inf_{\substack{a+b=u\\ a\in L^1(\Omega),b\in W^{2,1}(\Omega)}} \sum_{i\in I_{hk}} \big(\|a\eta_i\|_{L^1(\Omega)}+t\|b\eta_i\|_{W^{2,1}(\Omega)}\big)\\
&\le \kappa_1 \inf_{\substack{a+b=u\\ a\in L^1(\Omega),b\in W^{2,1}(\Omega)}} \big(\|a\|_{L^1(V_h\cup V_k)}+t\|b\|_{W^{2,1}(V_h\cup V_k)}\big)
\end{aligned}$$

where κ_1 depends on κ and $\overline{M}$. Summing up on h,k we get, by (A.12) and (A.13),

$$\sum_{h,k=1}^{+\infty}\sum_{i\in I_{hk}} K(t,u\eta_i)\le \kappa_1\, K(t,u)\,.$$

Then by (A.10), (A.14) and using the last estimate we get

$$\begin{aligned}
\int_{|x-y|<\rho}\frac{|u(x)-u(y)|}{|x-y|^{n+2\theta}}\,dxdy &\le \sum_{h,k=1}^{+\infty}\sum_{i\in I_{hk}}\kappa_0\int_0^{+\infty}\frac{1}{t^{1+\theta}}K(t,u\eta_i)dt\\
&\le \kappa_0\,\kappa_1\int_0^{+\infty}\frac{1}{t^{1+\theta}}K(t,u)dt=\kappa_0\,\kappa_1\|u\|_{X_\theta(\Omega)}\,,
\end{aligned}$$

whence $X_\theta(\Omega)\subset W^{2\theta,1}(\Omega)$. To prove the reverse inclusion, consider $\{\eta_h,U_h\}_h$ as before. First of all observe that, we can estimate for each $\rho>0$

$$[u\eta_h]_{W^{2\theta,1}(\Omega)}\le \frac{c}{\rho^{n+2\theta}}\|u\|_{L^1(U_h)}+\int_{|x-y|<\rho}\frac{|u(x)\eta_h(x)-u(y)\eta_h(y)|}{|x-y|^{n+2\theta}}\,dxdy\,.$$

where $c=2|U_h|$ is a positive constant independent on h since U_h are balls with fixed radius. Adding and subtracting $u(x)\eta_h(y)$ we can estimate

$$\begin{aligned}
&\int_{|x-y|<\rho}\frac{|u(x)\eta_h(x)-u(y)\eta_h(y)|}{|x-y|^{n+2\theta}}dxdy\\
&\le\int_{\Omega\times\Omega}\Big[\mathrm{Lip}(\eta_h)\frac{|u(x)|}{|x-y|^{n-1+2\theta}}\chi_{A_{h,\rho}}(x,y)+\frac{|u(x)-u(y)|}{|x-y|^{n+2\theta}}\chi_{\Omega\times U_h}(x,y)\Big]dxdy
\end{aligned}$$

where $A_{h,\rho}=(U_h\times\Omega\cup\Omega\times U_h)\cap\{(x,y)\in\Omega\times\Omega\,:\,|x-y|<\rho\}$. Then, choosing ρ small enough in order that the ρ-enlarged sets U_h^ρ have the same overlapping as the U_h's and $A_{h,\rho}\subset U_h^\rho\times U_h^\rho$, we get

$$\|u\eta_h\|_{W^{2\theta,1}(\Omega)}\le\kappa_2\|u\|_{L^1(U_h^\rho)}+\int_{U_h}dy\int_{B(y,\rho)}\frac{|u(x)-u(y)|}{|x-y|^{n+2\theta}}dx$$

where κ_2 depends (only) on $\|\eta_h\|_{W^{1,\infty}},\theta,\rho,n$. Since the overlapping is bounded we can find two constants κ_3,κ_4 such that

$$\sum_h\|u\eta_h\|_{W^{2\theta,1}(\Omega)}\le\kappa_3\Big[\|u\|_{L^1(\Omega)}+\int_\Omega dy\int_{B(y,\rho)}\frac{|u(x)-u(y)|}{|x-y|^{n+2\theta}}dx\Big]\le\kappa_4\|u\|_{W^{2\theta,1}(\Omega)}\,.$$

Then for each $\epsilon > 0$ we can find $\tilde{a}_h \in L^1(\Omega)$, $\tilde{b}_h \in W^{2,1}(\Omega)$ such that $\tilde{a}_h + \tilde{b}_h = u\eta_h$ and $\|\tilde{a}_h\|_{L^1(\Omega)} + t\|\tilde{b}_h\|_{W^{2,1}(\Omega)} \leq K(t, u\eta_h) + \epsilon 2^{-h}$. Define $a = \sum_h \tilde{a}_h$ and $b = \sum_h \tilde{b}_h$. Then $a + b = u$ and

$$K(t,u) \leq \|a\|_{L^1(\Omega)} + t\|b\|_{W^{2,1}(\Omega)} \leq \sum_h \|\tilde{a}_h\|_{L^1(\Omega)} + t\|\tilde{b}_h\|_{W^{2,1}(\Omega)} \leq \sum_h K(t, u\eta_h) + \epsilon$$

and then $K(t,u) \leq \sum_h K(t, u\eta_h)$. Now, as before, since the functions v_h are in $W^{2\theta,1}(\mathbf{R}^n_+)$ and in $\mathbf{R}^n_+$, the norms of $W^{2\theta,1}(\mathbf{R}^n_+)$ and $X_\theta(\mathbf{R}^n_+)$ are equivalent, there exists a constant κ_5, depending only on the norm of the embedding of $W^{2\theta,1}(\mathbf{R}^n_+)$ in $X_\theta(\mathbf{R}^n_+)$ and ψ_h, such that

$$\int_0^{+\infty} \frac{1}{t^{1+\theta}} K(t, u\eta_h) dt \leq \kappa_5 \int_\Omega dx \int_\Omega \frac{|u(x)\eta_h(x) - u(y)\eta_h(y)|}{|x-y|^{n+2\theta}} dy\,. \tag{A.15}$$

Therefore there is a constant κ_6 (depending only on κ_4 and κ_5)

$$\begin{aligned} \int_0^{+\infty} \frac{1}{t^{1+\theta}} K(t,u) dt &\leq \int_0^{+\infty} \frac{1}{t^{1+\theta}} \sum_{h=1}^{+\infty} K(t, u\eta_h) dt \\ &\leq \kappa_5 \sum_{h=1}^{+\infty} \|u\eta_h\|_{W^{2\theta,1}(\Omega)} \leq \kappa_6 \|u\|_{W^{2\theta,1}(\Omega)}. \end{aligned}$$

□

A.3 Complex interpolation spaces

The complex interpolation methods were introduced by J. L. Lions in [29], A. P. Calderón in [11] and [12]. We shall follow the treatment of [46]. Let Y, X be complex Banach spaces with $Y \hookrightarrow X$ and let S be the strip $\{z = x + iy \in \mathbf{C} : 0 \leq x \leq 1\}$. By the maximum principle for holomorphic functions defined on a strip, we get that if $F : S \to X$ is holomorphic in the interior of S, continuous and bounded in S, then for each $z \in S$

$$\|F(z)\|_X \leq \max\{\sup_{t\in\mathbf{R}} \|F(it)\|_X, \sup_{t\in\mathbf{R}} \|F(1+it)\|_X\}.$$

Definition A.3.1. *Denote by $\mathcal{H}(X,Y)$ the space consisting of all continuous and bounded functions $F : S \to X$ which are holomorphic in the interior of the strip such that $t \mapsto F(it) \in C(\mathbf{R}, X)$, $t \mapsto F(1+it) \in C(\mathbf{R}, Y)$ and such that*

$$\|F\|_{\mathcal{H}(X,Y)} = \max\{ \sup_{t\in\mathbf{R}} \|F(it)\|_X,\ \sup_{t\in\mathbf{R}} \|F(1+it)\|_Y \} < \infty.$$

By using the maximum principle, it is not hard to prove that $\mathcal{H}(X,Y)$ is a Banach space. The complex interpolation spaces are defined by means of functions in $\mathcal{H}(X,Y)$.

Definition A.3.2. *For every $\theta \in [0,1]$, we define*

$$[X,Y]_\theta = \{F(\theta) :\ F \in \mathcal{H}(X,Y)\},$$

with norm

$$\|f\|_{[X,Y]_\theta} = \inf_{F\in\mathcal{H}(X,Y),F(\theta)=f} \|F\|_{\mathcal{H}(X,Y)}$$

That $[X,Y]_\theta$ is a Banach space follows from the fact that $[X,Y]_\theta$ is isomorphic to the quotient space $\mathcal{H}(X,Y)/\mathcal{N}_\theta$ where $\mathcal{N}_\theta$ is the subset of $\mathcal{H}(X,Y)$ consisting of the functions which vanish at $z=\theta$. Since $\mathcal{N}_\theta$ is closed, the quotient space is a Banach space and so is $[X,Y]_\theta$. The Banach space $[X,Y]_\theta$ is indeed an intermediate space as the next proposition states.

Proposition A.3.3. *Let* $\theta \in (0,1)$*; then*

$$Y \hookrightarrow [X,Y]_\theta \hookrightarrow X.$$

Proof. Let $f\in Y$. The constant function $F(z)=f$ belongs to $\mathcal{H}(X,Y)$ and

$$\|F\|_{\mathcal{F}(X,Y)} = \max\{\|f\|_X, \|f\|_Y\} \le c\|f\|_Y$$

for some $c>0$. Therefore $f=F(\theta)\in[X,Y]_\theta$ and $\|f\|_{[X,Y]_\theta} \le c\|f\|_Y$. The other embedding is a consequence of the maximum principle. Indeed if $f=F(\theta)$ with $F\in\mathcal{H}(X,Y)$ then

$$\begin{aligned}\|f\|_X &\le \max\{\sup_{t\in\mathbf{R}}\|F(it)\|_X, \sup_{t\in\mathbf{R}}\|F(1+it)\|_X\}\\ &\le c\max\{\sup_{t\in\mathbf{R}}\|F(it)\|_X, \sup_{t\in\mathbf{R}}\|F(1+it)\|_Y\}\\ &= c\|F\|_{\mathcal{H}(X,Y)}\end{aligned}$$

so that $f\in X$ and $\|f\|_X \le c\|F\|_{\mathcal{H}(X,Y)}$. $\square$

In general $[X,Y]_\theta$ does not coincide with any $(X,Y)_{\theta,p}$. If X,Y are Hilbert spaces then the equality holds for $p=2$, that is

$$[X,Y]_\theta = (X,Y)_{\theta,2} \quad 0<\theta<1.$$

In the non Hilbertian case there are no general rules.

Two other useful facts are recalled here, one concerning the dual space of such complex interpolation spaces and the last proves that $[X,Y]_\theta$ are actually interpolation spaces.

Theorem A.3.4. *(Dual space) Let* $\theta\in(0,1)$*. If* Y *is dense in* X *and one of the two spaces* X *or* Y *is reflexive, then*

$$[X,Y]'_\theta = [Y',X']_{1-\theta}. \tag{A.16}$$

This theorem is a consequence of the results in A.P. Calderón [12]. For the proof we refer to [12].

Theorem A.3.5. *Let* $(X_1, Y_1), (X_2, Y_2)$ *be complex interpolation couples. Assume that* $T \in \mathcal{L}(X_1, X_2) \cap \mathcal{L}(Y_1, Y_2)$, *then the restriction of* $T_{|[X_1,Y_1]_\theta}$ *is in* $\mathcal{L}([X_1, Y_1]_\theta, [X_2, Y_2]_\theta)$ *for every* $\theta \in (0, 1)$. *Moreover,*

$$\|T\|_{\mathcal{L}([X_1,Y_1]_\theta,[X_2,Y_2]_\theta)} \leq (\|T\|_{\mathcal{L}[X_1,X_2]})^{1-\theta}(\|T\|_{\mathcal{L}[Y_1,Y_2]})^{\theta}.$$

For the proof and a complete analysis of these spaces we refer to [46].

Appendix B

Heat kernel estimates on domains

In this section we collect some upper and lower estimates for the integral kernel of the semigroup associated with the parabolic problem

$$\begin{cases} \partial_t w - \mathcal{A}w = 0 & \text{in } (0,\infty)\times\Omega \\ w(0) = u_0 & \text{in } \Omega \\ \langle ADw, \nu\rangle = 0 & \text{in } (0,\infty)\times\partial\Omega. \end{cases} \tag{B.1}$$

under the hypotheses summarized at the beginning of Chapter 5. Since we shall deal with several semigroups, the exponential notation seems to us to be clearer, as it emphasizes the relevant elliptic generator. In fact we consider

$$\mathcal{A}_0 = \operatorname{div}(A\cdot D), \quad \mathcal{A}' = \operatorname{div}(A\cdot D) + B\cdot D \quad \text{and} \quad \mathcal{A} = \operatorname{div}(A\cdot D) + B\cdot D + c$$

and the related semigroups $e^{-t\mathcal{A}_0}, e^{-t\mathcal{A}'}$ and $e^{-t\mathcal{A}}$ whose kernels p_0, p' and p are such that, e.g.,

$$e^{-t\mathcal{A}}f(x) = \int_\Omega p(t,x,y)f(y)\,dy$$

and the analogous expressions for $e^{-t\mathcal{A}_0}$ and $e^{-t\mathcal{A}'}$ hold.
We first recall upper estimates directly for p, that are well-known. On the contrary, lower estimates are known in the symmetric case, i.e., for p_0. After observing that there is no difficulty in passing from p' to p, we shall deduce lower estimates for p', deducing them from those on p_0 via a perturbation argument. The proofs in Section B.2.2 are due to G. Metafune, E.M. Ouhabaz and D. Pallara whom we thank for communicating the above results and allowing us to reproduce them here.

B.1 Gaussian upper bounds for heat kernels

We collect the known Gaussian upper bound results in the following statement and we refer to [45, Theorem 5.7] for the proof.

Theorem B.1.1. *(Kernel estimates)*
Let Ω be an open set of $\mathbf{R}^n$ uniformly regular of class C^2. Let $\mathcal{A}$, $\mathcal{B}$ be as in (2.3)–(2.7) *and let $(T(t))_{t\geq 0}$ be the analytic semigroup generated by the realization of $\mathcal{A}$ in $L^1(\Omega)$ with homogeneous boundary conditions $\mathcal{B}u = 0$; for the kernel $p : (0, +\infty) \times \Omega \times \Omega \to \mathbf{R}$ of the semigroup $(T(t))_{t\geq 0}$ the following estimates hold: there exist $b, c_1 > 0$, a real number ω such that for $|\alpha|, |\beta| < 2$, $x, y \in \Omega$ and $t > 0$*

$$|D_x^\alpha D_y^\beta p(t, x, y)| \leq \frac{c_1}{t^{\frac{n+|\alpha|+|\beta|}{2}}} e^{-b\frac{|x-y|^2}{t}} e^{\omega t}. \tag{B.2}$$

B.1.1 Some norm estimates

Immediate consequences of the Gaussian upper bound are the following $L^1 - L^p$ and $L^p - L^\infty$ estimates.

Proposition B.1.2. *Let $p \geq 1$ and let $e^{-t\mathcal{A}}$ be the semigroup generated by $\mathcal{A}$. Then there exist $c_2, c_3 > 0$ such that*

$$\|e^{-t\mathcal{A}}\|_{\mathcal{L}(L^1,L^p)} \leq c_2 t^{-\frac{n}{2}(1-\frac{1}{p})} \quad 0 < t < 1, \tag{B.3}$$

and

$$\|e^{-t\mathcal{A}}\|_{\mathcal{L}(L^p,L^\infty)} \leq c_3 t^{-\frac{n}{2p}} \quad 0 < t < 1. \tag{B.4}$$

Proof. Let $f \in L^1(\Omega)$; then, using (B.2) we get

$$\begin{aligned}
\|e^{-t\mathcal{A}} f\|_{L^p(\Omega)}^p &= \int_\Omega \Big| \int_\Omega p(t, x, y) f(y)\, dy\Big|^p dx \\
&\leq \|f\|_{L^1(\Omega)}^p \int_\Omega \|p(t, x, \cdot)\|_{L^\infty(\Omega)}^p dx \\
&\leq c_1 t^{-np/2} \|f\|_{L^1(\Omega)}^p \int_\Omega e^{-b\frac{|x-y|^2}{t}} dx \\
&= c_1' t^{-\frac{n}{2}(p-1)} \|f\|_{L^1(\Omega)}^p
\end{aligned}$$

Thus

$$\|e^{-t\mathcal{A}} f\|_{L^p(\Omega)} \leq c_2 t^{-\frac{n}{2}(1-\frac{1}{p})} \|f\|_{L^1(\Omega)}$$

and (B.3) is proved. Similarly, let $f \in L^p$, and $p' = p/(p-1)$, then again by (B.2)

$$\begin{aligned}
\|e^{-t\mathcal{A}} f\|_{L^\infty(\Omega)} &= \sup_{x\in\Omega} \Big| \int_\Omega p(t,x,y) f(y)\,dy \Big| \\
&\le \|f\|_{L^p(\Omega)} \sup_{x\in\Omega} \|p(t,x,\cdot)\|_{L^{p'}(\Omega)} \\
&\le c_1 \|f\|_{L^p(\Omega)} \sup_{x\in\Omega} \Big(t^{-\frac{n}{2}p'} \int_\Omega e^{-bp' \frac{|x-y|^2}{t}}\,dy \Big)^{1/p'} \\
&= c_3 t^{-\frac{n(p'-1)}{2p'}} \|f\|_{L^p(\Omega)} = c_3 t^{-\frac{n}{2p}} \|f\|_{L^p(\Omega)}.
\end{aligned}$$

□

B.2 Gaussian lower bounds

This section is devoted to obtain Gaussian lower bounds for $p(t,x,y)$. Such lower bounds in the symmetric case can be deduced from Gaussian upper bounds and Hölder continuity of the kernel.

Remark B.2.1. One can easily observe that if some Gaussian lower bounds are established for p', the same hold for p, more precisely $p(t,x,y) \ge e^{-\omega t} p'(t,x,y)$. Indeed, since $c \in L^\infty(\Omega)$, then there exists $\omega > 0$ such that $-\omega \le c(x) \le \omega$ a.e. $x \in \Omega$. Let $f \ge 0$ in Ω and consider u and v solutions respectively of the problems

$$\begin{cases} \partial_t u = \operatorname{div}(A\cdot Du) + B\cdot Du & \text{in } (0,+\infty)\times\Omega \\ u(0,x) = f(x) & \text{in } \Omega \\ \langle ADu, \nu\rangle = 0 & \text{in } (0,+\infty)\times\partial\Omega. \end{cases} \tag{B.5}$$

and

$$\begin{cases} \partial_t v = \operatorname{div}(A\cdot Dv) + B\cdot Dv + cv & \text{in } (0,+\infty)\times\Omega \\ v(0,x) = f(x) & \text{in } \Omega \\ \langle ADv, \nu\rangle = 0 & \text{in } (0,+\infty)\times\partial\Omega. \end{cases} \tag{B.6}$$

By the maximum principle we deduce that $u \ge 0$. We want to prove that $v \ge e^{-\omega t} u$, hence $p(t,x,y) \ge e^{-\omega t} p'(t,x,y)$ as announced. The problem satisfied by $z = v - w$, with $w = e^{-\omega t} u$, is

$$\begin{cases} \partial_t z - \mathcal{A} z = (c+\omega) w \ge 0 & \text{in } (0,+\infty)\times\Omega \\ z(0,x) = 0 & \text{in } \Omega \\ \langle ADz, \nu\rangle = 0 & \text{in } (0,+\infty)\times\partial\Omega. \end{cases} \tag{B.7}$$

Thus applying again the maximum principle we deduce $z \ge 0$, i.e.

$$p(t,x,y) \ge p'(t,x,y) e^{-\omega t}.$$

As a consequence of Remark B.2.1 we can restrict the study to the operator $\mathcal{A}' = \mathcal{A} - c$ and our aim will be to deduce Gaussian lower bound for p'.

B.2.1 The symmetric case

We first consider the symmetric case and show lower bounds for p_0 (more details are contained in [34]). Under our assumptions on the coefficients, p_0 is Hölder continuous, that is

$$|p_0(t,x,y) - p_0(t,x',y)| \le kt^{-n/2-\gamma/2}|x-x'|^\gamma, \quad \text{for all } x, x', y \in \Omega \tag{B.8}$$

for some $\gamma > 0$ and $k > 0$ independent on y. Moreover it satisfies the Gaussian upper bound in Theorem B.1.1 and the conservation property holds: $\displaystyle\int_\Omega p_0(t,x,y)\,dy = 1$ for all $t > 0$ and $x \in \Omega$.
The first step shows that an on-diagonal lower bound can be deduced from a Gaussian upper bound and the conservation property.

Proposition B.2.2. *There exists a constant $C > 0$ such that for all $t > 0$ and a.e. $x \in \Omega$*

$$p_0(t,x,x) \ge Ct^{-n/2} \tag{B.9}$$

Proof. Fix $\delta > 0$; we have

$$\begin{aligned}
\int_{\Omega\setminus B(x,\delta\sqrt{t})} p_0(t,x,y)\,dy &\le c_1 t^{-n/2} \int_{\Omega\setminus B(x,\delta\sqrt{t})} e^{-\frac{b}{2}\frac{|x-y|^2}{t}} e^{-\frac{b}{2}\frac{|x-y|^2}{t}}\,dy \\
&\le c_1 t^{-n/2} e^{-\frac{b}{2}\delta^2} \int_{\mathbf{R}^n} e^{-\frac{b}{2}\frac{|x-y|^2}{t}}\,dy \\
&\le ke^{-\frac{b}{2}\delta^2}.
\end{aligned}$$

Now, for δ large enough, $ke^{-\frac{b}{2}\delta^2} \le \frac{1}{2}$, thus a.e. $x \in \Omega$

$$\begin{aligned}
\int_{\Omega\cap B(x,\delta\sqrt{t})} p_0(t,x,y)\,dy &= 1 - \int_{\Omega\setminus B(x,\delta\sqrt{t})} p_0(t,x,y)\,dy \\
&\ge \frac{1}{2}.
\end{aligned}$$

It follows by the semigroup property and the symmetry of p_0 that

$$\begin{aligned}
p_0(t,x,x) &= \int_\Omega p_0(t/2,x,y)p_0(t/2,y,x)\,dy \\
&= \int_\Omega |p_0(t/2,x,y)|^2\,dy \\
&\ge \int_{\Omega\cap B(x,\delta\sqrt{t})} |p_0(t/2,x,y)|^2\,dy \\
&\ge \frac{1}{|\Omega\cap B(x,\delta\sqrt{t})|}\left(\int_{\Omega\cap B(x,\delta\sqrt{t})} p_0(t/2,x,y)\,dy\right)^2 \\
&\ge \frac{1}{4|B(x,\delta\sqrt{t})|} \ge Ct^{-n/2}
\end{aligned}$$

for some positive constant C. $\square$

The following step consists in deducing an off-diagonal Gaussian lower bound from the on-diagonal one, by exploiting the Hölder continuity of p_0.

Proposition B.2.3. *There exist positive constants C and η such that*

$$p_0(t,x,y) \geq Ct^{-n/2} \tag{B.10}$$

for all $x, y \in \Omega$ and $t > 0$, sufficiently small such that $|x-y| \leq \eta\sqrt{t}$.

Proof. Since by (B.8)

$$|p_0(t,x,y) - p_0(t,x',y)| \leq kt^{-\frac{n}{2}-\frac{\gamma}{2}}|x-x'|^\gamma$$

for all $x, x', y \in \Omega$ we have

$$p_0(t,x,y) \geq p_0(t,y,y) - kt^{-n/2-\gamma/2}|x-y|^\gamma$$

Thus, using estimate (B.9),

$$\begin{aligned} p_0(t,x,y) &\geq Ct^{-n/2} - kt^{-n/2}\Big(\frac{|x-y|}{\sqrt{t}}\Big)^\gamma \\ &= Ct^{-n/2}(1 - \Big(\frac{|x-y|}{\sqrt{t}}\Big)^\gamma) \\ &\geq Ct^{-n/2} \end{aligned}$$

for $|x-y| \leq \frac{1}{2}\sqrt{t}$, which shows (B.10). $\square$

Let us now extend the previous estimate to arbitrary x, y in Ω.

Theorem B.2.4. *Let $p_0(t,x,y)$ be the heat kernel of $\mathcal{A}_0$. There exist constants $c_0, C_0 > 0$ such that*

$$p_0(t,x,y) \geq C_0 t^{-n/2} e^{-c_0\frac{|x-y|^2}{t}} \tag{B.11}$$

for all $x, y \in \Omega$ and $t > 0$.

Proof. Let $x, y \in \Omega$. Fix $N \in \mathbf{N}$ and consider a finite sequence of points x_i, $0 \leq i \leq N$ in Ω such that $x_0 = x$, $x_N = y$, $[x_i, x_{i+1}] \subset \Omega$ and $|x_i - x_{i+1}| \leq K\frac{|x-y|}{N} =: r$ for all $i = 0, \dots, N-1$. Then by the semigroup property and the positivity of $p_0(t,x,y)$, we have

$$\begin{aligned} p_0(t,x,y) &= \int_\Omega \cdots \int_\Omega p_0(\frac{t}{N}, x, z_1)p_0(\frac{t}{N}, z_1, z_2)\dots p_0(\frac{t}{N}, z_{N-1}, y)\,dz_1\dots dz_{N-1} \\ &\geq \int_{B(x_1,r)\cap\Omega} \cdots \int_{B(x_{N-1},r)\cap\Omega} p_0(\frac{t}{N}, x, z_1)p_0(\frac{t}{N}, z_1, z_2)\dots p_0(\frac{t}{N}, z_{N-1}, y)\,dz_1\dots dz_{N-1} \end{aligned}$$

Let us observe that if $z_i \in B(x_i, r)$ and $z_{i+1} \in B(x_{i+1}, r)$ (where we have set $z_0 = x$ and $z_N = y$), then it holds that

$$|z_i - z_{i+1}| \leq |x_i - x_{i+1}| + 2r \leq (K+2)r \quad i = 0, \dots, N-1.$$

If $(K+2)|x-y| \leq \eta\sqrt{t}$ (η as in Proposition B.2.3) then $|x-y| \leq \eta\sqrt{t}$. In this case (B.11) follows from (B.9) and Proposition B.2.3.
If $(K+2)|x-y| > \eta\sqrt{t}$, we choose $N \geq 2$ to be the smallest integer such that

$$(K+2)\frac{|x-y|}{\sqrt{N}} \leq \eta\sqrt{t}.$$

this yields that $|z_i - z_{i+1}| \leq (K+2)\frac{|x-y|}{N} \leq \eta\sqrt{\frac{t}{N}}$ for $i = 0, \dots, N-1$. Then using Proposition B.2.3 in the above integrals we get

$$\begin{aligned} p_0(t,x,y) &\geq \int_{B(x_1,r)\cap\Omega} \cdots \int_{B(x_{N-1},r)\cap\Omega} p_0(\frac{t}{N}, x, z_1) \dots p_0(\frac{t}{N}, z_{N-1}, y)\, dz_1 \dots dz_{N-1} \\ &\geq C^N \Big[\Big(\frac{t}{N}\Big)^{-\frac{n}{2}}\Big]^N \int_{B(x_1,r)\cap\Omega} \cdots \int_{B(x_{N-1},r)\cap\Omega} dz_1 \dots dz_{N-1} \\ &\geq k(n,\Omega) C^N \Big[\Big(\frac{t}{N}\Big)^{-\frac{n}{2}}\Big]^N \Big[\Big(\frac{t}{N}\Big)^{\frac{n}{2}}\Big]^{N-1} \geq k(n,\Omega) e^{-C'N}\Big(\frac{t}{N}\Big)^{-\frac{n}{2}}, \end{aligned} \tag{B.12}$$

where we have used the regularity of Ω in order to say that there exists a constant $k(n,\Omega) > 0$ such that for all $x \in \overline{\Omega}$, $|\Omega \cap B(x,r)| \geq k(n,\Omega)|B(x,r)|$. Finally, by definition of N, we have $N - 1 \leq K_\gamma \frac{|x-y|^2}{t}$, thus from (B.12)

$$p_0(t,x,y) \geq C_0 t^{-n/2} e^{-c_0 \frac{|x-y|^2}{t}}.$$

This concludes the proof. □

B.2.2 The non-symmetric case

Notice that in the proof of Proposition B.2.3 symmetry has not been used. Therefore if $p'(t,\cdot,y)$ is Hölder continuous and $p'(t,x,x) \geq ct^{-n/2}$, using an argument similar to Proposition B.2.3 and Theorem B.2.4, we get Gaussian lower bound for $p'(t,x,y)$, too. Moreover Theorem B.2.4 holds also without assumptions of symmetry and Hölder continuity. Its proof uses only estimate (B.10).

Let us show that the $L^1 \to L^\infty$ norm of the difference $e^{-t\mathcal{A}'} - e^{-t\mathcal{A}_0}$ is relatively small. Now, we prove a result which allows us to conclude without assuming Hölder continuity for p'.

Proposition B.2.5. *There exists $C > 0$ such that*

$$\|e^{-t\mathcal{A}'} - e^{-t\mathcal{A}_0}\|_{\mathcal{L}(L^1,L^\infty)} \leq Ct^{-\frac{n}{2}+\frac{1}{2}} \tag{B.13}$$

Proof. The integral representation of the solution gives that

$$\begin{aligned} e^{-t\mathcal{A}'} - e^{-t\mathcal{A}_0} &= \int_0^t e^{-(t-s)\mathcal{A}'} B \cdot De^{-s\mathcal{A}_0} ds \\ &= \int_0^{t/2} e^{-(t-s)\mathcal{A}'} B \cdot De^{-s\mathcal{A}_0} ds + \int_{t/2}^t e^{-(t-s)\mathcal{A}'} B \cdot De^{-s\mathcal{A}_0} ds \end{aligned}$$

Now, by using (B.3), (B.4), the fact that $De^{-s\mathcal{A}_0} \in \mathcal{L}(L^p)$ for $1 < p \leq 2$ and that $\|De^{-\frac{s}{2}\mathcal{A}_0}\|_{\mathcal{L}(L^p)} \leq c_4 s^{-1/2}$, we get for $p > 1$ (close to 1) the following estimate

$$\begin{aligned}
&\Big\| \int_0^{t/2} e^{-(t-s)\mathcal{A}'} B \cdot De^{-s\mathcal{A}_0} f ds \Big\|_{L^\infty(\Omega)} \\
&\quad \leq \|B\|_\infty \|f\|_{L^1(\Omega)} \int_0^{t/2} \|e^{-(t-s)\mathcal{A}'}\|_{\mathcal{L}(L^p,L^\infty)} \|De^{-\frac{s}{2}\mathcal{A}_0}\|_{\mathcal{L}(L^p)} \|e^{-\frac{s}{2}\mathcal{A}_0}\|_{\mathcal{L}(L^1,L^p)}\, ds \\
&\quad \leq C\|f\|_{L^1(\Omega)} \int_0^{t/2} (t-s)^{-n/2p} s^{-1/2} s^{-\frac{n}{2}(1-\frac{1}{p})}\, ds \\
&\quad \leq Ct^{-n/2p}\|f\|_{L^1(\Omega)} \int_0^{t/2} s^{-1/2} s^{-\frac{n}{2}(1-\frac{1}{p})}\, ds \\
&\quad = Ct^{-\frac{n}{2}+\frac{1}{2}}\|f\|_{L^1(\Omega)}
\end{aligned} \tag{B.14}$$

where $C = C(c_2, c_3, c_4, \|B\|_\infty)$. Moreover from (B.2) we have that $\|De^{-\frac{s}{2}\mathcal{A}_0}\|_{\mathcal{L}(L^q,L^\infty)} \leq c_5 s^{-\frac{1}{2}-\frac{1}{q}}$. Thus, using an exponent q close to ∞ we get

$$\begin{aligned}
&\Big\| \int_{t/2}^{t} e^{-(t-s)\mathcal{A}'} B \cdot De^{-s\mathcal{A}_0} f ds \Big\|_{L^\infty(\Omega)} \\
&\quad \leq \|B\|_\infty \|f\|_{L^1(\Omega)} \int_{t/2}^{t} \|De^{-\frac{s}{2}\mathcal{A}_0}\|_{\mathcal{L}(L^q,L^\infty)} \|e^{-\frac{s}{2}\mathcal{A}_0}\|_{\mathcal{L}(L^1,L^q)} ds \\
&\quad \leq C\|f\|_{L^1(\Omega)} \int_{t/2}^{t} s^{-1/2} s^{-n/2}\, ds \\
&\quad \leq Ct^{-\frac{n}{2}+\frac{1}{2}}\|f\|_{L^1(\Omega)}
\end{aligned} \tag{B.15}$$

where $C = C(c_2, c_5, \|B\|_\infty)$. Summing up (B.14) and (B.15) we get the claim. □

As an immediate consequence we deduce a Gaussian lower bound for $p(t, x, y)$.

Theorem B.2.6. *Let $p'(t, x, y)$ be the fundamental solution of $\partial_t - \mathcal{A}'$. Then there exist positive constants C_1, c_1 such that*

$$p'(t, x, y) \geq C_1 t^{-n/2} e^{-c_1 \frac{|x-y|^2}{t}}$$

for all $x, y \in \Omega$ and $t > 0$, sufficiently small.

Proof. Since

$$\|e^{-t\mathcal{A}'} - e^{-t\mathcal{A}_0}\|_{L^1 \to L^\infty} \leq Ct^{-\frac{n}{2}+\frac{1}{2}} \tag{B.16}$$

by the Dunford-Pettis theorem (see [7] for a proof) we have

$$\sup_{x,y\in\Omega} |p'(t, x, y) - p_0(t, x, y)| \leq Ct^{-\frac{n}{2}+\frac{1}{2}}$$

whence, for $|x - y| \leq \eta\sqrt{t}$ (η as in Proposition B.2.3) we get

$$\begin{aligned}
p'(t, x, y) &\geq p_0(t, x, y) - Ct^{-\frac{n}{2}+\frac{1}{2}} \\
&\geq Ct^{-\frac{n}{2}}(1 - \sqrt{t}) \\
&\geq Ct^{-\frac{n}{2}}
\end{aligned}$$

for $t \leq \delta_0$ independent of x, y. Thus Proposition B.2.3 is true also for $p'(t,x,y)$ and proceeding as before we deduce (B.11) also for $p'(t,x,y)$. $\square$

From Remark B.2.1 we finally deduce the following.

Corollary B.2.7. *Let $p(t,x,y)$ be the heat kernel of $\partial_t - \mathcal{A}$. Then there exist constants $c_1, C_1 > 0$ such that*

$$p(t,x,y) \geq C_1 t^{-n/2} e^{-c_1 \frac{|x-y|^2}{t}} e^{-\omega t} \tag{B.17}$$

for all $x, y \in \Omega$ and $t > 0$ small.

List of symbols

Number sets and vector spaces	
$\mathbf{N}, \mathbf{Z}, \mathbf{Q}, \mathbf{R}, \mathbf{C}$	set of natural, integer, rational, real and complex numbers
$\mathbf{R}^n$	set of all real n-tuples
$\mathbf{S}^{n-1}$	unit sphere of $\mathbf{R}^n$
$\mathbf{R}^n_+$	$\mathbf{R}^n \cap \{x_n \geq 0\}$
$\mathbf{C}^n$	set of all complex n-tuples
$a \wedge b, a \vee b$	minimum and maximum of a and b
$\lvert\alpha\rvert$	the length of the multi-index α, i.e. $\lvert\alpha\rvert = \alpha_1 + \cdots + \alpha_n$
$\mathrm{Re}\,\lambda$, $\mathrm{Im}\,\lambda$	real and imaginary part of $\lambda \in \mathbf{C}$
$\#E$	the cardinality of the set E
Topological and metric space notation	
$\overline{E}$	topological closure of E
∂E	topological boundary of E
E^c	the complementary set of E in a domain Ω or in $\mathbf{R}^n$
$E \subset\subset F$	$\overline{E} \subset F, \overline{E}$ compact
$B(x_0, r)$	open ball with center x and radius r
$B^+(0, r)$	$B(0, r) \cap \mathbf{R}^n_+$
$\mathcal{L}(X, Y)$	set of bounded and linear operators from X to Y
$\mathcal{L}(X)$	$\mathcal{L}(X, X)$
X'	dual space of the Banach space X

Matrix and linear algebra	
I	the identity matrix
$\det B$	the determinant of the matrix B
e_i	i-th vector of the canonical basis of $\mathbf{R}^n$
$\mathrm{Tr} B$	the trace of the matrix B
$\Vert B\Vert_\infty$	the Euclidean norm of the matrix B, i.e. $(\sum_{i,j=1}^n b_{ij}^2)^{1/2}$
$\Vert B\Vert_{1,\infty}$	$(\sum_{i,j,h=1}^n \vert D_h b_{ij}\vert^2)^{1/2}$
$\Vert B\Vert_{2,\infty}$	$(\sum_{i,j,h,k=1}^n \vert D_{hk} b_{ij}\vert^2)^{1/2}$
$\langle\cdot,\cdot\rangle$ or $x\cdot y$	the Euclidean inner product between the vectors $x, y \in \mathbf{R}^n$
Function spaces: let $f : X \to Y$	
$f \llcorner E$ or $f_{\vert E}$	restriction of f to $E \subset X$
$\mathrm{supp}\, f$	closure of $\{x \in X : f(x) \neq 0\}$
χ_E	characteristic function of the set E
u_t	partial derivative with respect to t
D_i	partial derivative with respect to x_i
D_{ij}	$D_i D_j$
Du	space gradient of a real-valued function u
D^2u	Hessian matrix of a real-valued function u
Δu	$\mathrm{Tr}(D^2u)$
$C(X,Y)$	space of continuous functions from X into Y
$C(\Omega)$	space of continuous functions valued in $\mathbf{R}$ or $\mathbf{C}$
$C_c(\Omega)$	functions in $C(\Omega)$ with compact support in Ω
$C_0(\Omega)$	closure in the sup norm of $C_c(\Omega)$
$UC_b(\Omega)$	space of the uniformly continuous and bounded functions on Ω
$C_b^k(\overline{\Omega})$	space of k-times differentiable functions with $D^m f$ for $\vert m\vert \le k$ bounded and continuous up to the boundary
$C^\alpha(\Omega)$	space of α-Hölder continuous functions, $\alpha \in (0,1)$
$C^{k,\alpha}(\Omega)$	space of $f \in C^k(\Omega)$ with $D^m f \in C^\alpha(\Omega)$ for $\vert m\vert \le k$ and $\alpha \in (0,1)$
$\mathcal{S}(\mathbf{R}^n)$	Schwartz space of rapidly decreasing functions
$[u]_{C^\alpha(\Omega)}$	the seminorm $\sup_{x,y\in\Omega} \frac{\vert u(x)-u(y)\vert}{\vert x-y\vert^\alpha}$
$\Vert\cdot\Vert_{L^\infty(\Omega)}$	sup norm
$\Vert u\Vert_{C^{k,\alpha}(\Omega)}$	$\sum_{\vert\alpha\vert\le k} \Vert D^\alpha u\Vert_{L^\infty(\Omega)} + [D^k u]_{C^\alpha(\Omega)}$
$(L^p(\Omega), \Vert\cdot\Vert_{L^p(\Omega)})$	usual Lesbegue space
$(W^{k,p}(\Omega), \Vert\cdot\Vert_{W^{k,p}(\Omega)})$	usual Sobolev space
$W^{k,p}_{\mathrm{loc}}(\Omega)$	space of functions belonging to $W^{k,p}(\Omega')$ for every $\Omega' \subset\subset \Omega$
$W^{k,p}_0(\Omega)$	closure of $C_c^\infty(\Omega)$ in $W^{k,p}(\Omega)$
$W^{-m,p}(\Omega)$	dual space of $W_0^{m,p'}(\Omega)$ with $\frac{1}{p}+\frac{1}{p'}=1$
$BV(\Omega)$	functions with bounded variation in Ω

Operators	
$\mathcal{A}$	linear operator
$\mathcal{A}^*$	formal adjoint operator of $\mathcal{A}$
A	realization of $\mathcal{A}$ in a Banach space X
$D(A)$	the domain of A
$\rho(A)$	resolvent set of the linear operator A
$\sigma(A)$	spectrum of the linear operator A
I	identity operator
$[A, B]$	the operator $AB - BA$ defined in $D(AB) \cap D(BA)$
Measure theory and BV functions	
$\mathcal{B}(X)$	σ- algebra of Borel subsets of a topological space X
$[\mathcal{M}(X)]^m$	the $\mathbf{R}^m$-valued finite Radon measures on X
$\mathcal{M}^+(X)$	the space of positive finite measures on X
$\mathcal{L}^n$	Lebesgue measure in $\mathbf{R}^n$
ω_n	Lebesgue measure of $B(0,1)$ in $\mathbf{R}^n$
$\mathcal{H}^k$	k-dimensional Hausdorff measure
$\|E\|$ or $\mathcal{L}^n(E)$	the Lebesgue measure of the set E
$\|\mu\|$	total variation of the measure μ
$\mu \llcorner E$	restriction of the measure μ to the set E
Du	distributional derivative of u
$\mathcal{P}(E, \Omega)$	perimeter of E in Ω
$\mathcal{P}(E)$	perimeter of E in $\mathbf{R}^n$
ν_E	generalized inner normal to E
E^t	set of points of density t of E
$\mathcal{F}E, \partial^* E$	reduced and essential boundary of E

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