Uniqueness of the one-dimensional bounce problem as a generic property in $L^1([0,T];\mathbb{R})$.

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Sunto. - *Si prova l'esistenza di un sottoinsieme* M^*, *di seconda categoria in* $L^1([0,T];\mathbb{R})$, *tale che per ogni assegnata forza* f *in* M^* *il problema del rimbalzo unidimensionale ha unicità.*

Introduction.- In [1] we studied the problem of a material point moving on a straight line subject to a strength f depending on time and to a perfectly elastic bouncing law.

This problem consists in the following:

given $f \in L^1([0,T];\mathbb{R})$, $s < 0$ $b \in \mathbb{R}$ or $s = 0$, $b \leq 0$ (permissible data), find $u \leq 0$ such that it satisfies a lipschitz condition on $[0,T]$;

$$\int_0^T [u(t)\,\ddot{\phi}(t) - f(t)\phi(t)]\,dt \leq 0 \qquad \text{for every} \quad \phi \in C_0^\infty([0,T];\mathbb{R}^+);$$

for $u < 0$ one has $\displaystyle\int_0^T [u(t)\ddot{\phi}(t) - f(t)\phi(t)]\,dt = 0$ for every $\phi \in C_0^\infty([0,T];\mathbb{R})$;

for every $t \in]0,T[$ $\dot{u}^+(t)$ and $\dot{u}^-(t)$ exist; moreover $\dot{u}^+(0)$, $\dot{u}^-(T)$ exist and

$$\frac{1}{2}[\dot{u}^\pm(t)]^2 = \frac{1}{2}[\dot{u}^+(0)]^2 + \int_0^t f(\eta)\,\dot{u}(\eta)\,d\eta \qquad \text{for} \quad t \in [0,T];$$

$u(0) = s$, $\dot{u}^+(0) = b$ (energy conservation law).

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In [1] we have given a counterexample to uniqueness.
Prof. E. De Giorgi asked whether one could characterize the functions of $L^1([0,T];\mathbb{R})$ for which the corresponding bounce problem has a unique solution.

Also in [1] we gave conditions that ensured uniqueness (e.g. if f is a simple function (cfr. theorem 2 in [1])).

The aim of this paper is to show that uniqueness of the bounce problem is a generic property in $L^1([0,T];\mathbb{R})$.

Actually, following G. Vidossich [2], we prove the existence of a G_δ-set M^* of $L^1([0,T];\mathbb{R})$ that contains the dense set $M_0 = \{f \in L^1([0,T];\mathbb{R}) \mid f$ simple$\}$ of $L^1([0,T];\mathbb{R})$ such that for every $f \in M^*$ the solution to the Cauchy problem for the one-dimensional bounce is unique.

We recall that:

(i) a G_δ -set is a countable intersection of open sets;

(ii) a "generic property" about points of a topological space is a property that holds for all points of a subset of second category.

Since second category is the topological analogue of almost everywhere, a generic property is a property that is true for most points in the given space.

The relation between G_δ - sets and the concept of generic property is that a dense G_δ -set in a second category space is of second category (since the complement of a dense G_δ-set is a first category set, while in a second category space the complement of a first category set is of second category).

Results.

The first step is to prove that the set $M_o = \{f \in L^1([0,T];\mathbb{R}) \mid f \text{ simple}\}$ is not a G_δ - set in $L^1([0,T];\mathbb{R})$.

It suffices to remark that M_o is a set of first category because it is a subset of $\bigcup_{n\in N} C_n$ which is of first category, where

$C_n = \{f \in L^1([0,T];\mathbb{R}) \mid f$ constant on $(\alpha_n,\beta_n) \subset [0,T]$, α_n and β_n rational$\}$.

__Lemma.__ - __If__ $f \in M_o$, $(f_n)_n$ __is a sequence in__ $L^1([0,T];\mathbb{R})$ __converging to__ f __in__ $L^1([0,T];\mathbb{R})$, __and if__ u_n __is a solution to the Cauchy problem for bounce with strength__ f_n, __then__

$$\lim_{n\to+\infty} u_n = u_f \qquad \text{\underline{uniformly in}} \quad [0,T] ,$$

__where__ u_f __is the unique solution to the Cauchy problem for bounce with strength__ f.

Proof. For every $n \in N$ we have

(j) $u_n \in \text{Lip}([0,T];\mathbb{R})$, $u_n \leq 0$ in $[0,T]$;

(jj) $\int_0^T [u_n(t)\, \ddot{\phi}(t) - f_n(t)\phi(t)]\, dt \leq 0$ for every $\phi \in C_o^\infty([0,T];R^+)$;

(jjj) for $u_n < 0$ $\int_0^T [u_n(t)\, \ddot{\phi}(t) - f_n(t)\phi(t)]\, dt = 0$ for every $\phi \in C_o^\infty([0,T];\mathbb{R})$;

(jv) for every $t \in]0,T[$ $\dot{u}_n^+(t)$ and $\dot{u}_n^-(t)$ exist; moreover $\dot{u}_n^+(o), \dot{u}_n^-(T)$ exist and $\frac{1}{2}[\dot{u}_n^\pm(t)]^2 = \frac{1}{2}[\dot{u}_n^+(0)]^2 + \int_o^t f_n(\eta)\, \dot{u}_n(\eta) d\eta$ for $t \in [0,T]$

(v) $u_n(0) = s \quad , \dot{u}_n^+(0) = b$.

From (jv) and (v) it follows that

$\| \dot{u}_n \|_{L^\infty([0,T];\mathbb{R})} \leq$ constant (indipendent of n).

It follows, because bouncing points for u_n(where $\dot{u}_n^+ = -\dot{u}_n^+ \neq 0$) are at most countable, that the sequence $(u_n)_n$ is equibounded and equilipschitz (equicontinuous) and therefore a subsequence $(u_{n_k})_k$ exists such that

$$\lim_{k\to+\infty} u_{n_k} = v \text{ uniformly in } [0,T].$$

We claim that $v = u_f$. Indeed if we prove that v is solution of the Cauchy problem for bounce with strength f, $v = u_f$ and $\lim\limits_{n\to+\infty} u_n = u_f$ uniformly in $[0,T]$ shall follow, on account of uniqueness.

Because $(u_{n_k})_k$ is equilipschitz, $\lim\limits_{k\to+\infty} u_{n_k} = v$ uniformly in $[0,T]$ and $\lim\limits_{k\to+\infty} f_{n_k} = f$ in $L^1([0,T];\mathbb{R})$, from (j),(jj),(jjj) and (v) follow respectively

$v \in Lip([0,T];\mathbb{R}) \quad , v \leq 0 \quad$ on $\quad [0,T]$;

$$\int_0^T [v(t)\ \ddot{\phi}(t) - f(t)\phi(t)]\ dt \leq 0 \qquad \text{for every} \qquad \phi \in C_0^\infty([0,T];\mathbb{R}^+);$$

for $v < 0$ one has $\int_0^T [v(t)\ \ddot{\phi}(t) - f(t)\ \phi(t)]dt = 0$ for every $\phi \in C_0^\infty([0,T];\mathbb{R}$

$v(0) = s,\ \dot{v}^+(0) = b.$

Moreover $\dot{v}^+(t)$ and $\dot{v}^-(t)$ exist for $t \in]0,T[$ and $\lim_{k\to+\infty} \dot{u}_{n_k} = \dot{v}$ almost everywhere in $[0,T]$.

Indeed the function $w(t) = v(t) - \int_0^t (\int_0^\eta f(\xi)d\xi)d\eta$ is the uniform limit of the sequence of concave funtions (see (jj))

$$w_{n_k}(t) = u_{n_k}(t) - \int_0^t (\int_0^\eta f_{n_k}(\xi)d\xi)d\eta \ .$$

Therefore w is concave, almost everywhere in $[0,T]$ differentiable and $\lim_{k\to+\infty} \dot{w}_{n_k} = \dot{w}$ almost everywhere in $[0,T]$.

This implies $\lim_{k\to+\infty} \dot{u}_{n_k} = \dot{v}$ almost everywhere in $[0,T]$, because

$$\lim_{k\to+\infty} f_{n_k} = f \quad \text{in} \quad L^1([0,T];\mathbb{R}).$$

Moreover $\lim_{k\to+\infty} \int_0^t f_{n_k}(\eta)\dot{u}_{n_k}(\eta)d\eta = \int_0^t f(\eta)\,\dot{v}(\eta)d\eta$.

Then, from (jv), as $k \to +\infty$:

$\frac{1}{2}[\dot{v}^\pm(t)]^2 = \frac{1}{2}b^2 + \int_0^t f(\eta)\,\dot{v}(\eta)d\eta$ almost everywhere in $[0,T]$, and also

$\frac{1}{2}[\dot{v}^\pm(t)]^2 = \frac{1}{2}b^2 + \int_0^t f(\eta)\,\dot{v}(\eta)d\eta$ for every $t \in [0,T]$, by a continuation of $[\dot{v}^\pm]^2$ at those points where v has no derivate.

Hence $v = u_f$ and $\lim_{n\to+\infty} u_n = u_f$ uniformly in $[0,T]$. ∎

Proposition.-

There exists a G_δ -set M^* such that

$M_o \subset M^* \subset L^1([0,T];\mathbb{R})$ and

for every $f \in M^*$ the solution to the Cauchy problem for the one-dimensional bounce is unique.

Proof. For every $f \in L^1([0,T];\mathbb{R})$ let

$$S(f) = \{u \in Lip([0,T];\mathbb{R}) \mid u \text{ solution to the Cauchy problem for the one-dimensional bounce with strength } f\}.$$

Define $D : L^1([0,T];\mathbb{R}) \to R$ by

$$D(f) = \sup_{v,w \in S(f)} \|v-w\|_{C^\circ([0,T];\mathbb{R})}.$$

We note that, if $f \in D^{-1}(0)$ then $S(f)$ is made by only one function. The proof (cfr. [2]) rests on the following statement:

(a) For every $f \in M_o$ and every $n \in N$ there exists an open neighborhood $\mathcal{J}_n^f$ of f in $L^1([0,T];\mathbb{R})$ such that $D(g) < \frac{1}{n}$ for every $g \in \mathcal{J}_n^f$.

In order to prove (a), we proceed ab absurdo. Therefore we assume the statement to be false.

Then there are $\bar{f} \in M_o$, $\bar{n} \in N$ and a sequence $(f_k)_k$ in $L^1([0,T];\mathbb{R})$ such that $\lim_{k\to+\infty} f_k = \bar{f}$ in $L^1([0,T];\mathbb{R})$ and $D(f_k) \geq \frac{1}{\bar{n}}$ for every k.

By $D(f_k) \geq \frac{1}{\bar{n}}$, for every k there are $v_k, w_k \in S(f_k)$ such that

$$\| v_k - w_k \|_{C^\circ([0,T];\mathbb{R})} \geq \frac{1}{n} .$$

Since $\lim_{k\to+\infty} f_k = \bar{f}$ in $L^1([0,T];\mathbb{R})$ and $\bar{f} \in M_0$, the lemma implies $\lim_{k\to+\infty} v_k = \lim_{k\to+\infty} w_k = u_{\bar{f}}$ uniformly in $[0,T]$, a contradiction.

Then $\mathcal{T}_n = \bigcup_{f\in M_0} \mathcal{T}_n^f$ is an open subset of $L^1([0,T];\mathbb{R})$. Therefore $M^* = \bigcap_{n=1}^{\infty} \mathcal{T}_n$ is a G_δ - set in $L^1([0,T];\mathbb{R})$.

For every $f \in M^*$ the Cauchy problem for the one-dimensional bounce with strength f has an unique solution since $M^* \subseteq D^{-1}(0)$ by (a). ■

Theorem. - The uniqueness of solutions to the Cauchy problem for the one-dimensional bounce is a generic property in $L^1([0,T];\mathbb{R})$.

Proof. -

The assertion follows at once (in view of what was said at the end of the introduction), because M_0 is dense in $L^1([0,T];\mathbb{R})$ and $L^1([0,T];\mathbb{R})$ is of second category, since it is a complete metric space (Baire's theorem). ■

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R E F E R E N C E S

[1] M. CARRIERO - E. PASCALI, *Il problema del rimbalzo unidimensionale e sue approssimazioni con penalizzazioni non convesse*, in corso di stampa sui Rendiconti di Matematica, Roma, fasc. IV, Vol.13, Serie VI (1980).

[2] G. VIDOSSICH, *Existence, Uniqueness and Approximation of Fixed Points as a Generic Property*, Universidade de Brasília, Brasília - Brasil, pp.17-29 (1973).

[3] H.L.ROYDEN, *Real Analysis*.