R.1.- Let f be an o-regular function from a normal topological space S to a finite directed graph G and Y a closed subset of S. Then, if for $a \in G$ we have $A^{f} \cap Y = \phi$ and $A^{f} \cap Y \neq \phi$ there exists an o-regular function $g: S \rightarrow G$, which is o-homotopic to f and such that $A^{g} \cap Y = \phi$ (See Lemma 6).

R.2 - In the construction of R.1, if there exist n vertices $p_1, \ldots, p_n \in G$, such that $p_1^f \cap \ldots \cap p_n^f = \phi$ then also it follows $p_1^{g_1} \cap \ldots \cap p_n^{g_n} = \phi$. (See Corollary 7).

R.3.- Let f be an o-regular function from S,S' to G,G', where S is a normal topological space, S' a closed subspace of S, Y a closed subset of S',G a finite directed graph and G' a subgraph of G. Then, if for $a \in G$ we have $A^{f} \cap Y = \phi$ and $A^{f} \cap Y = \phi$, there exists an o-regular function $g: S,S' \rightarrow G,G'$, which is o-homotopic to f and such that $\overline{A^{g}} \cap Y = \phi$. (See Lemma 11).

R.4.- In the construction of R. 3, if there exist n vertices P_1, \ldots, P_n . and m vertices $q_1, \ldots, q_m \in G'$, such that $\overline{P_1^f} \cap \ldots \cap \overline{P_n^f} \cap Q_1^{\overline{f'}} \cap \ldots \cap Q_m^{\overline{f'}} = \phi$, then also it follows that $\overline{P_1^g} \cap \ldots \cap \overline{P_n^g} \cap Q_1^{\overline{g'}} \cap \ldots \cap Q_m^{\overline{g'}} = \phi$. While, from $\overline{P_1^f} \cap \ldots \cap \overline{P_n^f} \cap \ldots = \phi$, it results $\overline{P_1^g} \cap \ldots \cap \overline{P_n^g} \cap S' = \phi$. (See Corollary 12).

By Duality Principle, the results dual to the previous ones are also true for o^{*}-regular functions.

1) Headed and totally headed subsets of a graph

DEFINITION 1.- Let G be a directed graph and X a non-empty subset of G. A vertex of X is called a head (resp. a tail) of X in G, if it is a

predecessor (resp. a successor) of all the other vertices of X. We denote by $H_{G}(X)$ (resp. $T_{G}(X)$) or, more simply, by H(X) (resp. T(X)) the set of the heads (resp. tails) of X in G. Then X is called beaded (resp. tailed) if $H(X) \neq \phi$ (resp. $T(X) \neq \phi$), otherwise, X is called non-headed (resp. non-tailed).

Finally X is called totally headed (resp. totally tailed), if all the non-empty subsets of X are headed (resp. tailed).

REMARK 1. - If X is a singleton, we agree to say that H(X) = T(X) = X, then X is totally headed and also totally tailed. If X is a pair, X headed $\iff X$ totally headed $\iff X$ tailed $\iff X$ totally tailed.

REMARK 2. - This definition and the following ones can be extended to undirected graphs. (See Proposition 6).

REMARK 3. - The concepts of head and tail (headed and tailed subset, etc.) are dual to each other.

DEFINITION 2. - A non-headed (resp. non-tailed) subset X is called minimal if all its non-empty proper subsets are headed (resp. tailed).

DEFINITION 3. - A finite directed graph G is called almost complete if the set of its vertices is totally headed.

REMARK. - A complete finite undirected graph is also almost complete.

PROPOSITION 1. - A finite directed graph G is almost complete iff the diagram (*) of the relation (\rightarrow) includes the diagram of a totally

ordered relation (<) in G.

(*) We use the term diagram rather than graph because graph is already used in another sense.

Proof. - i) Since *G* is almost complete, we can choose a vertex $v_1 \in G$, which is a predecessor of all the other vertices of *G*, as the first one; then a vertex $v_2 \in G - \{v_1\}$, predecessor of all the other vertices of $G - \{v_1\}$, as the second one; and so on.

ii) Since the diagram of the relation (\rightarrow) includes the diagram of a totally ordered relation (<) in G, we can totally order the vertices of G. Then every vertex of $_G$ is a predecessor of the vertices subsequent in the order relation.

Hence G is almost complete.

REMARK. - By ordering the vertices of G as in b) of Proposition 1, we say that the order relation (<) of G is compatible with the relation (\rightarrow)

of G.

PROPOSITION 2. - Let G be an almost complete graph. Then the dually directed graph G^* is also almost complete.

Proof. - Let (<) be a totally order relation, compatible with the relation (\rightarrow) of G.Then the dual order relation (>) is compatible with the relation (+) of the dually directed graph G^* .

DEFINITION 4. - Let G be a directed graph and X a subset of G. We call maximal subgraph induced by X the subgraph of G consisting of those directed edges of G, whose vertices are in X.

PROPOSITION 3. - A subset X of G is totally headed iff the maximal subgraph induced by X is almost complete.

PROPOSITION 4. - A subset X of G is totally headed iff it is totally



Proof. - By Remark 3 to Definition 1 and by Proposition 2,3 we have:

X totally headed in $G \iff$ the maximal subgraph induced by X is almost complete \iff the dually directed graph of the maximal subgraph induced by X is almost complete \iff X is totally headed in $G^* \iff X$ is totally tailed in G.

PROPOSITION 5. - A subset X of G is non-headed minimal iff it is non-tailed minimal.

Proof. - Since all the subsets of X are totally headed, by Proposition 4, they are also totally tailed. If we assume that X is tailed, then, by Definition 1, it is totally tailed. Hence, by Proposition 4, it is also totally

headed. Contradiction.

REMARK. - Then "almost complete graph, totally headed subset, non-headed minimal subset are selfdual concepts, while it does not follow for headed or tailed subset.

PROPOSITION 6. - In an undirected graph there does not exist any non-headed minimal n-tuple X with n > 2.

Proof. If all the pairs of vertices of X are headed (i.e. they are vertices of edges), then the maximal subgraph induced by X is complete. Hence it is also totally headed.

EXAMPLES.

1) Let $G = \{a, b, c, d, e\}$ be the graph with the edges $a \rightarrow b$, $a \rightarrow c$, $a \rightarrow d$,

 $b \rightarrow d$, $b \rightarrow e$, $c \rightarrow d$. Then the subset {a,b,e} is non-headed and non-tailed,

but it is not minimal non-headed (i.e.minimal non-tailed); {a,b,c} is headed

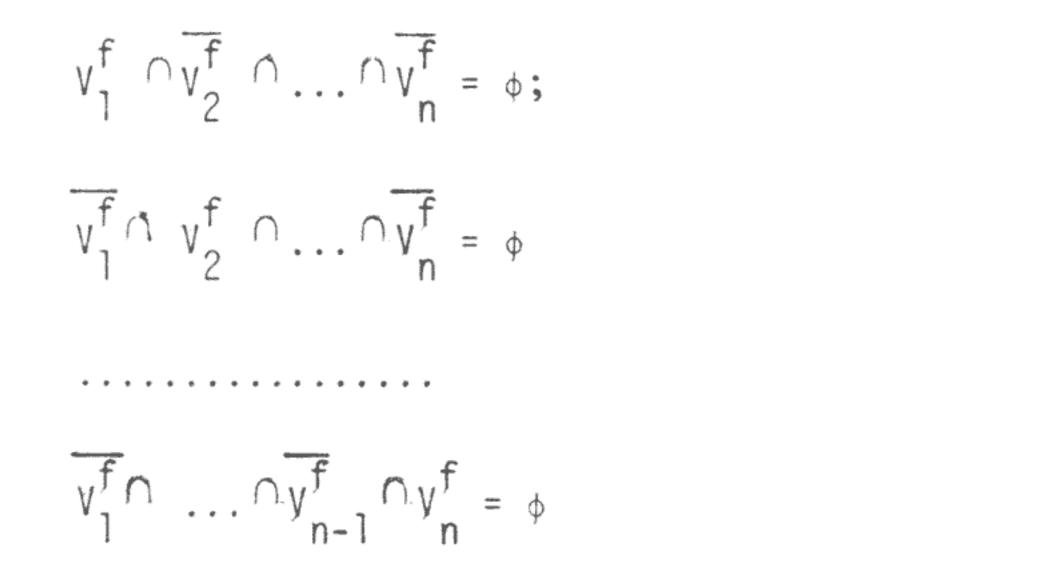
and non-tailed; {b,c,d} is non-headed and tailed; {a,b,c,d} is headed and

tailed, but not totally headed (tailed).

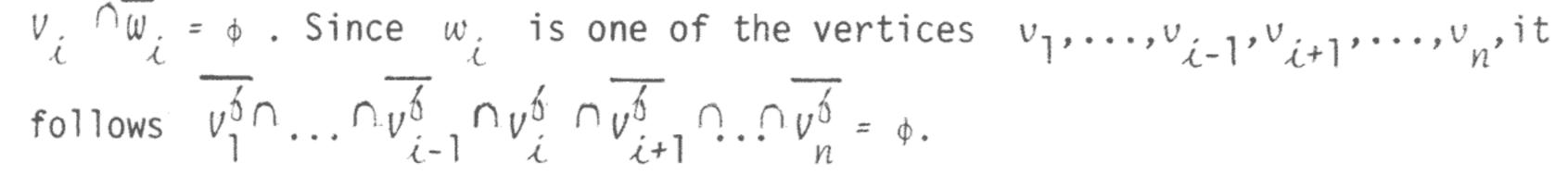
2) The graphs $G = \{u, v, w\}$ with the edges $u \rightarrow v, u \rightarrow w, v \rightarrow w$ and $G' = \{q, \pi, s, t\}$ with edges $q \rightarrow \pi, q \rightarrow s, q \rightarrow t, \pi \rightarrow s, \pi \rightarrow t, s \rightarrow t$ are examples of almost complete graphs. Moreover, the sets $\{u, v, w\}$, $\{q, \pi, s, t\}$ are examples of totally headed (i.e. totally tailed) subsets. Their compatible orders are, respectively, $u < v < w, q < \pi < s < t$. 3) In the graphs $G = \{f, g, h\}$ with the edges $f \rightarrow g, g \rightarrow h, h \rightarrow f$ and $G' = \{\ell, m, n, p\}$ with the edges $\ell \rightarrow m, \ell \rightarrow n, m \rightarrow n, m \rightarrow p, n \rightarrow \ell, n \rightarrow p,$ $p \rightarrow \ell, p \rightarrow m$ the sets $\{f, g, h\}$ and $\{\ell, m, n, p\}$ are examples of non-headed minimal (i.e. non-tailed minimal) subsets.

2) Singularities of a regular function.

PROPOSITION 7. - Let S be a topological space, G a finite directed graph, $f: S \rightarrow G$ and o-regular function from S to G and $X = \{v_1, v_2, \dots, v_n\}$ a non-headed subset of G ($n \ge 2$). Then it holds:



Proof. - Since X is a non-headed subset, there is no vertex v_i , which is a predecessor of all the other *n*-1 vertices. Then, for every i = 1, ..., nlet w_i be a vertex such that $v_i \neq w_i$. From o-regularity of f it is



DEFINITION 5. - Let S be a topological space, G a finite directed