one has

$$P_r(Q_i) = 0 \qquad 2r > \max(n_1,\ldots,n_k).$$

Using Bott's "Vanishing Theorem" one can, in a certain sense invert the preceding result obtaining

THEOREM B

Let M be a riemannian smooth orientable manifold of dimension n. Furthermore let M have k complementary distributions $E_i \subset TM$ and let the bundle $Q_i = TM/E_i$ have fibre of dimension q_i with

$$q_i = n - n_i \qquad \text{and} \qquad n_1+\ldots+n_k = n.$$

If Q_i is integrable, then for $2r > \max(n_1,\ldots,n_k,4q_i)$

$$P_h(Q_i)p_s(E_i) = 0 \qquad \forall h,s \geq 1 \quad \text{and} \quad h+s = r$$

holds.

In order to give a self-contained presentation we recall the necessary preliminaries.

1. Preliminaries

Let M be a smooth (paracompact) manifold and let E be a vector $\mathbb{R}^q$-bundle on M. As is well known, the total (real) Pontrjagin class p(E) of E is defined by

$$p(E) = 1+p_1(E)+\ldots+p_{\left[\frac{q}{2}\right]}(E) = \left[\det\left(I-\frac{1}{2\pi}\Omega\right)\right]$$

where Ω is the curvature of an arbitrary connection on E and $p_r(E) \in H^{4r}(M;\mathbb{R})$

where $H^*(M; \mathbb{R})$ is the de Rham cohomology ring of M. $P_r(E)$ is called the r-the Pontrjagin class of E.

Clearly $P_r(E) = 0$ for $4r > n$ if $n = \dim M$. Moreover if E is an oriented bundle of even dimension q, then the class $p_{q/2}(E)$, which is locally represented by the closed form $(2\pi)^{-q}$ det Ω, equals the square of the Euler class; this latter is strictly connected with the Euler-Poincaré characteristic of the manifold under consideration, if E = TM.

If E is the tangent bundle TM, then the classes are also called Pontrjagin class of M and are often denoted by $P_r(M)$.

Let Ω is the curvature form of a connection on the principal fibre bundle of orthonormal frames, then the explicit expression of Pontrjagin classes is give by[1)]

(1.1) $$P_r(TM) = \left[\frac{[(2r)!]^2}{(2^r\, r!)^2 (2\pi)^{2r}} \sum_{(i)} \Theta^{(2r)}_{i_1 \dots i_{2r}} \wedge \Theta^{(2r)}_{i_1 \dots i_{2r}} \right]$$

where

(1.2) $$\Theta^{(s)}_{i_1 \dots i_s} = \frac{1}{s!} \sum_{(i)} \delta(i_1,\dots,i_s; j_1,\dots j_s) \Omega_{j_1 j_2} \wedge \dots \wedge \Omega_{j_{s-1} j_s}$$

s is an even integer and $\delta(i_1,\dots,i_s; j_1,\dots,j_s)$ is the generalysed Kronecker symbol.

For n even, the n-form

(1.3) $$\Omega^{(n)} = \left[2^n \pi^{n/2} \left(\frac{n}{2}\right)! \right]^{-1} \sum \varepsilon_{i_1 \dots i_n} \wedge \Omega_{i_1 i_2} \dots \wedge \Omega_{i_{n-1} i_n}$$

called Gauss curvature form of M, is a representative of the Euler class of

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1)See J.A. Thorpe [6]

TM and if M is compact and orientable, the Gauss-Bonnet theorem says that

$$\int_M f^* \Omega^{(n)} = \chi(M)$$

where $\chi(M)$ is the Euler-Poincaré characteristic of M and $f: M \to \tilde{G}_n(M)$ is an orientation of M, $\tilde{G}_n(M)$ being the Grassmann bundle of the oriented n-planes tangent to M.

2. A vanishing theorem

We can now prove the following theorem ([3])

(2.1). Theorem

Let M be a riemannian smooth orientable manifold of dimension n which admits k complementary (smooth) distributions of oriented n_i-planes $(i = 1,\dots,k)$. Then the real Pontrjagin classes $P_r(M)$ are null for $2r>\max(n_1,\dots,n_k)$

Proof.

Let $\tilde{E}$ be the principal fibre bundle of the orthonormal frames (associated to the tangent). Its structural group is $\tilde{G} = SO(n)$ (the rotation group). We recall that the Lie algebra $\tilde{\mathcal{G}}$ of SO(n) can be identified with space of the skew-symmetric matrices of order n.

Let us consider the subbundle E of $\tilde{E}$ formed of the frames "adapted" to the distributions, viz. the orthogonal frames $\{e_i\}(i=1,\dots,n)$ so that the vectors

$$\{e_{\alpha_j}\} \qquad \alpha_j = n_1+\dots+n_{j-1}+1,\dots,n_1+\dots+n_j \qquad (n_\circ = 0)$$

form a basis for T^j. The bundle E can be regarded as having structural group