3. Local mild solution.

In order to prove that (10) has a unique local solution, we consider the space $Y = C([0,\bar{t}], X_o)$ with the usual norm $||u;Y|| = \max\{||u(t)||, t\epsilon[0,\bar{t}]\}$ and the non linear operator:

$$(12) \quad [P(u)](t) = u_1(t) + \int_0^t Z_o(t-s)\, F(u(s))ds \qquad D(P) \subset Y .$$

Then the equation (10) becomes

$$(13) \qquad u = P(u) .$$

As $D(F) \neq X_o$ also $D(P)$ is different from Y and so it is natural to introduce the following sets:

$$Y_\infty = C([0,\bar{t}] ; X_o \cap X_\infty) \quad \text{and}$$

$$S(r) = \{f\epsilon Y_\infty : ||f(t)||_\infty \leq r \ \text{ for } \ t \in [0,\bar{t}\,]\}.$$

$S(r)$ is a closed subset of Y.

We propose to prove that if $u_o \epsilon X_o \cap X_\infty$ and $r > ||u_o||_\infty$ then P maps $S(r)$ into itself and is strictly contractive over $S(r)$. So we will be able to assert the existence of a unique soluction.

We will need the following lemmas.

Lemma (4). If $X'_o = L^1(R \times \bar{V}^2)$ and $X'_\infty = L^\infty(R \times \bar{V}^2)$ then

: $J_1, J_2, J_3 \ J_1 \in B(X'_o) \cap B(X'_\infty)$ and

$$||J_1 f|| \leq c\, ||f|| \ ; \ ||J_2 f|| \leq c^2/2\, ||f|| \ ; \ ||J_3 J_1 f|| \leq \frac{d}{2}\, ||f||$$

$$||J_1 f||_\infty \leq c\, ||f\,||_\infty \ ; \ ||J_2 f||_\infty \leq c^2/2\, ||f||_\infty \ ; \ ||J_3 J_1 f||_\infty \leq \frac{d}{2}\, ||f||_\infty$$

where $c = v_2 - v_1$.

PROOF.

If follows easily from the definitions ∎

Lemma (5). F is a locally Lipschitz operator over $X_\circ \cap X_\infty$ and satisfies the following inequalities:

(a) $||F(f) - F(g)|| \leq d\,(||f||_\infty ||g||_\infty\ ||f-g||$

(b) $||F(f)||_\infty \leq d\ ||f\ ||_\infty^2$.

PROOF

If $f,g \in X_\circ \cap X_\infty$ then $f,g \in X'_\circ \cap X'_\infty$ and we can consider the operators J (that define F) as operators over $X'_\circ$ and X'_∞. So using lemma (4) the result follows from the following inequalities

$$||f(f) - F(g)|| \leq ||J_1 f \cdot J_2(f-g)|| + ||J_1(f-g) \cdot J_2 g|| + ||gJ_3J_1(f-g)|| +$$

$$+ ||(g-f)J_3J_1 f|| \leq ||J_1 f||_\infty \cdot ||J_2(f-g)|| + ||J_1(f-g)|| \cdot ||J_2 g||_\infty + ||g||_\infty \cdot ||J_3J_1(f-g)|| +$$

$$+ ||g-f|| \cdot ||J_3J_1 f||$$ ∎

Lemma (6). (a) If $g \in Y_\infty$ with $||g(s)||_\infty \leq \alpha(s)$ for $s \in [0,\bar{t}]$ and $\alpha(s)$ is continuous then

$$||\int_0^t g(s)ds||_\infty \leq \int_0^t \alpha(s)ds \qquad \text{for } t\in[0,\bar{t}]$$

(b) $$||Z_\circ(t)f||_\infty = e^{t/T}||f\ ||_\infty \qquad \text{for } f\in X_\circ \cap X_\infty \quad \text{and} \quad t \geq 0$$

PROOF

The integral $\beta(t) = \int_0^t g(s)ds$ is a strong Riemann integral in $X_\circ$ and so it is the strong limit of the corresponding Riemann sums:

$$B_n = \sum_{i=1}^{2^n} (s_{n,i} - s_{n,i-1})\ g\ (\bar{s}_{n,i}) \qquad n = 1,2,3,...$$

where $s_{n,i} = i\ \ t/2^n \qquad i = 1,2,...,2^n$

$$s_{n,i-1} \leq \bar{s}_{n,i} \leq s_{n,i}\ .$$

Now note that

$$||B_n||_\infty \leq \sum_{i=1}^{2^n}(s_{n,i} - s_{n,i-1})\alpha(\bar{s}_{n,i}) \leq \int_0^t \alpha(s)ds$$

if we choose the $\bar{s}_{n,i}$ so that

$$\alpha(\bar{s}_{n,i}) = \min\{\alpha(s);\ s_{n,i-1} \leq s \leq s_{n,i}\}$$

The assertion now follows because $s(\int_0^t \alpha(s)ds)$ is closed in X_o.

(b) follows from the definition of $Z_o(t)$.

∎

Lemma (7). If $u_o \in X_o \cap X_\infty$ and $r > ||u_o||_\infty$ then:

(a) $Y_\infty \subset D(P)$

(b) $||(P_u)(t)||_\infty \leq c_1(\bar{t})r$ for $u \in S(r)$ where

$$c_1(\bar{t}) = \frac{||u_o||_\infty}{r} + (1+d\ r\ T)(e^{\bar{t}/T} - 1)$$

(c) $||P(u) - P(w);\ Y|| \leq c_2(\bar{t})||u-w;Y||$ for $u,w \in S(r)$ where $c_2(\bar{t}) = 2\ d\ r\ \bar{t}$.

PROOF

(a) If $u_o \in X_o \cap X_\infty$ then $u_1 \in Y_\infty$ and from lemma (5) it follows that $F(u) \in Y$ if $u \in Y_\infty$.

So $P\ u \in Y$.

(b) If $u \in S(r)$ we have, by Lemmas (5) and (6)

$$||P(u)(t)||_\infty \leq e^{t/T}||u_o||_\infty + d\ r^2T(e^{t/T} - 1) \leq ||u_o||_\infty + (||u_o||_\infty + d\ r^2T)(e^{t/T} - 1)$$

(c) follows directly from (a) of Lemma (5) and from (b) of Lemma (2).

∎

THEOREM (1). If $u_o \in X_o \cap X_\infty$ and $r > ||u_o||_\infty$ then the equation (13) has a unique local solution $u \in S(r) \subset Y_\infty$.

PROOF. If $c(\bar{t}) = \max\{c_1(\bar{t}), c_2(\bar{t})\}$ then we can choose $\bar{t}$ so that $c(\bar{t}) < 1$. Then, Lemma (7) shows that P maps $S(r)$ into itself and that P is strictly contractive over $S(r)$.

Remark (2). The nonlinear operator F is not Fréchet differentiable, contrary to what happens in the papers [1] and [2], where this fact allowed the assertion that the mild solution was also the strict solution of the problem (see [6]).

The results are so different because in paper [1] the operator F is mollified and in paper [2] we used the space $X = U.C.B.(R^3)$ and $X_o = \{f : f \in X \ , \ f(x,v,w) = 0 \ \text{if} \ (v,w) \notin \bar{V}^2\}$

4. Positivity of the solution.

In this section we propose to prove that the solution of the problem (13) is positive if the initial condition u_o is positive.

This result is important from a physical point of view, since $u(x,v,w;t)dxdvdw$ gives the expected number of vehicles that, at time t, have (i) position between x and $x+dx$
(ii) speed between v and $v+dv$, (iii) desired speed between w and $w+dw$.

Introduce the following closed positive cones:

$X_o^+ = \{f \in X_o : f(x,v,w) \geq 0 \quad \text{for a.e.} \ (x,v,w) \in R \times \bar{V}^2\}$

$Y^+ = \{u \in Y : u(t) \in X_o^+ \quad \text{for} \ t \in [0,\bar{t}]\}$

and the relatively closed subsets:

$s^+(r) = s(r) \cap X_o^+$

$S^+(r) = S(r) \cap Y^+$.

Moreover define: