1. Introduction.

In this note we are examining again the model proposed by S. Paveri-Fontana in [5] and studied in various papers, in particular [1] and [2].

The problem of evolution, connected with such a model is

$$(1)\quad \begin{cases} \left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right)u + \frac{\partial}{\partial v}\left(\frac{w-v}{T}u\right) = F(u) & x\in R;\ t>0;\ v,w\in(v_1,v_2) = V \\ & (0\le v_1<v_2<+\infty) \\ u(x,v,w;0) = u_o(x,v,w) & x\in R;\ v,w\in\bar{V} \\ u(x,v,w;t) = 0 & t\ge 0;\ x\in R;\ v,w\notin\bar{V} \end{cases}$$

where, if $f = f(x,v,w)$,

$$(2)\quad F(f) = q\left[(J_1f)\cdot(J_2f) - f\,J_3J_1f\right] \qquad q \text{ constant in } [0,1]$$

$$J_1f = \int_{v_1}^{v_2} f(x,v,w')\,dw'$$

$$J_2f = \int_{v}^{v_2} (v'-v)f(x,v',w)\,dv'$$

$$J_3f = \int_{v'}^{v} (v-v')f(x,v',w)\,dv' .$$

The meaning of the symbols can be found in [5], [1] and [2]. In [2], the problem (1) is studied when u belongs to the space of the uniformly continuous and bounded functions $X = U.C.B.(R^3)$ and the existence and uniqueness of the local (in time) strict solution is proved. Noted that $u = u(x,v,w;t)$ is a car density and that

$$\int_{-\infty}^{+\infty} dx \int_{v_1}^{v_2} dv \int_{v_1}^{v_2} u(x,v,w;t)dt$$

gives the total number of cars on the motorway at the time t, the most natural space to study the problem (1) is $L^1(R^3)$. In [1], mollifying the non-linear part of the equation, i.e. F, we obtained the existence and uniqueness of the global strictsolution. Mollifyng, in our case, means replacing F with

(3) $F_\varepsilon(f)(x,v,w) = q[K_\varepsilon(J_1 f)\cdot(J_2 f) - f\, K_\varepsilon J_3 J_1 f]$

where

(4) $(K_\varepsilon f)(x,v,w) = \int_x^{+\infty} k_\varepsilon(x'-x) f(x',v,w)dx'$

and

(5) $k_\varepsilon \in L^\infty(0,+\infty)$; $k_\varepsilon(y) \geq 0$; $k_\varepsilon(y) = 0$ if $y \notin (0,\varepsilon)$; $\int_0^\infty k_\varepsilon(y)dy = 1$.

The aim of this work is to study the original problem, i.e.(1) , in L^1 and to find the connexion between the solution $u(t)$ of (1) and the solution $u_\varepsilon(t)$ of the mollified problem.

Precisely we prove that if $u_o \in L^1 \cap L^\infty$ then (1) has a unique local "mild" solution, i.e. the integral version of (1) has a unique local solution. If $[0,\bar{t}]$ is the existence time interval of such solution $u(t)$, we have

$$\lim_{\varepsilon\to o+} ||u_\varepsilon(t) - u(t)|| = 0$$

uniformly respect to t in $[0,\bar{t}]$. $||\cdot||$ is the usual norm in L^1.

We shall use the well-known results of linear semigroup theory for which we refer to [4] Chapter 9. For the results on the non linear evolution equations (in particular for semi-linear ones) we refer to [3],[6] and [8].

2. THE ABSTRACT PROBLEM.

Denote $X = \{f = f(x,v,w);\ f \in L^1(R^2 x\bar{V})\}$ and $X_o = \{f; f \in X,\ f(x,v,x) = 0$ a.e. if $v \notin V\}$ X_o is a closed subspace of X and we use it to get the third relation in (1).

Define

(6)
$$\begin{cases} A_1 f = v\, f_x - \frac{w-v}{T} f_v + \frac{1}{T} f \\ D(A_1) = \{f \in X_o;\ \exists f_x, f_v,\ v\, f_x + \frac{w-v}{T} f_v \in X_o\} \end{cases}$$

where $f_x = \frac{\partial f}{\partial x}$, $f_v = \frac{\partial f}{\partial v}$ are distributional derivatives.

If we consider the linear homogeneous problem connected with (1) and use the method of characteristics, we have

(7) $u(x,v,w;t) = \exp\frac{t}{T}\, u_o\,(\bar{x}(t),\ \bar{v}(t),w)$

where

$$\bar{x}(t) = \bar{x}(x,v,w;t) = x-wt+(w-v)T(\exp\frac{t}{T} - 1)$$

$$\bar{v}(t) = \bar{v}(x,w;t) = w-(w-v)\exp\frac{t}{T} \quad .$$

If we denote

(8) $[Z(t)f]\ (x,v,w) = \exp\frac{t}{T}\, f(\bar{x}(t),\bar{v}(t),w)$ $\qquad t\in R$

then we have as in [1] .

Lemma (1) . (a) $\{Z(t);\ t\in R\} \subset \mathcal{B}(X)$; (b) $||Z(t)f||=||f||$ for $f\in X$; (c) $\{Z(t);\ t\in R\}$ is a group.

If $Z_o(t)$ is the restriction of $Z(t)$ to the subspace X_o, $Z_o(t)$ maps X_o into itself for $t \geq 0$ and we have

Lemma (2). (a) $\{Z_o(t);\ t \geq 0\} \subset \mathcal{B}(X_o)$ and is a semigroup
(b) $||Z_o(t)f|| = ||f||$, for $f\in X_o$; (c) $Z_o(t)$ is strougly continuous in t for $t\geq 0$

If we denote by A_o the infinitesimal generator of $Z_o(t)$ ([4]) Chapeter 9) it is easy to prove that A_1 is the restriction of A_o to the set $D(A_1) \subset D(A_o)$ and that $Z_o(t)[D(A_1)] \subset D(A_1)$ (see [1]).

The natural domain of F is

$$D(F) = \{f : f \in X_o \ , \ F(f) \in X_o$$

and because this is not the whole X_o it is useful to introduce the following sets

$$X_\infty = L^\infty(R^2 \times \bar{V}) \quad \text{and}$$

$$s(r) = \{f : f \in X_o \cap X_\infty \; ; \; ||f||_\infty \leq r\}$$

where r is a positive constant and

$$||f||_\infty = \text{ess sup } \{|f(x,v,x)| \; : \; (x,v,w) \in R^2 \times \bar{V}\}.$$

We have

<u>Lemma</u> (3). (a) $X_o \cap X_\infty \subset D(F)$; (b) $||F(f)|| \leq q \, d ||f|| \; ||f||_\infty$ if $f \in X_o \cap X_\infty$, where $d = (v_2 - v_1)^3$; (c) $s(r)$ is closed in X_o .

<u>PROOF</u>.

(a),(b): If $f \in X_o \cap X_\infty$ and $v \notin V$ then $F(f)(x,v,w) = 0$ a.e.

(c) If we suppose that $f_n \in s(r)$, $||f_n - f|| \to 0$ as $n \to \infty$, but $f \notin s(r)$, then we obtain a contradiction

$\blacksquare$

<u>Remark</u> (1). It is useful to introduce $s(r)$ because $X_o \cap X_\infty$ is not closed in X_o.

With the preceding notation, the problem (1) assumes the abstract form

$$(9) \qquad \frac{du}{dt} = A_o u(t) + F(u(t)) \quad t>0; \; \lim_{t \to 0+} u(t) = u_o \in D(A_o)$$

where $u : [0,+\infty) \to X_o$ and $\frac{d}{dt}$ is a strong derivative . The integral version of the problem (9) is

$$(10) \qquad u(t) = u_1(t) + \int_0^t Z_o(t-s)F(u(s)) \, ds \qquad t > 0$$

where

$$(11) \qquad u_1(t) = Z_o(t)u_o$$

and from (b) of Lemma (2) $||u_1(t)|| = ||u_o||$

Every solution of (9) is also a solution of (10), but the converse is not generally true. For this reason every solution of (10) is said to be a "mild" solution of (9) (see [3]).

3. Local mild solution.

In order to prove that (10) has a unique local solution, we consider the space $Y = C([0,\bar{t}], X_o)$ with the usual norm $||u;Y|| = \max\{||u(t)||, t\in[0,\bar{t}]\}$ and the non linear operator:

$$(12) \quad [P(u)](t) = u_1(t) + \int_0^t Z_o(t-s)\, F(u(s))ds \qquad D(P) \subset Y .$$

Then the equation (10) becomes

$$(13) \qquad u = P(u) .$$

As $D(F) \neq X_o$ also $D(P)$ is different from Y and so it is natural to introduce the following sets:

$$Y_\infty = C([0,\bar{t}] ; X_o \cap X_\infty) \quad \text{and}$$

$$S(r) = \{f\in Y_\infty : ||f(t)||_\infty \leq r \quad \text{for} \quad t \in [0,\bar{t}]\}.$$

$S(r)$ is a closed subset of Y.

We propose to prove that if $u_o \in X_o \cap X_\infty$ and $r > ||u_o||_\infty$ then P maps $S(r)$ into itself and is strictly contractive over $S(r)$. So we will be able to assert the existence of a unique soluction.

We will need the following lemmas.

Lemma (4). If $X_o' = L^1(R \times \bar{V}^2)$ and $X_\infty' = L^\infty(R \times \bar{V}^2)$ then

: J_1, J_2, J_3 $J_1 \in B(X_o') \cap B(X_\infty')$ and

$$||J_1 f|| \leq c\, ||f|| \quad ; \quad ||J_2 f|| \leq c^2/2 ||f|| \quad ; \quad ||J_3 J_1 f|| \leq \frac{d}{2} ||f||$$

$$||J_1 f||_\infty \leq c\, ||f||_\infty \quad ; \quad ||J_2 f||_\infty \leq c^2/2 ||f||_\infty \quad ; \quad ||J_3 J_1 f||_\infty \leq \frac{d}{2} ||f||_\infty$$

where $c = v_2 - v_1$.

PROOF.

If follows easily from the definitions ∎

Lemma (5). F is a locally Lipschitz operator over $X_o \cap X_\infty$ and satisfies the following inequalities:

(a) $||F(f) - F(g)|| \leq d\,(||f||_\infty ||g||_\infty\ ||f-g||$

(b) $||F(f)||_\infty \leq d\ ||f\ ||_\infty^2$.

PROOF

If $f,g \in X_o \cap X_\infty$ then $f,g \in X'_o \cap X'_\infty$ and we can consider the operators J (that define F) as operators over X'_o and X'_∞. So using lemma (4) the result follows from the following inequalities

$$||f(f) - F(g)|| \leq ||J_1 f \cdot J_2(f-g)|| + ||J_1(f-g) \cdot J_2 g|| + ||gJ_3J_1(f-g)|| +$$

$$+ ||(g-f)J_3J_1 f|| \leq ||J_1 f||_\infty ||J_2(f-g)|| + ||J_1(f-g)|| \cdot ||J_2 g||_\infty + ||g||_\infty ||J_3J_1(f-g)|| +$$

$$+ ||g-f|| \cdot ||J_3J_1 f||$$ ∎

Lemma (6). (a) If $g \in Y_\infty$ with $||g(s)||_\infty \leq \alpha(s)$ for $s \in [0,\bar{t}]$ and $\alpha(s)$ is continuous then

$$|| \int_0^t g(s)ds ||_\infty \leq \int_0^t \alpha(s)ds \qquad \text{for } t \in [0,\bar{t}]$$

(b) $$||Z_o(t)f||_\infty = e^{t/T} ||f\ ||_\infty \qquad \text{for } f \in X_o \cap X_\infty \text{ and } t \geq 0$$

PROOF

The integral $\beta(t) = \int_0^t g(s)ds$ is a strong Riemann integral in X_o and so it is the strong limit of the corresponding Riemann sums:

$$B_n = \sum_{i=1}^{2^n} (s_{n,i} - s_{n,i-1})\ g\ (\bar{s}_{n,i}) \qquad n = 1,2,3,\ldots$$

where $s_{n,i} = i\ \ t/2^n$ $\qquad i = 1,2,\ldots,2^n$

$s_{n,i-1} \leq \bar{s}_{n,i} \leq s_{n,i}$.

Now note that

$$||B_n||_\infty \le \sum_{i=1}^{2^n}(s_{n,i} - s_{n,i-1})\alpha(\bar{s}_{n,i}) \le \int_0^t \alpha(s)ds$$

if we choose the $\bar{s}_{n,i}$ so that

$$\alpha(\bar{s}_{n,i}) = \min\{\alpha(s);\ s_{n,i-1} \le s \le s_{n,i}\}$$

The assertion now follows because $s(\int_0^t \alpha(s)ds)$ is closed in X_o.

(b) follows from the definition of $Z_o(t)$. ∎

Lemma (7). If $u_o \in X_o \cap X_\infty$ and $r > ||u_o||_\infty$ then:

(a) $Y_\infty \subset D(P)$

(b) $||(Pu)(t)||_\infty \le c_1(\bar{t})r$ for $u \in S(r)$ where

$$c_1(\bar{t}) = \frac{||u_o||_\infty}{r} + (1 + d\ r\ T)(e^{\bar{t}/T} - 1)$$

(c) $||P(u) - P(w);\ Y|| \le c_2(\bar{t})||u-w;Y||$ for $u,w \in S(r)$ where $c_2(\bar{t}) = 2\ d\ r\ \bar{t}$.

PROOF

(a) If $u_o \in X_o \cap X_\infty$ then $u_1 \in Y_\infty$ and from lemma (5) it follows that $F(u) \in Y$ if $u \in Y_\infty$.

So $P\ u \in Y$.

(b) If $u \in S(r)$ we have, by Lemmas (5) and (6)

$$||P(u)(t)||_\infty \le e^{t/T}||u_o||_\infty + d\ r^2 T(e^{t/T} - 1) \le ||u_o||_\infty + (||u_o||_\infty + d\ r^2 T)(e^{t/T} - 1)$$

(c) follows directly from (a) of Lemma (5) and from (b) of Lemma (2). ∎

THEOREM (1). If $u_o \in X_o \cap X_\infty$ and $r > ||u_o||_\infty$ then the equation (13) has a unique local solution $u \in S(r) \subset Y_\infty$.

PROOF. If $c(\bar{t}) = \max\{c_1(\bar{t}), c_2(\bar{t})\}$ then we can choose $\bar{t}$ so that $c(\bar{t}) < 1$. Then, Lemma (7) shows that P maps $S(r)$ into itself and that P is strictly contractive over $S(r)$.

Remark (2). The nonlinear operator F is not Fréchet differentiable, contrary to what happens in the papers [1] and [2], where this fact allowed the assertion that the mild solution was also the strict solution of the problem (see [6]).

The results are so different because in paper [1] the operator F is mollified and in paper [2] we used the space $X = U.C.B.(R^3)$ and
$X_0 = \{f : f \in X\ ,\ f(x,v,w) = 0 \text{ if } (v,w) \notin \bar{V}^2\}$

4. Positivity of the solution.

In this section we propose to prove that the solution of the problem (13) is positive if the initial condition u_0 is positive.

This result is important from a physical point of view, since $u(x,v,w;t)dxdvdw$ gives the expected number of vehicles that, at time t, have (i) position between x and $x+dx$
(ii) speed between v and $v+dv$, (iii) desired speed between w and $w+dw$.

Introduce the following closed positive cones:

$X_0^+ = \{f \in X_0 : f(x,v,w) \geq 0 \quad \text{for a.e. } (x,v,w) \in R \times \bar{V}^2\}$

$Y^+ = \{u \in Y : u(t) \in X_0^+ \quad \text{for } t \in [0,\bar{t}]\}$

and the relatively closed subsets:
$s^+(r) = s(r) \cap X_0^+$
$S^+(r) = S(r) \cap Y^+$.

Moreover define:

$Y_\infty^+ = Y_\infty \cap Y^+$.

Note that $Z_o(t)\ [X_o^+] \subset X_o^+$ but F does not map $D(F) \cap X_o^+$ into X_o^+. If this last condition was satisfied it would easily follow that $u(\cdot) \in S^+(r)$ locally, when $u_o \in X_o^+ \cap X_\infty$ and $r > ||u_o||_\infty$.

In order to prove that the solution is positive it is sufficient to prove that:

(14) there exists $a > 0$ such that $F_1(u) = (a\ I + F)(u) \in X_o^+$ for $u \in s^+(r)$ and that if we define

$$T(t) = e^{-at} Z_o(t) \qquad \text{and}$$

$$(P_1 g)(t) = T(t)u_o + \int_0^t T(t-s)F_1(g(s))ds$$

(15) P_1 maps $S(r)$ into itself.

These sufficient conditions are in [7], but for the reader's convenience we prove them in the appendix and we seize the opportunity to generalize some results.

Hence we have

Lemma (8). The assertions (14) and (15) are true.

PROOF

$au + F(u) = q\ J_1 u, J_2 u + (a - q\ J_3 J_1 u)u$, so if $u \in s^+(r)$ in order to prove $a\ u + F(u) \geq 0$ a.e. it is sufficient to prove $q\ J_3\ J_1 u \leq a$. Note that $q\ J_3 J_1 u \leq ||u||_\infty \frac{d}{2} q \leq r \frac{d}{2} q$ for $u \in s(r)$ so the condition (14) follows if we take $a \geq r \frac{d}{2} q$.

To prove the condition (15) we put $b = \frac{1}{T} - a$ then, if $g \in S(r)$, we have

$$||(P_1 g)(t)||_\infty \leq e^{bt}\ ||u_o||_\infty + d\ r^2 \int_0^t e^{b(t-s)} ds$$

and thus

$$||P_1 g)(t)||_\infty \leq c'(\bar{t})r$$

where

$$c'(\bar{t}) = \begin{cases} \frac{||u_o||_\infty}{r} + \left(\frac{||u_o||_\infty}{r} b + d\, r \right) \frac{e^{b\bar{t}}-1}{b} & \text{if} \quad b \neq 0 \\ \frac{||u_o||_\infty}{r} + d\, r\, \bar{t} & \text{if} \quad b = 0 \end{cases}$$

In each case we can have $c'(\bar{t}) < 1$ provided $r > ||u_o||_\infty$. ∎

Hence we can conclude with the following

THEOREM (2). If $u_o \in X_o^+ \cap X_\infty$ and $r > ||u_o||_\infty$ then the equation (13) has a unique local solution $u \in S^+(r)$.

Remark (3)For fixed $\bar{t}$ and $r = ||u_o||_\infty$ we can always choose a so as P_1 maps $S(r)$ into itself. In fact if $g \in S(r)$ we have

$$||P_1 g)(t)||_\infty \leq r + r(b+dr) \frac{e^{b\bar{t}}-1}{b} \qquad \text{for} \quad t \in [0,\bar{t}]$$

and so

$$||(P_1 g)\ (t)||_\infty \leq r \qquad \text{for} \quad t \in [0,\bar{t}]$$

when

$$b+dr \leq 0 \quad , \text{ i.e.} \qquad a \geq \frac{1}{T} + dr \ .$$

Nevertheless this result does not enable us to improve theorem (2) by removing the condition $r > ||u_o||_\infty$, because it was used in theorem (1), from which the theorem (2) comes.

In other words given $[0,\bar{t}]$ and $r = ||u_o||_\infty$, P_1 maps $S(r)$ into itself but P_1 can be noncontractive.

5. Global mild solution.

As in [1] we introduce the functional

$$J\, f = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dv \int_{V_1}^{V_2} f(x,v,w)dw \qquad \text{for} \quad f \in \mathfrak{X}.$$

We have:

Lemma (9). (a) $|Jf| \leq ||f||$ and so $J \in X^* = B(X;R)$

(b) $J Z_o(t)f = J f$ for $f \in X_o$ and $t \geq 0$

(c) $J F(f) = 0$ for $f \in X_o \cap X_\infty$.

Applying the functional J to (10) we have

$Ju(t) = J u_o$ for $t \in [0,\bar{t}]$

and because the solution is positive if $u_o \in X_o^+ \cap X_\infty$

we have

(16) $||u(t)|| = ||u_o||$ for $t \in [0,\bar{t}]$.

The physical meaning of this result is the invariability of the total number of the vehicles on the motorway with respect to the time. This is natural because the motorway is supposed to have no entrances or exits.
This fact allowed us to obtain the global solution in [1], but here we need information about $||u(t)||_\infty$ and not about $||u(t)||$.
We can prove that P is strictly contractive over S(r) also with respect to the norm $||\cdot||_\infty$, using the inequalities
$||F(f) - F(g)||_\infty \leq \frac{c^3+d}{2}(||f||_\infty+||g||_\infty)||f-g||_\infty$ for $f,g \in X_o \cap X_\infty$ and for $u,w \in S(r)$
$||P(u)-P(w)||_\infty \leq ||u-w||_\infty (c^3+d)r\ T(e^{t/T}-1)$. Then by well-known techniques (see e.g. [8] pag. 48) we can obtain that

$$||u(t)||_\infty \leq \frac{||u_o||_\infty\, e^{\bar{t}/T}}{1 - d\,T\,||u_o||_\infty\,(e^{\bar{t}/T}-1)} \qquad \text{for} \quad t \in [0,\bar{t}]$$

provided that

$$e^{\bar{t}/T} < \frac{1}{d\,T||u_o||_\infty} + 1 \qquad \text{and} \qquad ||u_o||_\infty \neq 0 \ .$$

From this last inequality it is clear that the time interval $[0,\bar{t}]$ will increase when $||u_o||_\infty$ decreases.

This result is also justified from the physical point of view, because once the are more than a suitable number of vehicles at each point of the motorway there will be a traffic jam.

In particular for $||u_o||_\infty = 0$ (i.e. $u_o(x,v,w) = 0$ a.e.) we have the global solution (i.e. for $t \in [0,\infty)$)

6. Connexion with the mollified problem.

As we said in the introduction, the mollified version of the problem (1) has a unique strict global solution $u_\varepsilon(t)$ and if $u_\circ \in X_\circ^+ \cap X_\infty$ we have

$$u_\varepsilon(t) = Z_\circ(t)u_\circ + \int_0^t Z_\circ(t-s)\, F_\varepsilon(u_\varepsilon(s))ds \qquad \text{for } t \geq 0$$

If $[0,\bar{t}]$ is the existence interval for the solution of the problem (13), we have for $t \in [0,\bar{t}]$:

$$(17)\quad ||u_\varepsilon(t) - u(t)|| \leq \int_0^t ||F_\varepsilon(u_\varepsilon(s)) - F(u(s))||ds.$$

The aim of this section is to prove the following

THEOREM (3). If $u_\circ \in X_\circ^+ \cap X_\infty$, $u(t)$ is the mild solution of the problem (1) in the interval $[0,\bar{t}]$ and $u_\varepsilon(t)$ is the strict global solution of the mollified version of the problem (1), then we have

$$(18)\quad \lim_{\varepsilon\to 0+} ||u_\varepsilon(t)-u(t)|| = 0 \qquad \text{uniformly in } t \in [0,\bar{t}]\ .$$

PROOF

If $f,g \in X_\circ \cap X_\infty$ then

$$(19)\quad ||F_\varepsilon(f) - F(g)|| \leq ||F_\varepsilon(f) - F_\varepsilon(g)||+||F_\varepsilon(g)-F(g)|| \leq 2\delta(||f|| + ||g||)||f-g|| + ||F_\varepsilon(g)-F(g)||$$

where $\delta = (v_2-v_1)\, ||k_\varepsilon||_\infty$ (see [1]).

Since we proved that the norm of the solution is invariable both in [1] and in this paper (see (16)), we have

$$||u_\varepsilon(t)|| = ||u_\circ|| = ||u(t)|| \qquad \text{for } t \in [0,\bar{t}]$$

and then, from (17) and (19)

$$||u_\varepsilon(t) - u(t)|| \leq \int_0^t ||F_\varepsilon(u(s)) - F(u(s))||ds + 4\delta||u_\circ||\int_0^t ||u_\varepsilon(s)-u(s)||ds.$$

If we suppose that we have proved that

$$(20) \qquad \lim_{\varepsilon\to 0+} \int_0^t ||F_\varepsilon(u(s))-F(u(s))||ds = 0 \quad \text{uniformly in } t\in[0,\bar{t}],$$

and $\eta > 0$ is given, then a suitable $\delta > 0$ can be found such that

$$||u_\varepsilon(t) - u(t)|| \leq \eta + 4\delta\, ||u_\circ|| \int_0^t ||u_\varepsilon(s)-u(s)||ds$$

for each $\varepsilon \in (0,\delta)$ and for $t \in [0,\bar{t}]$. Hence

$$||u_\varepsilon(t) - u(t)|| \leq \eta\, e^{4\delta||u_\circ||\bar{t}} \qquad \text{for } t \in [0,\bar{t}]$$

by Gronwall's Lemma. So the theorem is proved as soon as we have proved (20).
Define, for brevity

$$f(\varepsilon,s) = ||F_\varepsilon(u(s)) - F(u(s))||$$

and note that $f(\varepsilon,\cdot)$ is continuous because $F_\varepsilon(\cdot)$, $F(\cdot)$ and $u(\cdot)$ are continuous.
By Lebesgue's bounded convergence theorem to prove (20) it is sufficient to prove

$$(21) \qquad \lim_{\varepsilon\to 0+} f(\varepsilon,s) = 0 \qquad \text{for } s \in[0,\bar{t}]$$

$$(22) \qquad f(\varepsilon,s) \leq g(s) \qquad \text{for } s \in[0,\bar{t}],\ \varepsilon > 0$$

where $g(s)$ is a summable function independent of ε. We will prove (21) and (22) with the help of some lemmas. But first note that

$$(23) \quad F_\varepsilon(g) - F(g) = q[(K_\varepsilon-I)J_1gJ_2g - g(K_\varepsilon-I)J_3J_1g]$$

<u>Lemma</u> (10)

(a) $K_\varepsilon \in B(X_\circ)$, $\quad ||K_\varepsilon|| \leq 1$

(b) $\lim_{\varepsilon\to 0+} ||K_\varepsilon f - f|| = 0 \qquad$ for $\qquad f \in X_\circ$

PROOF

(a) $$||K_\varepsilon f|| \leq \int_{-\infty}^{+\infty} dv \int_{v_1}^{v_2} dw \int_{-\infty}^{+\infty} dx \int_{x}^{+\infty} k_\varepsilon(x'-x)|f(x',v,w)|dx' =$$

$$= \int_{-\infty}^{+\infty} dv \int_{v_1}^{v_2} dw \int_{-\infty}^{+\infty} dx' |f(x',v,w)| \int_{-\infty}^{x'} k_\varepsilon(x'-x)dx = ||f||$$

because $$\int_{-\infty}^{x'} k_\varepsilon(x'-x)dx = \int_{0}^{+\infty} k_\varepsilon(y)dy = 1$$

(b) $$||K_\varepsilon f-f|| = || \int_{x}^{+\infty} dx' \; k_\varepsilon(x'-x)[f(x',v,w)-f(x,v,w)]|| =$$

$$= || \int_0^\varepsilon dy \; k_\varepsilon(y)[f(x+y,v,w)-f(x,v,w)]|| \leq \int_0^\varepsilon dy \; k_\varepsilon(y)||f(x+y,y,w)-f(x,v,w)||.$$

Since $f \in X_o$ we have $\lim_{y\to o} ||f(x+y,v,w)-f(x,v,w)|| = 0$ and because $y \to 0$ as $\varepsilon \to o+$ the thesis follows. ∎

COROLLARY (1). If $g \in X_o \cap X_\infty$ then:

(a) $||F_\varepsilon(g) - F(g)|| \leq 2d \; ||g|| \; ||g||_\infty$

(b) $\lim_{\varepsilon\to o+} ||F_\varepsilon(g) - F(g)|| = 0$

PROOF

(a) By (a) of Lemma (10) we have $||k_\varepsilon - I|| \leq 2$ and by (23)

$$|| F_\varepsilon(g) - F(g)|| \leq 2(||J_1 g J_2 g|| + ||g||_\infty ||J_3 J_1 g||).$$

Now the assertion follows by Lemma (4).

(b) Define $g_1 = J_1 g J_2 g$, $g_2 = J_3 J_1 g$ then note that

$$||F_\varepsilon (g) - F(g)|| \leq ||K_\varepsilon g_1 - g_1|| + ||g||_\infty ||K_\varepsilon g_2 - g_2||$$

by (23) and finally the assertion follows by (b) of Lemma (10). ∎

Now, if $u_o \in X_o^+ \cap X_\infty$ then $u(s) \in X_o \cap X_\infty$ for $s \in [0,\bar{t}]$ and (21) follows by (b) of Corollary (1). By the results of § 5, the e exists $M > 0$ such that $||u(s)||_\infty \leq M$ for $s \in [0,\bar{t}]$ and (22) follows by (a) of Corollary (1) ∎

APPENDIX

Let X be a real Banach space and in this space consider the semi-linear problem

(A1) $$\frac{du}{dt} = A\,u + F(u) \qquad u(0) = u_o \in D(A)$$

where A is linear and generates the semigroup $Z(t)$ while F is non linear but at least continuous. The integral version of (A1) is the equation

(A2) $$u(t) = Z(t)u_o + \int_0^t Z(t-s)F(u(s))ds.$$

A solution of (A2) is called a mild solution of (A1) but as we know it is not in general a strict solution of (A1). On the other hand a solution of (A1) is also a solution of (A2).

Under suitable conditions such as the Fréchet-differentiablity of F and the continuity of the derivative a solution of (A2) with $u_o \in D(A)$ is also a solution of (A1) (see [6]).

Let c be a closed cone of X, i.e. c is a closed subset of X that satisfies the condition

$$x,y \in c\ ,\quad \alpha \geq 0 \Longrightarrow x + y \in c\ ,\quad \alpha x \in c\ .$$

If we set

$Y = C([0,\bar{t}];\ X)$, $C = \{u \in Y; u(t) \in c \ \text{ for } t \in [0,\bar{t}]\}$ s a closed subset of X contained in $D(F)$

$s' = s \cap c$

$S = \{u \in Y;\ u(t) \in s \ \text{ for } t \in [0,\bar{t}]\}$

$S' = S \cap C$

$$(P\,u)(t) = Z(t)u_o + \int_0^t Z(t-s)F(u(s))ds$$

we have the following

PROPOSITION. If

(a) $u_\circ \in s'$

(b) $\mathcal{Z}(t)\, u \in c$ for $u \in c$

(c) $F(u) \in c$ for $u \in s'$

(d) $P : S \to S$ and is strictly contractive then the unique solution u of $u = P\,u$ belongs to S'.

PROOF

The hypothesis ensure that $P : S' \to S'$ and so $u \in S'$ when (c) is not satisfied the follwing is useful.

THEOREM. With the same hypothesis (a), (b) and (d) of the preceding proposition, if

(c') $a > 0$ exists such that $F_1(u) \in c$ for $u \in s'$ where $F_1 = F + a\,I$

(d') If we set

$$(P_1 v)(t) = T(t)u_\circ + \int_0^t T(t-s)F_1(v(s))ds$$

where $T(t) = e^{-at}\, Z(t)$, P_1 maps S into itself, then we have the same conclusion of the preceding proposition, for a suitable $\bar{t}$.

In order to prove the theorem define the linear operator

$$(\mathcal{Z} u)(t) = a \int_0^t Z(t-s)u(s)ds \qquad \text{for } u \in Y \text{ and prove the following.}$$

LEMMA. $\mathcal{Z} \in B(Y)$ and for a suitable $\bar{t}$ the operator $I + \mathcal{Z}$ is invertible.

The operators P and P_1 are connected by the following equality

(A3) $$P_1 = (I + \mathcal{Z})^{-1}(P + \mathcal{Z})$$

PROOF.

$$(-a) \int_0^t Z(t-t')(P_1 u)(t')dt' = (-a) \int_0^t e^{a(t-t')} T(t-t')\cdot$$

$$\cdot\left[T(t')u_\circ + \int_0^{t'} T(t'-s)\, F_1(u(s))ds\right]dt' =$$

$$= (-a)\int_0^t e^{a(t-t')}\left[T(t)u_\circ + \int_0^{t'} T(t-s)F_1(u(s))ds\right]dt' =$$

$$= \left[1 - e^{at}\right] T(t)u_\circ + \int_0^t \left[1 - e^{a(t-s)}\right] T(t-s)F_1(u(s))ds =$$

$$= P_1 u - \left[Z(t)u_\circ + \int_0^t Z(t-s)(F(u(s)) + a\, u(s))ds\right]$$

So

$$(P_1 u)(t) = Z(t)u_\circ + \int_0^t Z(t-s)\{F(u(s)) + a\left[u(s) - P_1(u(s))\right]\}ds$$

and then

$$(P_1 u)(t) = (Pu)(t) + a\int_0^t Z(t-s)\left[u(s) - P_1(u(s))\right]ds$$

i.e.

(A4) $P_1 u = P\,u + \mathcal{Z}\cdot(u - P_1 u)$.

By last equality we have:

(A5) $(I + \mathcal{Z})\, P_1 u = (P + \mathcal{Z})u$.

Note that if $||Z(t)|| \leq M\, e^{bt}$ it follows that

$$||\mathcal{Z}|| \leq \begin{cases} a\, M\, \bar{t} & \text{if } b = 0 \\ a\, M\, \dfrac{e^{b\bar{t}} - 1}{b} & \text{if } b \neq 0 \end{cases}$$

. It is clear that this quantity is less than 1 for a suitable $\bar{t}$ and thus $I + \mathcal{Z}$ is invertible . So the assertion is true.

∎

REMARK (A1). (A4) and (A5) are valid in $[0,\bar{t}]$ for every $\bar{t}$, while (A3) is valid just for a suitable $\bar{t}$ (such that $||\mathcal{Z}|| < 1$).

COROLLARY (a) If u is a solution of $u = P_1 u$ in $[0,\bar{t}']$ then u is also solution of $u = Pu$ in the same interval. (b) If P is contractive then so is P_1 but in general not for the same $\bar{t}$. The converse is also true. (c) If u is a solution of $u = Pu$ in $[0,\bar{t}]$ then it is also solution of $u = P_1 u$ but, in general, in a smaller interval.

PROOF

(a) follows by (A4) and by remark (A1)

(b) By the lemma it follows that

$$||P_1|| \leq \frac{||P|| + ||\mathfrak{Z}||}{1 - ||\mathfrak{Z}||}$$

where $||\cdot||$ is the usual seminorm defined for Lipschitz operators

(i.e. $||P|| = \sup \{\frac{||P(u) - P(v)||}{||u-v||};\ u,v \in D(P)\}$).

We have $||P_1|| < 1$ if $||P|| + 2\,||\mathfrak{Z}|| < 1$ and this is true for a suitable $\bar{t}$. The inverse follows by (A4); in fact

$$||P|| \leq ||I + \mathfrak{Z}||\cdot||P_1|| + ||\mathfrak{Z}|| \quad .$$

(c) follows by (A3) and by remark (A1)

∎

PROOF OF THEOREM.

By hypothesis and by the Corollary it follows that

(b') $T(t)\, u \in c$ if $u \in c$

(c') $F_1(u) \in c$ if $u \in s'$

(d') P_1 maps S into itself and is strictly contractive, so by the Proposition it follows that a unique solution u of $u = Pu$ exists for a suitable $\bar{t}$ and it belongs to S'.

∎

Remark (A2)

The preceding results are especially useful in the cases where the mild solution is not in general the strict solution. Otherwise the preceding result is trivial because the problems

$$\frac{du}{dt} = A\,u + F(u) \quad ; \qquad u(0) = u_o \qquad \text{and}$$

$$\frac{dv}{dt} = (A-aI)v + F_1(v) \qquad v(o) = u_o$$

coincide.

ACKNOWLEDGMENT. - *The author wishes to thank Dr. Janet Dyson for her help during his stay in Oxford.*

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Accettato per la pubblicazione su proposta di A.BELLENI-MORANTE.