ON SOME TOPOLOGICAL-TYPE PROPERTIES OF CERTAIN ELEMENTS OF A PARTIALLY ORDERED SET

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SOMMARIO. - D. DRAKE e W.J. THRON hanno dato in [1] una caratterizzazione degli elementi v-irriducibili e degli elementi fortemente v-irriducibili di un reticolo distributivo $(L,\leq)$. Tra l'altro in [1] è stato provato che un elemento $c \in L$ è irriducibile se e solo se si può identificare $(L,\leq)$, tramite un isomorfismo reticolare f, con un sottoreticolo $(L',\leq)$ del reticolo delle parti $\mathcal{P}(X)$ di un opportuno insieme X in tal modo che $f(c)$ è la chiusura in L' di un certo elemento $x \in X$ (cioè $f(c)$ è il più piccolo elemento di L', rispetto all'inclusione insiemistica, cui appartiene x).

Come è ben noto un elemento di un reticolo distributivo è v-irriducibile se e solo se esso è v-primo. Questa proprietà è usata in maniera essenziale in [1]. In questo lavoro noi prendiamo lo spunto da questa proprietà per dare una caratterizzazione degli elementi v-primi e degli elementi fortemente v-primi di un qualsiasi insieme parzialmente ordinato (in particolare di un qualsiasi reticolo $(S,\leq)$). Precisiamo che qui, in analogia con una caratterizzazione degli elementi v-primi e degli elementi fortemente v-primi di un reticolo, un elemento c di un insieme parzialmente ordinato $(S,\leq)$ è detto v-primo se il sottoinsieme $D_c =\{s\in S: c\nleq s\}$ è v-diretto, cioè $D_c = \emptyset$ oppure per ogni $x_1, x_2 \in D_c$ (per ogni $x_1,\ldots,x_n \in D_c$) esiste $t \in D_c$ tale che $x_1 \leq t$ e $x_2 \leq t$ ($x_i \leq t$ per ogni $i = 1,\ldots,n$); inoltre c è detto fortemente v-primo se $D_c=\emptyset$ oppure D_c è dotato di massimo. Allora noi proviamo che un elemento $c \in S$ è v-primo in $(S,\leq)$ se e solo se possiamo identificare $(S,\leq)$, tramite un isomorfismo f rispetto all'ordinamento, con un insieme di insiemi (non necessariamente un reticolo di insiemi) del tipo di [1] in modo tale che $f(c)$ è la chiusura in $(f(S),\leq)$ di un elemento di $\cup f(S)$; inoltre proviamo che c è fortemente v-primo in $(S,\leq)$ se e solo se per ogni isomorfismo f del tipo su menzionato l'insieme $f(c)$ è la chiusura in $(f(S),\leq)$ di un punto di $\cup f(S)$.

SUMMARY. - A characterization of v-irreducible elements and of strongly v-irreducible elements of a distributive lattice $(L,\leq)$ was given by D.DRAKE and W.J. THRON in [1]. Among other things in [1] it was proven that an element $c \in L$ is v-irreducible iff one can identify $(L,\leq)$, by means of a lattice isomorphism f, with a sublattice $(L',\leq)$ of the power set P(X) of a suitable set X, in such a way that f(c) is the closure in L' of an element $x \in X$(i.e.F(c) is the minimum element in L', with respect to the set inclusion, including x).

As is well-known an element of a distributive lattice is v-irreducible iff it is v-prime. This property is exploited is an essential manner in [1]. Now then in our paper we took this property as a starting point for a characterization of v-prime and of strongly v-prime elements of any partially ordered set (in particular of any lattice). Here, on the analogy of some characterization of v-prime elements and of strongly v-prime elements of a lattice, an element c of a partially ordered set (shortly "poset" $(S,\leq)$ is said v-prime iff the subset $D_c = \{s \in S : c \nleq s\}$ is v-directed, i.e. $D_c = \emptyset$ or for every $x_1, x_2 \in D_c$ (for every $x_1,\ldots,x_n \in D_c$) there exists $t \in D_c$ such that $x_1 \leq t$ and $x_2 \leq t$ ($x_i \leq t$ for every i = 1,...n); moreover c is said strongly v-prime if $D_c = \emptyset$ or D_c has maximum element. Then we prove than an element $c \in S$ is v-prime in $(S,\leq)$ iff we can identify $(S,\leq)$ by means of an order isomorphism f, with a set (but not necessarily a lattice) of sets of the type of [1] in such a way that f(c) is the closure in $(f(S),\leq)$ of an element of $\cup f(S)$; moreover we prove that c is strongly v-prime in $(S,\leq)$ iff for all function f of the above type the set f(c) is the closure in $(f(S),\leq)$ of an element of $\cup f(S)$.

N. 1 PRELIMINARY CONSIDERATIONS.

We recal that a lattice is said a set lattice (see [1] p. 57) iff its elements are subsets of a suitable set X and the order relation is the set inclusion; in particular if the lattice is a sublattice of the power set $\mathcal{P}(X)$ then it is called a proper set lattice.

More generally we shall say that a set lattice $(L',\subseteq)$ is a "U-proper set lattice" iff the lattice join is equal to the set union.

We recall also that a proper set representation of a lattice $(L,\leq)$

is an ordered pair $((L',\subseteq),f)$, where $(L',\subseteq)$ is a proper set lattice and f is an isomorphism from $(L,\leq)$ onto $(L',\subseteq)$. If $(L',\subseteq)$ is a $\cup$-proper set we shall call $((L',\subseteq),f)$ a "$\cup$-proper set representation". We want to extend the previous definitions to the case of an arbitrary partially ordered set.

In the meantime we observe that the lattice join is equal to the set union in a set lattice $(L',\subseteq)$ iff the following property holds:

(i) For every $A_1,\ldots,A_n \in L'$ $\bigcup_{i=1}^{n} A_i$ is equal to the set intersection of all the elements of L' which include every $A_1,\ldots,A_n$.

As a consequence of this fact we shall say that a poset is a $\cup$-proper set poset iff its elements are subsets of a suitable set X, the order relation is the set inclusion and property i) holds[1]; thus we shall say that the ordered pair $((S',\subseteq),f)$, where $(S',\subseteq)$ is a $\cup$-proper set poset and f is a function, is a $\cup$-proper set representation of a poset $(S,\leq)$ iff f is an order isomorphism from S onto S'(i.e. f is a bijective isotone function from S onto S' and f^{-1} is also isotone).

In the following we shall prove the next properties:

1) An element c of a poset $(S,\leq)$ is v-prime iff a $\cup$-proper set representation $((f(S),\subseteq),f)$ of $(S,\leq)$, exists such that $f(c)$ is a point closure $f(S)$. Moreover if $(S,\leq)$ has at least a v-prime element then a $\cup$-proper set representation $((f(S),\subseteq),f)$ of $(S,\leq)$ exists such that f maps every v-prime element of $(S,\leq)$ in a point closure in $f(S)$.

2) An element c of the poset $(S,\leq)$ is strongly v-prime iff for every

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([1]) In this case if a common upper bound of $A_1,\ldots,A_n$ does not exist in L' then we put the above mentioned set intersection equal to $\bigcup_{Y\in L'} Y$.

$\mathcal{U}$-proper set representation $((f(S), \subseteq, f)$ of $(S,\leq)$[2] $f(c)$ is a point closure.

N. 2. A BRIEF REVIEW OF PREORDERED SETS.

Let S be a set and $\lesssim$ a preorder relation for S(i.e. $\lesssim$ exhibits the transitive and reflexive properties). All the most important notions about a poset can be extended to a preordered set (e.g. upper bound, lower bound, maximum, minimum, ℓ.u.b., g.ℓ.b., etc.); thus a right tail of a preordered set $(S,\lesssim)$ will be every $Y \subseteq S$ such that $\forall\ x,y \in S: x \in Y$ and $x \lesssim y \Longrightarrow y \in Y$.

We observe that if $Y_1 \subseteq S$ then the set $r(Y_1) = \{x \in S : x$ is an upper bound of $Y_1\}$ is a rigth tail of $(S,\lesssim)$; in particular the principal filter $r(y) = r(\{y\})$ generated by $y \in S$ is a right tail of $(S,\lesssim)$. Moreover

(ii) $r(X) = \bigcap_{x \in X} r(x)$ and if X is a right tail then $X = \bigcup_{x \in X} r(x)$.

Now let $\mathcal{C}$ be a subset of $\mathcal{P}(S)$ (the power set of S), x an element of S and $\mathcal{C}_x = \{X \in \mathcal{C} : x \in X\}$. Then we define, for every $x,y \in S$

(j) $$x \lesssim y(\mathcal{C}) \text{ iff } \mathcal{C}_x \subseteq \mathcal{C}_y .$$

Clearly the defined relation is a preorder relation. Moreover if $\mathcal{C}'$ is the set of set complements of the elements of $\mathcal{C}$ it follows, since $\mathcal{C}_x \subseteq \mathcal{C}_y$ iff $\mathcal{C}'_y \subseteq \mathcal{C}'_x$,

(jj) $$x \lesssim y(\mathcal{C}) \text{ iff } y \lesssim x(\mathcal{C}') .$$

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(2) We shall prove that there exists at least a $\mathcal{U}$-proper set representation of $(S,\leq)$.

One can easily verify that every element of $\mathcal{C}$ is a right tail of S preordered with respect to relation defined in (j). On the other hand if $(S,\lesssim)$ is a preordered set and $\mathcal{R}$ is the set of the right tails of $(S,\lesssim)$ then for every $x,y \in S$ the following properties hold

(e) $x \lesssim y$ iff $x \lesssim y(\mathcal{R})$ according to (j), and $x \lesssim y$ iff $x \lesssim y(\mathcal{R}_0)$,

where $\mathcal{R}_0$ is the set of the principal filters of $(S,\leq)$ generated by an elment of S. And hence, as a consequence of (jj), the following property holds:

(ee) $x \leq y$ iff $y \leq x\ (\mathcal{R}'_0)$,

where $\mathcal{R}'_0$ is the set of all set complements in S of the elements of $\mathcal{R}_0$.

Now we can give the following

THEOREM 1. Let $\mathcal{C} \leq \mathcal{P}(S)$ be, $Y \in \mathcal{C}$ and $y \in Y$. Then Y is a point closure in $\mathcal{C}$ with respect to y iff y is minimum in Y with respect to relation defined in (j).

PROOF. In fact: y is minimum in $Y \Longleftrightarrow \forall x \in Y : \mathcal{C}_y \subseteq \mathcal{C}_x \Longleftrightarrow \forall x \in Y$, $\forall Z \in C_y : Z \in \mathcal{C}_x \Longleftrightarrow \forall x \in Y, \forall Z \in \mathcal{C}_y : x \in Z \Longleftrightarrow \forall Z \in \mathcal{C}_y : Y \subseteq Z$.

Q.E.D.

N. 3. A CHARACTERIZATION OF V-PRIME AND STRONGLY V-PRIME ELEMENTS OF A POSET.

Henceforth let $(S,\leq)$ be a poset. Then we can consider the function $g : S \to \mathcal{P}(S)$ mapping an element $x \in S$ into the set $g(x) = \{y \in S : x \nleq y\} = S - r(x)$. Clearly f is an injective function; moreover $\forall x,y \in S$ $x \leq y$ iff $g(x) \leq$

$\leq g(y)$, in fact $x \leq y$ iff the principal filter $r(x)$ includes the principal filter $r(y)$, and hence $x \leq y$ iff $g(x) = S - r(x) \subseteq S - r(y) = g(y)$; thus g is an isomorphism from $(S,\leq)$ onto $(g(S),\subseteq)$. Now we want to prove the following.

THEOREM 2. The ordered pair $((g(S),\subseteq),g)$ is a U-proper set representation of $(S,\leq)$.

PROOF. Since g is an isomorphism from $(S,\leq)$ onto $(g(S),\subseteq)$ we need only prove that $(g(S),\subseteq)$ is a U-proper set poset. Thus if $Y \subseteq S$ then $\bigcap_{y\in Y} r(y) = r(Y) = \bigcup_{x\in r(Y)} r(x)$ (see (ii) of N. 2). Whence $\bigcup_{y\in Y}(S-r(y))= S-r(Y)=$

$$= \bigcap_{x\in r(Y)}(S-r(x)) = \bigcap_{x\in r(Y)} g(x).$$

In particular if $Y = \{y_1,\ldots,y_n\}$ and $A_i = g(y_i) = S-r(y_i)$ $(i=1,\ldots,n)$

then

$$\bigcup_{i=1}^{n} A_i = \bigcap_{x\in r(Y)} g(x)$$

and hence $\bigcup_{i=1}^{n} A_i$ is equal to the set intersection of all $g(x)$ such that

$g(x) \supseteq \bigcup_{i=1}^{n} A_i$. Q.E.D.

Now we can give the following

THEOREM 3. Let c be an element of S and let c be not minimum in $(S,\leq)$. Then c is strongly v-prime in $(S,\leq)$ iff for ever U-proper set representation $((f(S),\subseteq),f)$ of $(S,\leq)$ $f(c)$ is a point closure in $f(S)$.

PROOF. Let $f(c)$ be a point closure in $f(S)$ for every U-proper set representation $((f(S),\subseteq),f)$. Now then we consider the U-proper set representation of theorem 2; thus, as a consequence of the fact that c is not minimum in $(S,\leq)$, the set $\{y\in S: c\nleq y\}$ is different from $\emptyset$ and it is a point closure in $g(S)$.

Hence, as a consequence of theorem 1 and of (ee) in N. 2, g(c) has a maximum in $(S,\leq)$, therefore c is a strongly v-prime element of $(S,\leq)$.

Conversely let c be a strongly v-prime element in $(S,\leq)$; in such a case the set $\{y\in S: c\nleq y\}$ has a maximum b. Now let $((f(S),\subseteq),f)$ be an arbitrary u-proper set representation of $(S,\leq)$. Since $c\nleq b$ then $f(c)\nsubseteq f(b)$ and hence we can consider an element $x\in f(c)-f(b)$. We want to prove that f(c) is a point closure with regard to x. In fact if Z is an element of f(S) such that $x\in Z$ then there exists $z\in Z$ such that $f(z) = Z$. Now then $c\leq z$; in fact assume, ab absurdo, that $c\nleq z$, then $z\leq b$, thus $Z = f(z) \subseteq f(b)$ and hence $x\in f(b)$. Since $c\leq z$ iff $f(c) \subseteq f(z) = Z$, the theorem follows.

Q.E.D.

THEOREM 4. An element $c \in S$ is v-prime in $(S,\leq)$ iff a $\mathbb{U}$-proper set representation $((f(S),\leq),f)$ of $(S,\leq)$ exists such that f(c) is a point closure in f(S).

PROOF. Let $((f(S),\leq),f)$ be a $\mathbb{U}$-proper set representation of $(S,\leq)$ such that f(c) is a point closure in f(S) with regard to x; moreover let $y,z \in S$ such that $c\nleq y$ and $c\nleq z$. Then $(f(c) \nsubseteq f(y)$ and $f(c) \nsubseteq f(z)$, thus $x\notin f(y)$ and $x\notin f(z)$; moreover (since $(f(S),\subseteq)$ is a $\mathbb{U}$-proper set poset) $f(y)\cup f(z)$ is equal to set intersection of all f(t) including f(y) and f(z). As a consequence an element $t \in S$ exists such that $f(t) \supseteq f(z)\cup f(y)$ and $x\notin f(t)$, thus $f(c)\nsubseteq f(t)$ and hence $c\nleq t$ but $y\leq t$ and $z\leq t$. That means that the subset $\{s \in S : c\nleq s\}$ is v-directed .

Convensely let c be a v-prime element in $(S,\leq)$ and $((f(S),\subseteq),f)$ a $\mathbb{U}$-proper set representation of $(S,\leq)$. If f(c) is a point closure in f(S) we have nothing to prove. If not let us consider the set $X' = X\cup\{X\}$, ((cfr. [1], p. 62, proof of theorem 31), where $X = \bigcup_{t \in S} f(t)$, and the function $f' : S \to \mathcal{P}(X')$ such that for every $s \in S$ $f'(s) = f(S)$ iff $c \not\leq s$ and $f'(s) = f(s) \cup \{X\}$ iff $c \leq f(s)$. Clearly f' is an injective function and for every, $s_1, s_2 \in S$ $s_1 \leq s_2$ iff $f'(s_1) \subseteq f'(s_2)$, moreover $f'(c)$ is point closure with regard to X in $f'(S)$. Then we must only prove that $((f'(S), \subseteq), f')$ is a $\bigcup$-proper set representation of $(S, \leq)$.

Now if $s_1, \ldots, s_n \in S$ then $\bigcup_{i=1}^{n} f(s_i)$ is equal to the set intersection of all the elements of $f(S)$ that include every $f(s_i)$. Hence let's consider the following two cases:

CASE 1 : for every $i = 1, \ldots, n$: $c \not\leq s_i$. Then for every $i = 1, \ldots, n$ $f'(s_i) = f(s_i)$, moreover (since c is v-prime in $(S, \leq)$) an element $s \in S$ exists such that $c \not\leq s$ and for every $i = 1, \ldots n$ $s_i \leq s$, thus $f'(s) = f(s)$ hence $\bigcup_{i=1}^{n} f'(s_i) (= \bigcup_{i=1}^{n} f(s_i))$ is equal to the set intersection of all the elements of $f'(S)$ that include every $f'(s_i) (= f(s_i))$.

CASE 2 : for some i: $c \leq x_i$. Then $f'(s_i) = f(s_i) \cup \{X\}$; moreover for every $s \in S$ such that $f'(s)$ includes every $f'(s_i)$ one has $f'(s) \supseteq f'(s_i)$, thus $c \leq s_i \leq s$ and hence $f'(s) = f(s) \cup \{X\}$. Then in this case too we can conclude that $\bigcup_{i=1}^{n} f'(s_i) (= \bigcup_{i=1} f(s_i) \cup \{x\})$ is equal to the set intersection of all the element of $f'(S)$ including every $f(s_i)$.

Q.E.D.

REMARK.

We observe that if $(S, \leq)$ has at least a v-prime element then a $\bigcup$-proper set representation $((f(S), \subseteq), f)$ of $(S, \leq)$ exists such that f maps every v-prime element of $(S, \leq)$ in a point closure. In fact let A be the set of all v-prime elements of $(S, \leq)$, $((f(S), \subseteq), f)$ a $\bigcup$-proper set representation of $(S, \leq)$ and B the set of all the elements of A

mapped into a point closure. If $B = A$ we have nothing to prove.

Now we suppose that $B \neq A$ and

$$(m) \qquad (\bigcup_{s\in S} f(s)) \cap (A-B) = \emptyset \quad (3).$$

Then we consider the function f' that maps every $s \in S$ into the set $f(s) \cup A_s$, where $A_s = \{y \in A - B : y \leq s\}$. Clearly f' is an injective function and for every $s_1, s_2 \in S$ $s_1 \leq s_2$ iff $f(s_1) \cup A_{s_1} \subseteq f(s_2) \cup A_{s_2}$. Moreover if $s_1, \ldots, s_n$ are arbitrary elements of S then

$$\bigcup_{i=1}^{n}(f(s_i) \cup A_{s_i}) = (\bigcup_{i=1}^{n} f(s_i)) \cup (\bigcup_{i=1}^{n} A_{s_i}).$$

Now let Z be the set of all upper bounds of $\{s_1, \ldots, s_n\}$ in $(S, \leq)$.

We want to prove that $\bigcap_{z\in Z} (f(z) \cup A_z) = \bigcup_{i=1}^{n}(f(s_i) \cup A_{s_i}) = \bigcup_{i=1}^{n}(f(s_i)) \cup (\bigcup_{i=1}^{n} A_i)$.

As a consequence of condition (m) $\bigcap_{z\in Z}(f(z) \cup A_z) = (\bigcap_{z\in Z} f(z)) \cup (\bigcap_{z\in Z} A_z)$;

moreover we already know that $\bigcap_{z\in Z} f(z) = \bigcup_{i=1}^{n} f(s_i)$ and $\bigcap_{z\in Z} A_z \supseteq \bigcup_{i=1}^{n} A_{s_i}$; then it is sufficient to prove that $\bigcap_{z\in Z} A_z \subseteq \bigcup_{i=1}^{n} A_{s_i}$.

Now if $x \in \bigcap_{z\in Z} A_z$ then x is a v-prime element of $(S, \leq)$ such that $x \leq z$ for every $z \in Z$. Moreover Z is the set of all upper bounds of $\{s_1, \ldots, s_n\}$, then as a consequence of the definition of v-prime element $x \leq s_i$ for some $i \in \{1, \ldots, n\}$, thus $x \in \bigcup_{i=1}^{n} A_{s_i}$ and hence $\bigcap_{z\in Z} A_z \subseteq \bigcup_{i=1}^{n} A_{s_i}$.

From this the enounced assertion follows.

REFERENCE

[1] D.DRAKE and W.J.THRON "On the representations of an abstract lattice as the family of closed sets of a topological space". Trans. of Amer. Math. Soc. 120(1965), 57-71.

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(3) The case $(\bigcup_{s\in S} f(s)) \cap (A-B) \neq \emptyset$ can easily be reconducted to condition (m)