Operata la sostituzione precedente, è immediato verificare che:

$z(x)$ appartiene a $C^1([0,x_\circ],\mathbb{R}^2)$ ed è soluzione di (3),(4), se e solo se: $z(x)$ appartiene a $C^\circ([0,x_\circ]\mathbb{R}^2)$, ed è soluzione di

$$
(5)\qquad \begin{cases} z_1(x) = \displaystyle\int_0^x z_2(t)dt + u_\circ \\[2ex] z_2(x) = -\dfrac{1}{a_2}\left[a_1\displaystyle\int_0^x z_2(t)dt + \int_0^x \psi(t)z_1(t)dt + \int_0^x z_1(\xi)d\xi \int_0^\xi z_1(t)\phi\left(\frac{t}{\xi}\right)dt\right] + u_1 \end{cases}
$$

1. UNICITA' DEL PROBLEMA (3), (4)

Siano $z(x) = (z_1(x),z_2(x))$ e $\bar{z}(x) = (\bar{z}_1(x),\bar{z}_2(x))$ soluzioni del problema (3),(4), con i dati rispettivamente, $u_\circ,u_1,\psi(x)$ e $\bar{u}_\circ,\bar{u}_1,\bar{\psi}(x)$, con $z(x)$ e $\bar{z}(x)$ appartenenti a $C^1([0,x_\circ],\mathbb{R}^2)$.

Tenendo conto delle (5) si ha:

$$
(6)\qquad \begin{cases} z_1(x)-\bar{z}_1(x) = \displaystyle\int_0^x [z_2(t) - \bar{z}_2(t)]dt + u_\circ - \bar{u}_\circ \\[2ex] z_2(x)-\bar{z}_2(x) = -\dfrac{1}{a_2}\left\{a_1\displaystyle\int_0^x \left[z_2(t)-\bar{z}_2(t)\right]dt + \int_0^x\left[\psi(t)z_1(t)-\bar{\psi}(t)\bar{z}_1(t)\right]dt\right. + \end{cases}
$$

$$
\left. + \int_0^x d\xi\left[z_1(\xi)\int_0^\xi z_1(t)\phi\left(\frac{t}{\xi}\right)dt - \bar{z}_1(\xi)\int_0^\xi \bar{z}_1(t)\phi\left(\frac{t}{\xi}\right)dt\right]\right\} + u_1 - \bar{u}_1 .
$$

Sommando e sottraendo, nella seconda delle (6),

$$
\int_0^x \bar{z}_1(t)\psi(t)dt + \int_0^x \bar{z}_1(\xi)d\xi\int_0^\xi z_1(t)\phi\left(\frac{t}{\xi}\right)dt, \quad \text{risulta:}
$$

$$
z_2(x)-\bar{z}_2(x) = -\frac{1}{a_2}\left\{a_1\int_0^x\left[z_2(t)-\bar{z}_2(t)\right]dt + \int_0^x\left[z_1(t)-\bar{z}_1(t)\right]\psi(t)dt\right. +
$$

$$+\int_0^x \bar{z}_1(t)\left[\psi(t)-\bar{\psi}(t)\right]dt+\int_0^x\left[z_1(\xi)-\bar{z}_1(\xi)\right]d\xi\int_0^\xi z_1(t)\phi(\frac{t}{\xi})dt\ +$$

$$+\int_0^x \bar{z}_1(\xi)d\xi\int_0^\xi\left[z_1(t)-\bar{z}_1(t)\right]\phi(\frac{t}{\xi})dt\Bigg\}+u_1-\bar{u}_1.$$

Applicando all'ultimo integrale la formula di inversione di Dirichlet si trae:

$$z_2(x)-\bar{z}_2(x)=-\frac{1}{a_2}\Bigg\{a_1\int_0^x\left[z_2(t)-\bar{z}_2(t)\right]dt+\int_0^x\left[z_1(t)-\bar{z}_1(t)\right]\psi(t)dt\ +$$

$$+\int_0^x \bar{z}_1(t)\left[\psi(t)-\bar{\psi}(t)\right]dt+\int_0^x\left[z_1(t)-\bar{z}_1(t)\right]dt\int_0^t z_1(\xi)\phi(\frac{\xi}{t})d\xi\ +$$

$$+\int_0^x\left[z_1(t)-\bar{z}_1(t)\right]dt\int_t^x \bar{z}_1(\xi)\phi(\frac{t}{\xi})d\xi\Bigg\}+u_1-\bar{u}_1\ .$$

Ponendo

$$F(x,t)=\psi(t)+\int_0^t z_1(\xi)\phi(\ \frac{\xi}{t}\ )d\xi+\int_t^x \bar{z}_1(\xi)\phi(\frac{t}{\xi})d\xi$$

si ha

$$z_2(x)-\bar{z}_2(x)=-\frac{1}{a_2}\Bigg\{a_1\int_0^x\left[z_2(t)-\bar{z}_2(t)\right]dt+\int_0^x\left[z_1(t)-\bar{z}_1(t)\right]F(x,t)dt\ +$$

$$+\int_0^x \bar{z}_1(t)\left[\psi(t)-\bar{\psi}(t)\right]dt\Bigg\}+u_1-\bar{u}_1\ .$$

Tenendo conto della I^a equazione del sistema (6), si ottiene

$$z_2(x)-\bar{z}_2(x) = -\frac{1}{a_2}\left\{a_1\int_0^x\left[z_2(t)-\bar{z}_2(t)\right]dt+\int_0^x F(x,t)dt\int_0^t\left[z_2(\xi)-\bar{z}_2(\xi)\right]d\xi\right. +$$

$$\left.+\int_0^x\left[u_\circ-\bar{u}_\circ\right]F(x,t)dt+\int_0^x\bar{z}_1(t)\left[\psi(t)-\bar{\psi}(t)\right]dt\right\}+u_1-\bar{u}_1 .$$

Infine, applicando al II° integrale la formula di inversione di Dirichlet, il sistema (6) diventa

$$\begin{cases} z_1(x)-\bar{z}_1(x) = \displaystyle\int_0^x\left[z_2(t)-\bar{z}_2(t)\right]dt + u_\circ - \bar{u}_\circ \\ z_2(x)-\bar{z}_2(x) = -\dfrac{1}{a_2}\left\{a_1\displaystyle\int_0^x\left[z_2(\xi)-\bar{z}_2(\xi)\right]d\xi+\int_0^x\left[z_2(\xi)-\bar{z}_2(\xi)\right]d\xi\int_\xi^x F(x,t)dt+\right. \end{cases}$$

$$\left.+\int_0^x\left[u_\circ-\bar{u}_\circ\right]F(x,t)dt+\int_0^x\bar{z}_1(t)\left[\psi(t)-\bar{\psi}(t)\right]dt\right\}+u_1-\bar{u}_1 .$$

Pertanto si ha:

$$(7)\quad\begin{cases} |z_1(x)-\bar{z}_1(x)|\leq\displaystyle\int_0^x|z_2(t)-\bar{z}_2(t)|dt + |u_\circ - \bar{u}_\circ| \\ |z_2(x)-\bar{z}_2(x)| \leq \dfrac{1}{|a_2|}\cdot\left\{\displaystyle\int_0^x|z_2(\xi)-\bar{z}_2(\xi)|d\xi\left[|a_1|+\int_\xi^x|F(x,t)|dt\right]+\right. \end{cases}$$

$$\left.+\int_0^x|u_\circ-\bar{u}_\circ|\cdot|F(x,t)|dt+\int_0^x|\bar{z}_1(t)|\cdot|\psi(t)-\bar{\psi}(t)|dt\right\}+|u_1-\bar{u}_1| .$$

Posto

$$\lambda(\xi) = \frac{1}{|a_2|}\left\{|a_1| + \int_\xi^{x_\circ}|F(x_\circ,t)|dt\right\} \qquad \text{e}$$

$$g(x) = \frac{1}{|a_2|}\left\{\int_0^x |u_\circ-\bar{u}_\circ|\cdot|F(x,t)|dt+\int_0^x |\bar{z}_1(t)|\cdot|\psi(t)-\bar{\psi}(t)|dt\right\}+|u_1-\bar{u}_1|$$

dalla II^a delle (7) si ha

$$|z_2(x)-\bar{z}_2(x)| \lesssim \int_0^x |z_2(\xi)-\bar{z}_2(\xi)|\;\lambda(\xi)d\xi + g(x).$$

Per il lemma di Gronwall generalizzato, risulta

$$(8)\qquad |z_2(x)-\bar{z}_2(x)| \lesssim |u_1-\bar{u}_1|\exp\left\{\int_0^x \lambda(\xi)d\xi\right\} +$$

$$+\int_0^x \left(\frac{1}{|a_2|}\left\{|u_\circ-\bar{u}_\circ|\cdot|F(x,t)|+|\bar{z}_1(t)|\cdot|\psi(t)-\bar{\psi}(t)|\right\}\exp\left\{\int_t^x \lambda(\xi)d\xi\right\}\right)dt.$$

Dalla (8), ponendo $u_\circ = \bar{u}_\circ$, $u_1 = \bar{u}_1$, $\psi(t)\equiv\bar{\psi}(t)$, risulta

$$z_2(x) = \bar{z}_2(x) \qquad 0 \leq x \leq x_\circ ,$$

e, per la I^a delle (7)

$$z_1(x) = \bar{z}_1(x) \qquad 0 \leq x \leq x_\circ .$$

Pertanto è dimostrato che:

I) il problema (3),(4), ammette in $C^1([0,x_\circ])$ al più una soluzione.

2. DIPENDENZA CONTINUA DAI DATI.

Sussiste il seguente teorema:

II) se esistono $R > 0$ e $x_\circ > 0$, tali che, dati $u_\circ, u_1$ e $\psi(x)$, con $|u_\circ| \lesssim R$, $|u_1| \lesssim R$, $\psi(x) \in C^\circ([0,x_\circ])$, esiste la soluzione $u(x) \in C^2([0,x_\circ])$ di (1),(2), e sia tale che $|u(x)| \lesssim R$, $0 \leq x \leq x_\circ$, allora $u(x)$ dipende