Proof. By putting n=1 in (5.5), we obtain

$$\int_x^\infty dt\, e^{-t}\, \psi(\alpha,\beta,\gamma;t) = e^{-x}\, \psi(\alpha,\beta,\gamma;x)-\psi(\alpha+\beta,\beta,\gamma;x) +$$

$$+\psi(\alpha+\beta,\beta,\gamma+1;x)-\Gamma(\alpha,x). \tag{5.11}$$

In virtue of Proposition 1.1 the relation (5.11) is valid also when x=0. Using then (5.4) the assertion is proved.

6. SOME FUNCTIONS AND RELATIONS CONNECTED WITH THE ψ-FUNCTION.

a) "Case" $\gamma = 0$.

Obviously one has $\psi(\alpha,\beta,0;x) = 0$.

b) "Case" $\gamma = 1$.

For $\gamma = 1$ the function (1.6) specializes to the incomplete Γ-function. In fact, we have

$$\psi(\alpha,\beta,1;x) = \int_x^\infty dt\; t^{\alpha-\beta-1} e^{-t} = \Gamma(\alpha-\beta,x). \tag{6.1}$$

c) "Case" $\gamma = n$ (positive integer).

As we have already noted (see Sec.3), the function (1.6) can be expressed as a finite sum of incomplete Γ-functions.

d) "Case" $\gamma = -1$, $\alpha = n+1$, $\beta = 0$.

For $\gamma = -1$ the function (1.6) becomes

$$\psi(\alpha,\beta,-1;x) = -\int_x^\infty dt\; \frac{t^{\alpha-1}}{e^t\; t^{\beta}-1} \quad . \tag{6.2}$$

Putting in (6.2) $\alpha = n + 1$ (n positive integer) and $\beta = 0$ we get

$$(6.3) \qquad \psi(n+1, 0, -1;x) = -D(n, x) ,$$

where

$$(6.4) \qquad D(n,x) = \int_x^\infty dt \; \frac{t^n}{e^t-1}$$

is a function introduced by Debye in his theory of specific heat of solids [18]. From now on, we shall call (6.4) <u>the incomplete Debye function.</u>

<u>Remark 6.1</u>. For $x = 0$ and $n \geq 1$ the function (6.3) becomes

$$(6.5) \quad \psi(n+1,0,-1;0) = -D(n,0) = -\int_0^\infty dt \; \frac{t^n}{e^t-1} = -n! \; \zeta(n+1) ,$$

where $\zeta(z)$ is the Riemann zeta function.

More generally, from (6.2) we deduce that

$$(6.6) \quad \psi(\alpha,0,-1;0) = -\int_0^\infty dt \; \frac{t^{\alpha-1}}{e^t-1} = -\Gamma(\alpha)\zeta(\alpha),$$

for $\alpha > 1$.

<u>Remark 6.2</u>. We shall call <u>generalized incomplete Debye function</u>, the integral

$$(6.7) \qquad \int_x^\infty dt \; \frac{t^{\alpha-1}}{e^t-1} \; ,$$

which appears on the right-hand side of (6.2). Using the symbol $D(\alpha-1, x)$ to denote (6.7), we have

$$(6.8) \qquad \psi(\alpha, 0, -1; x) = -D(\alpha, x) \quad ,$$

from (6.2).

Remark (6.3). Let us point out that one is able to evaluate the sum of the series on the right of (3.3), for any $\gamma = -n$, which is also an arbitrary negative integer, in terms of a combination of incomplete Debye functions and other known functions. In the special case $\gamma = -1$, taking into account (6.8) we obtain

$$(6.9) \qquad D(\alpha-1, x) = \sum_{n=1}^{\infty} \frac{\Gamma(\alpha, nx)}{n^\alpha} \; ,$$

for each $x > 0$, from (3.3).

Furthermore, in view of (6.6) and (6.8), from (3.3) we find the known expansion for the Riemann zeta function:

$$(6.10) \qquad \zeta(\alpha) \equiv -\frac{1}{\Gamma(\alpha)} \psi(\alpha, 0, -1; 0) = \sum_{n=1}^{\infty} \frac{1}{n^\alpha} \; ,$$

for $\alpha > 1$.

To conclude the case d), we notice that (5.5), for

$\gamma = -1$, $\beta = 0$, $x = 0$ and $\alpha = n + 1$ (n positive integer) provides an integral representation for the finite sum $\sum_{j=1}^{m} \frac{1}{j^{n+1}}$, in terms of the incomplete Debye function (6.4), namely

$$\sum_{j=1}^{m} \frac{1}{j^{n+1}} = \frac{m}{n!}\int_{0}^{\infty} dt\ e^{-mt} D(n,t).$$

e) "Case" $\gamma = -\frac{1}{2}$, $\beta = 0$.

For $\beta = 0$ and $\gamma = -\frac{1}{2}$, it also exists the integral on the right of (1.6) for any $\alpha > +\frac{1}{2}$ when $x = 0$. Furthermore using the series expansion (3.1) one has

$$\psi(\alpha,0,-\tfrac{1}{2};0) = -\Gamma(\alpha) \sum_{n=1}^{\infty} \frac{(2n-1)!!}{(2n)!!} \frac{1}{n^{\alpha}}. \tag{6.11}$$

If we now define the function

$$Z(\alpha) = -\frac{1}{\Gamma(\alpha)} \psi(\alpha,0,-\tfrac{1}{2};0), \tag{6.12}$$

the relation (6.11) gives

$$Z(\alpha) = \sum_{n=1}^{\infty} \frac{(2n-1)!!}{(2n)!!} \frac{1}{n^{\alpha}},$$

for $\alpha > \frac{1}{2}$.

f) "Case" $\beta = 0$ and $\alpha > \max(0, -\gamma)$ $(\gamma \neq 0,1,2,\ldots)$.

Both the series on the right of (6.10) and (6.13) can be considered as special cases of the more general series:

$$(6.14) \qquad \sum_{n=1}^{\infty} \frac{\Gamma(n-\gamma)}{\Gamma(-\gamma)n!} \; \frac{1}{n^{\alpha}} \;,$$

which converges for any $\alpha > \max(0, -\gamma)$. From (3.3), we deduce that the sum of this series is given by the function

$$- \frac{1}{\Gamma(\alpha)} \; \psi(\alpha,0,\gamma;0).$$

In order to show that the series (6.14) is convergent, let us determine the asymptotic expansion of $\frac{\Gamma(n-\gamma)}{n!}$ for large n. In doing so, it is enough to recall that [19]

$$(6.15) \qquad \Gamma(az+b) \sim \sqrt{2\pi}\; e^{-az}(az)^{az+b-\frac{1}{2}} \;,$$

for $z \to \infty$, $|\arg z| < \pi$ and $a > 0$.

Using (6.15), we thus have

$$(6.16) \qquad \frac{\Gamma(n-\gamma)}{n!} \sim O(n^{-\gamma-1}) \;,$$

for large values of n.

Therefore, the convergence of the series (6.14) is assured if $\alpha > -\gamma$.

g) "A functional relation for the polygamma functions".

The properties of the function $\psi(\alpha,\beta,\gamma;x)$ defined according to (1.6) can be usefully exploited in order to re-derive a well-known functional relation for the polygamma functions:

$$\psi^{(n)}(x) = \frac{d^{n+1}}{dx^{n+1}} \ell n \quad \Gamma(x) \ , \tag{6.17}$$

where n = 1,2,3, ... and $x \neq 0,-1,-2, \ldots$.

More specifically, we will show that

PROPOSITION 6.5. "The following functional relation holds:

$$\psi^{(n)}(m+1) = (-1)^n n! \ \{-\zeta(n+1) + \sum_{j=1}^{m} \frac{1}{j^{n+1}}\} \ , \tag{6.18}$$

where m is a non-negative integer and $\zeta(n+1)$ is the zeta Riemann function".

In doing so, let us start off with the integral representation [20]

$$\psi^{(n)}(m+1) = (-1)^{n+1} \int_0^\infty dt \ \frac{t^n \ e^{-(m+1)t}}{1- \ e^{-t}} \ . \tag{6.19}$$

Since

$$\frac{d}{dt} \quad \psi(\alpha,0,-1;t) = \frac{t^{\alpha-1} \ e^{-t}}{1- \ e^{-t}} \ , \tag{6.20}$$

from (6.19) we have

$$\psi^{(n)}(m+1) = (-1)^{n+1} \int_0^\infty dt \ e^{-mt} \frac{d}{dt} \ \psi(n+1,0,-1;t) \ , \tag{6.21}$$

for $\alpha = n+1$.

Integrating by parts, Eq.(6.21) yields

$$\psi^{(n)}(m+1)=(-1)^{n+1}\left[-\psi(n+1,0,-1;0)+m\int_0^\infty dt \ e^{-mt}\psi(n+1,0,-1;t)\right] \tag{6.22}$$

In virtue of corollary 5.4 and theorem 5.5, the integral on the right of (6.22) reads

$$(6.23)\qquad m\int_0^\infty dt\ e^{-mt}\ \psi(n+1,0,-1;t) =$$

$$= \psi(n+1,0,-1;0) - \sum_{k=0}^{m} (-1)^k \binom{m}{k} \left[\psi(n+1,0,k-1;0)-\psi(n+1,0,k;0)\right].$$

If we set apart the term of (6.23) corresponding to k=0, having in mind that $\psi(n-1,0,0;0) = 0$ and resorting to the recurrence relation (4.1) for $\alpha = n$ and $\gamma = k$, Eq.(6.23) reads

$$(6.24)\qquad m\int_0^\infty dt\ e^{-mt}\psi(n+1,0,-1;t)= \sum_{k=1}^{m} (-1)^k \binom{m}{k} \frac{n}{k}\ \psi(n,0,k;0).$$

Since (see (3.4))

$$(6.25)\qquad \psi(n,0,k;0) = (n-1)!\ \sum_{j=1}^{k} (-1)^{j+1} \binom{k}{j} \frac{1}{j^n}\quad ,$$

Eq. (6.24) becomes

$$(6.26)\qquad m\int_0^\infty dt\ e^{-mt}\ \psi(n+1,0,-1;t) =$$

$$= n!\ \sum_{k=1}^{m} \frac{(-1)^k}{k} \binom{m}{k} \sum_{j=1}^{k} (-1)^{j+1} \binom{k}{j} \frac{1}{j^n}\ .$$

By interchanging the summations in (6.26), with the help of the identity

$$\frac{1}{k}\binom{k}{j} = \frac{1}{j}\binom{k-1}{j-1} ,$$

we are led to the expression

$$(6.27) \quad m\int_0^\infty dt\ e^{-mt}\psi(n+1,0,-1;t) =$$

$$= -n! \sum_{j=1}^{m} \frac{(-1)^{j+1}}{j^{n+1}} \sum_{k=j}^{m} (-1)^{k+1}\binom{m}{k}\binom{k-1}{j-1}$$

At this point, we need to show two lemmas, namely:

Lemma 6.5

"Suppose that j and m are positive integers such that $1 \leq j \leq m - 1$. Then one has

$$(6.28) \qquad \sum_{k=j}^{m} (-1)^{k+1}\binom{m}{k}\binom{k}{j} = 0."$$

Proof. Notice that

$$(6.29) \quad \binom{m}{k}\binom{k}{j} = \binom{m}{j}\binom{m-j}{k-j} .$$

As a consequence, we can write

$$(6.30) \qquad \sum_{k=j}^{m} (-1)^{k+1}\binom{m}{k}\binom{k}{j} = \binom{m}{j} \sum_{k=j}^{m} (-1)^{k+1}\binom{m-j}{k-j} ,$$

from which, by putting $h = k-j$ and taking into account the hypothesis $m - j \geq 1$, one finally gets

$$(6.31) \qquad \sum_{k=j}^{m} (-1)^{k+1}\binom{m}{k}\binom{k}{j} = (-1)^{j+1}\binom{m}{j}\sum_{h=0}^{m-j}(-1)^{h}\binom{m-j}{h} = 0.$$

Lemma 6.6

"Let j and m be any pair of positive integers such that $1 \le j \le m$. Then one has

$$(6.32) \qquad \sum_{k=j}^{m} (-1)^{k+1}\binom{m}{k}\binom{k-1}{j-1} = (-1)^{j+1}"$$

Proof. By putting

$$(6.33) \qquad f(j) = \sum_{k=j}^{m} (-1)^{k+1}\binom{m}{k}\binom{k-1}{j-1},$$

for convenience, we can write

$$(6.34) \quad f(j)+f(j+1) = (-1)^{j+1}\binom{m}{j} + \sum_{k=j+1}^{m}(-1)^{k+1}\binom{m}{k}\left[\binom{k-1}{j-1}+\binom{k-1}{j}\right]$$

$$= \sum_{k=j}^{m}(-1)^{k+1}\binom{m}{k}\binom{k}{j},$$

where we have used the identity

$$(6.35) \qquad \binom{k-1}{j-1} + \binom{k-1}{j} = \binom{k}{j}.$$

On the basis of Lemma 6.5 from Eq. (6.34) we find

$$(6.36) \qquad f(j) + f(j+1) = 0,$$

for $1 \le j \le m-1$.

Since f(1)=1, Eq. (6.36) tells us that

$$f(j) = (-1)^{j+1} , \tag{6.37}$$

where j = 1,2,..., m-1.

We complete the proof observing that the relation (6.37) also holds for j=m. In fact, putting j=m-1 we have from (6.36) and (6.37):

$$f(m) = -f(m-1) = (-1)^{m+1}.$$

Now let us go back to Eq. (6.28). Using the result (6.32), Eq. (6.28) becomes

$$m\int_0^\infty dt\; e^{-mt}\; \psi(n+1,0,-1;t) = -n!\sum_{j=1}^{m} \frac{1}{j^{n+1}} . \tag{6.38}$$

Then, making the substitution (6.38) into Eq. (6.23), we obtain

$$\psi^{(n)}(m+1)=(-1)^n\left[\psi(n+1,0,-1;0) + n!\sum_{j=1}^{m}\frac{1}{j^{n+1}}\right]. \tag{6.39}$$

Recalling that (see (6.10)):

$$\zeta(n+1) = -\frac{1}{n!}\,\psi(n+1,0,-1;0) ,$$

Eq. (6.39) finally produces the relation (6.18).

Remark 6.7

Using the identity

$$\binom{k}{j} = \frac{k}{j}\binom{k-1}{j-1} ,$$

Eq. (6.32) reads

$$(6.40) \qquad \sum_{k=j}^{m} \frac{(-1)^{k+1}}{k}\binom{m}{k}\binom{k}{j} = \frac{(-1)^{j+1}}{j} .$$

By putting h=k-j into (6.40), with the help of (6.29) one has

$$(6.41) \qquad (-1)^{j+1}\binom{m}{j}\sum_{h=0}^{m-j} (-1)^{h} \frac{1}{h+j}\binom{m-j}{h} = \frac{(-1)^{j+1}}{j} .$$

By putting in (6.41) n = m-j, we are led to the formula

$$(6.42) \qquad \sum_{h=0}^{n} (-1)^{h} \frac{1}{h+j}\binom{n}{h} = \frac{1}{j} \frac{1}{\binom{n+j}{j}} ,$$

which may be considered as a generalization of the well-known formula

$$\sum_{h=0}^{n} \frac{(-1)^{h}}{h+1}\binom{n}{h} = \frac{1}{n+1} ,$$

deducible from (6.42) when j = 1.