The extension of the function (1.6) and its basic relations to complex variables will be given elsewhere.

Let us close this section with a brief remark. As a wide class of known functions can be interpreted in the light of group theory (see, for example, [13]), so one might investigate whether the same happens for the function (1.6). We shall be concerned with this challenging prospect in the near future.

2. SELF-SIMILAR SOLUTIONS FOR THE NONLINEAR WAVE EQUATION $u_{tx} = ae^{-u} + bu^{\beta-1}$.

Consider the nonlinear partial differential equation (1.7) in 1 + 1 space-time coordinate system, where $u = u(x,t)$ and a, b and $\beta \neq 0$ are real parameters.

We shall look for self-similar solutions of (1.7). In doing so, let us put $u = u(\xi)$ in (1.7) where $\xi = x + vt$. Then (1.7) transforms into the ordinary differential equation (reduced form of (1.7)):

$$\frac{1}{2} v u_\xi^2 = -ae^{-u} + \frac{b}{\beta} u^\beta + c, \tag{2.1}$$

c being an integration constant.

By choosing c = 0 and

$$\frac{1}{2a} v = k > 0, \tag{2.2}$$

$$\frac{b}{\beta a} = 1, \tag{2.3}$$

(2.1) yields

$$(2.4)\quad \lambda(\xi - \xi_o) = \pm \int_{u_o}^{u} dt\, t^{-\frac{\beta}{2}} \left(1 - \frac{e^{-t}}{t^{\beta}}\right)^{-\frac{1}{2}} ,$$

where $u_o \equiv u(\xi_o)$, $\lambda = k^{-\frac{1}{2}}$ and ξ_o is a constant. (In the following, we shall select the positive sign in front of the integral (2.4)).

First let us deal with the case $\beta \neq 2$. Using a simple trick, (2.4) reads

$$(2.5)\quad \lambda\xi = \frac{2}{2-\beta} u^{\frac{2-\beta}{2}} + \int_{u}^{\infty} dt\, t^{-\frac{\beta}{2}} \left[1-\left(1-\frac{e^{-t}}{t^{\beta}}\right)^{-\frac{1}{2}}\right] + \text{const.}$$

Taking account of (1.6), (2.5) can be written as

$$(2.6)\quad \lambda\xi = \frac{2}{2-\beta} u^{\frac{2-\beta}{2}} + \psi\left(\frac{2-\beta}{2}, \beta, -\tfrac{1}{2}; u\right) + \text{const.}$$

At this stage it is instructive to treat some special cases of (1.7), namely:

a) <u>Case</u> $\beta = 2$.

Equations (1.7) and (2.1) become respectively:

$$(2.7)\quad u_{tx} = ae^{-u} + bu ,$$

and

$$(2.8)\quad \tfrac{1}{2} vu_{\xi}^2 = -ae^{-u} + \tfrac{1}{2} bu^2 + c.$$

From (2.8) we obtain for $c = 0$:

$$(2.9)\quad \lambda(\xi - \xi_o) = \int_{u_o}^{u} dt\, t^{-1} \left(1 - \frac{e^{-t}}{t^{\beta}}\right)^{-\frac{1}{2}} ,$$

where $b = 2a$ and ξ_o is a constant.

Following the same procedure previously used, we are led to the expression

$$(2.10) \qquad \lambda\xi = \ln u + \int_u^\infty dt\, t^{-1} \left[1 - \left(1 - \frac{e^{-t}}{t^2}\right)^{-\frac{1}{2}} \right] + \text{const.}$$

Equation (2.10) reads also

$$(2.11) \qquad \lambda\xi = \ln u + \psi(0,2, -\tfrac{1}{2} ; u) + \text{const},$$

where $\psi(0,2, -\frac{1}{2}; u)$ is defined by (1.6).

Remark 2.1. An explicit solution of the form (2.11) holds also for the equation

$$(2.12) \qquad w_{tx} = p\, e^{-w} + q\, w + r ,$$

p, q and r being constants.

In fact, by carrying out the substitution $w = u - \frac{r}{q}$, (2.12) transforms into (2.7), where $a = p\, e^{\frac{r}{q}}$ and $b = q$.

b) Case $\beta = 1$.

Equations (1.7) and (2.1) read respecitvely

$$(2.13) \qquad u_{tx} = a\, e^{-u} + b$$

and

$$(2.14) \qquad \tfrac{1}{2}\, v\, u_\xi^2 = -a\, e^{-u} + bu + c.$$

The change of variable $u = w - \frac{c}{b}$ transforms (2.14) into the equation

$$(2.15) \qquad \tfrac{1}{2}\, v\, w_\xi^2 = -a\, e^{\frac{c}{b}}\, e^{-w} + bw.$$

Choosing then $c = b \ln \frac{b}{a}$, from (2.15) one has

$$(2.16) \qquad h\, w_{\xi}^{2} = w - e^{-w} ,$$

where $h = \frac{v}{2b}$.

Equation (2.16) provides

$$(2.17) \qquad \mu(\xi - \xi_o) = \int_{w_o}^{w} dt\; t^{-\frac{1}{2}} \left(1 - \frac{e^{-t}}{t} \right)^{-\frac{1}{2}} ,$$

where $\mu = h^{-\frac{1}{2}}$.

From (2.17) one gets

$$(2.18) \qquad \lambda\xi = 2w^{\frac{1}{2}} + \int_{w}^{\infty} dt\; t^{-\frac{1}{2}} \left[1 - \left(1 - \frac{e^{-t}}{t}\right)^{-\frac{1}{2}} \right] + \text{const},$$

where $w = u + \ln \frac{b}{a}$.

Finally, taking account of (1.6), (2.18) can be written as

$$(2.19) \qquad \lambda\xi = 2w^{\frac{1}{2}} + \psi(\tfrac{1}{2}, 1, -\tfrac{1}{2}; w) + \text{const}.$$

c) <u>Case</u> $b = 0$.

In this case (1.7) specializes to Liouville's equation (1.4), whilst (2.1) becomes

$$(2.20) \qquad k\, u_{\xi}^{2} = -e^{-u} + 1,$$

where k is given by (2.2) and c has been chosen equal to a.

Equation (2.20) yields

$$\lambda(\xi - \xi_0) = \int_{u_0}^{u} dt \, (1 - e^{-t})^{-\frac{1}{2}} . \tag{2.21}$$

The integral on the right of (2.21) can be expressed in terms of the function (1.6) as follows

$$\int_{u_0}^{u} dt \, (1 - e^{-t})^{-\frac{1}{2}} = u + \psi(1,0,-\tfrac{1}{2};u) - u_0 - (1,0,-\tfrac{1}{2};u_0) . \tag{2.22}$$

Using (2.22), from (2.21) we obtain

$$\lambda\xi = u + \psi(1,0,-\tfrac{1}{2};u) + \text{const.} \tag{2.23}$$

The function $\psi(1,0\ -\tfrac{1}{2};u)$ can be explicitly determined in terms of elementary functions. In fact, since

$$\int_{u_0}^{u} dt(1-e^{-t})^{-\frac{1}{2}} = -2 \ln\left[1-(1-e^{-u})^{\frac{1}{2}}\right] - u + 2 \ln\left[1-(1-e^{-u_0})^{\frac{1}{2}}\right] + u_0 , \tag{2.24}$$

from (2.22) and (2.24) we obtain

$$\psi(1,0,-\tfrac{1}{2};u) = -2\ln\left[1-(1-e^{-u})^{\frac{1}{2}}\right] - 2u + \text{const}, \tag{2.25}$$

where the constant on the right is given by

$$2 \lim_{t \to +\infty} \left\{ t + \ln\left[1-(1-e^{-t})^{\frac{1}{2}}\right]\right\} = -2 \ln 2. \tag{2.26}$$

Now let us go back to (2.23). In view of (2.25) we have

$$(2.27) \qquad A\, e^{\lambda(\xi-\xi_0)} = \frac{e^{-u}}{\left[1-(1-e^{-u})^{\frac{1}{2}}\right]^2} ,$$

where

$$(2.28) \qquad A = \frac{e^{-u_0}}{\left[1-(1-e^{-u_0})^{\frac{1}{2}}\right]^2} .$$

Finally, by means of simple calculatinns, (2.27) allows us to obtain the following expression of u in terms of ξ :

$$(2.29) \qquad u = \ell n \frac{\left[1 + 2Ae^{\lambda(\xi-\xi_0)}\right]^2}{4\, Ae^{\lambda(\xi-\xi_0)}\left[1+e^{\lambda(\xi-\xi_0)}\right]} .$$

3. SERIES REPRESENTATION OF $\Psi(\alpha,\beta,\gamma;x)$ IN TERMS OF INCOMPLETE GAMMA FUNCTIONS.

Let us consider the binomial expansion

$$(3.1) \qquad \left(1 - \frac{e^{-t}}{t^{\beta}}\right)^{\gamma} = 1 + \frac{1}{\Gamma(-\gamma)} \sum_{n=1}^{\infty} \frac{\Gamma(n-\gamma)}{n!} \frac{e^{-nt}}{t^{n\beta}} ,$$

where the series on the right is uniformly convergent for $t \geqslant x$, x being any fixed number such that $e^{-x} < x^{\beta}$.

We can thus write

$$(3.2)\ \Psi(\alpha,\beta,\gamma;x) = -\frac{1}{\Gamma(-\gamma)} \sum_{n=1}^{\infty} \frac{\Gamma(n-\gamma)}{n!} \int_x^{\infty} dt\ t^{\alpha-n\beta-1} e^{-nt} .$$