1. INTRODUCTION

It is not often that we can formulate evolution equations pertinent to nonlinear phenomena admitting exact special solutions.

Among these equations the following are, for instance, of great interest both in physical applications and from a mathematical point of view (see review articles [1] , [2] and [3]):

i) The Korteweg-de Vries equation [4,5]

(1.1) $$u_t + u_x u + \beta u_{xxx} = 0$$

and its modified form [6]

(1.2) $$u_t + u_x u^2 + \beta u_{xxx} = 0,$$

where $u=u(x,t)$, β is the dispersive parameter [7] and subscripts indicate partial derivatives.

ii) The so-called sine-Gordon equation [8,1]

(1.3) $$u_{tx} = a \sin \sigma u,$$

and Liouville's equation [9] :

(1.4) $$u_{tx} = a\, e^{\sigma u},$$

where a and σ are constants.

All the aforesaid nonlinear differential equations afford exact self-similar solutions, i.e. solutions of the form $u=u(\xi)$, where $\xi = x + v t$ and v is a (real) constant.

When suitable asymptotic conditions on $u(\xi)$ are fulfilled, $u(\xi)$ is usually called a solitary wave solution [1] and plays a central role in many branches of science, as for example solid state and plasma physics, and biological systems ([1,3] and [10]).

Besides the abovementioned equations, in recent years other interesting, but analytically more intractable, evolution equations referred to nonlinear phenomena have been introduced [3], as the modified sine-Gordon equation

$$(1.5) \qquad u_{tx} = a \sin\sigma u + bu + c,$$

which is pertinent to the so-called massive Schwinger model (see [11] and references quoted therein).

In most cases, where (1.5) is a particular example, exact self-similar solutions for nonlinear evolution equations cannot be obtained in terms of known functions. However such solutions may be given sometimes provided that of course new functions are defined. But this is a ticklish question, since introducing new functions is generally satisfying only if their use goes beyond the specific context we are concerned with, that is at present the problem of finding special solutions of certain nonlinear differential equations.

Adopting this philosophy, in this paper we have introduced the new function

$$(1.6) \qquad \psi(\alpha,\beta,\gamma;x) = \int_x^\infty dt\ t^{\alpha-1} \left[1 - \left(1 - \frac{e^{-t}}{t^\beta}\right)^\gamma\right],$$

where α, β and γ are free parameters, which arises in a natural way when looking for an exact self-similar solution of the nonlinear wave equation

$$(1.7) \qquad u_{tx} = a\ e^{-u} + bu^{\beta-1},$$

a, b and β being constants, It should be remarked that (1.7) may be regarded as an extended form of Licuville's equation (1.4).

Notice that from (1.6) one has also

$$(1.8)\qquad \psi(\alpha,\beta,\gamma;x) = x^{\alpha}\int_{1}^{\infty} dz\; z^{\alpha-1}\left\{1-\left[1-\frac{e^{-xz}}{(xz)^{\beta}}\right]^{\gamma}\right\},$$

and

$$(1.9)\qquad \psi(\alpha,\beta,\gamma;x) = x^{\alpha}\int_{0}^{\infty} dy(1+y)^{\alpha-1}\left\{1-\left[1-\frac{e^{-x(1+y)}}{x^{\beta}(1+y)^{\beta}}\right]^{\gamma}\right\}.$$

These formulae will prove helpful later.

One of the main characteristics of the function (1.6), which has strongly affected the present investigation, is that of covering a series of both known and new special functions and certain functional relations connected with them. As for instance, when $\gamma = 1$ (1.6) reduces to the in - complete Gamma function [12] and for $\gamma = -1$, $\beta = 0$ (1.6) becomes the so-called Debye function (see §6). In other words, studying the properties of the function (1.6) means providing insight into the properties of a whole family of (old and new) special functions of physical and mathematical interest. Specifically, the aim of this work is to derive some of most significant relations concerning the function (1.6), laying stress on what these become when the parameters α, β and γ are suitably specialized.

For simplicity's sake, here we have assumed that α,β,γ and x are real, being understood that the latter is non-negative. Furthermore, we have restricted ourselves to consider only real values of $\psi(\alpha,\beta,\gamma;x)$. This implies that $e^{-t} < t^{\beta}$ for any t verifying $x \leq t < +\infty$. To this end, we show below that the integral (1.6) can also be extended to the interval $(0, +\infty)$ open on the left. We have the following

PROPOSITION 1.1. Suppose that $-e < \beta < 0$ and $\alpha > -|\beta|$. Then the integral

$$(1.10) \qquad \int_0^\infty dt\, t^{\alpha-1}\left[1 - \left(1 - \frac{e^{-t}}{t^{\beta}}\right)^{\gamma}\right],$$

exists for any (real) value of the parameter γ.

Proof. As we have previously said, here we are interested only in dealing with real values of the function (1.6). In order that this occurs for any γ, we should require that

$$(1.11) \qquad t^{-\beta} e^{-t} < 1.$$

Since the function $t^{-\beta}e^{-t}$ takes its maximum value at $t = -\beta > 0$, the limitation (1.11) implies that

$$(1.12) \qquad -e < \beta < 0.$$

Furthermore, since

$$t^{\alpha}\left[1 - (1 - t^{|\beta|}e^{-t})^{\gamma}\right] \sim O(t^{\alpha+|\beta|}).$$

as $t \to 0^+$, the assertion is proved.

PROPOSITION 1.2. When $\beta = 0$, the integral (1.10) exists for $\gamma > 0, \alpha > 0$ and for $\gamma < 0, \alpha > |\gamma|$.

Proof. The first part of the lemma is obvious. The second part arises from

$$t^{\alpha}(1 - e^{-t})^{-|\gamma|} \sim t^{\alpha-|\gamma|}\left[1 + O(t)\right]$$

as $t \to 0^+$.

The extension of the function (1.6) and its basic relations to complex variables will be given elsewhere.

Let us close this section with a brief remark. As a wide class of known functions can be interpreted in the light of group theory (see, for example, [13]), so one might investigate whether the same happens for the function (1.6). We shall be concerned with this challenging prospect in the near future.

2. SELF-SIMILAR SOLUTIONS FOR THE NONLINEAR WAVE EQUATION $u_{tx} = ae^{-u} + bu^{\beta-1}$.

Consider the nonlinear partial differential equation (1.7) in 1 + 1 space-time coordinate system, where $u = u(x,t)$ and a, b and $\beta \neq 0$ are real parameters.

We shall look for self-similar solutions of (1.7). In doing so, let us put $u = u(\xi)$ in (1.7) where $\xi = x + vt$. Then (1.7) transforms into the ordinary differential equation (reduced form of (1.7)):

$$\frac{1}{2} v u_\xi^2 = -ae^{-u} + \frac{b}{\beta} u^\beta + c, \tag{2.1}$$

c being an integration constant.

By choosing $c = 0$ and

$$\frac{1}{2a} v = k > 0, \tag{2.2}$$

$$\frac{b}{\beta a} = 1, \tag{2.3}$$

(2.1) yields