

Istituto di Geometria. Università di Torino.

DUALITY THEOREMS FOR REGULAR HOMOTOPY
OF FINITE DIRECTED GRAPHS. (*)

RIASSUNTO. - *Dati uno spazio topologico normale e numerabilmente paracompatto S ed un grafo finito ed orientato G si prova che tra gli insiemi $Q(S,G)$ e $Q^*(S,G)$ delle classi di o -omotopia e di o^* -omotopia esiste una biiezione naturale. Nelle stesse condizioni, se S' è un sottospazio chiuso di S e G' un sottografo di G , esiste ancora una biiezione naturale tra gli insiemi $Q(S,S';G,G')$ e $Q^*(S,S';G,G')$ delle classi di omotopia. Si mostra infine che in condizioni meno restrittive per lo spazio S le precedenti biiezioni possono non sussistere.*

INTRODUCTION

In the extension from the undirected graphs to the directed ones, we have two possible definitions of regular function. In fact, given a topological space S and a finite directed graph G , a function $f: S \rightarrow G$ is called *o -regular* (resp. *o^* -regular*) if for all $v, w \in G$ such that $v \neq w$ and $v \nrightarrow w$, it is $\overline{f^{-1}(v)} \cap f^{-1}(w) = \phi$ (resp. $f^{-1}(v) \cap \overline{f^{-1}(w)} = \phi$). Therefore we can deal with two different homotopies, the o -homotopy and the o^* -homotopy. Hence we examine the problem

(*) Work performed under the auspices of the *Consiglio Nazionale delle Ricerche* (CNR, GNSAGA), Italy.

of seeing if, under suitable conditions for the space S , the o -homotopy and the o^* -homotopy get to coincide necessarily, i.e. if there exists a natural bijection between the sets of homotopy classes $Q(S, G)$ and $Q^*(S, G)$. As we observed in [2], by the Duality Principle the o -homotopy and o^* -homotopy are interchanged by replacing the graph G by the dually directed graph G^* ; thus we can identify the four sets $Q(S, G)$, $Q^*(S, G)$, $Q(S, G^*)$, $Q^*(S, G^*)$ at the same time.

Briefly we show how to solve the foregoing statements. In Part one, at first, we just consider functions and homotopies that are *completely regular*, i.e. without singularities; hence we examine the sets of complete o -homotopy classes $Q_c(S, G)$ and the ones of complete o^* -homotopy classes $Q_c^*(S, G)$. Then we obtain some properties which characterize the regular and completely regular functions (§ 1) and we give the definition of *pattern*, by which we construct a relation from the set of completely o -regular functions to the one of completely o^* -regular functions. Consequently, we have (§ 3) the Duality Theorem for complete homotopy classes (Theorem 9): "*There exists a natural bijection between the sets of complete homotopy classes $Q_c(S, G)$ and $Q_c^*(S, G)$* ".

Now we recall the results obtained in [3], Theorems 12, 12*, 16, 16*:

- i) If the space S is normal (*), in every class of $Q(S, G)$ (resp. $Q^*(S, G)$) there exists a completely o -regular (resp. o^* -regular) function.
- ii) If $S \times I$ is normal, two completely o -regular (resp. completely o^* -regular) functions, which are homotopic, are also completely homotopic.

Hence it follows (§ 4) that if S and $S \times I$ are normal spaces, there exists a natural bijection from $Q_c(S, G)$ to $Q(S, G)$ and from $Q_c^*(S, G)$ to $Q^*(S, G)$. From here and Theorem 9 the Duality Theorem follows. Now if we recall that a normal space S such that the product $S \times I$ is normal, is said a *countably paracompact normal*

(*) We distinguish between normal space and T_4 -space, according to whether it is a T_2 -space or not.

space (see [12], pp.168-169) we can enunciate the Duality Theorem (Theorem 11): "If S is a countably paracompact normal space, then there exists a natural bijection from $Q(S,G)$ to $Q^*(S,G)$ ".

In Part two we consider the same problem for couples of topological spaces (S,S') and of directed graphs (G,G') . That is not a trivial generalization of Part one, because new difficulties rise. In general, indeed, we cannot construct patterns of completely o -regular functions, then we must add the further condition that the completely regular functions are *balanced* in S' as regards S (§ 5), i.e. such that for all $x' \in S'$, for all $v \in G$, $x' \in \overline{f^{-1}(v)}$ implies that $x' \in \overline{f^{-1}(v) \cap S'}$. Thus we can repeat the construction of patterns (§ 6).

A second difficulty rises in that the so constructed patterns are not in general balanced functions. Hence we must choose as subspace S' an *open subspace* (§ 7) and under this condition the duality for complete homotopy is solved.

Unfortunately we cannot deduce the Duality Theorem since the Normalization Theorems proved in [3] for S and $S \times I$ normal spaces hold only if S' is a closed set. We eliminate this last difficulty (§ 8,9) by considering the *decreasingly filtered set* of open subspaces including S' and the *inductive limit* of the functions balanced in any open neighbourhood of S' . Thus by proceeding as in Part one we obtain the Duality Theorem (Theorem 32): "If S is a countably paracompact normal space and S' a closed subspace of S , then there exists a natural bijection from the set of o -homotopy classes $Q(S,S';G,G')$ to the one of o^* -homotopy classes $Q^*(S,S';G,G')$ ".

In § 11 we generalize the Duality Theorem to the case of $(n+1)$ -tuples of topological spaces and of $(n+1)$ -tuples of graphs. In § 12 we obtain the Duality Theorem for *absolute and relative homotopy groups* and we prove that the natural bijections are isomorphisms. At last in § 13 we give some counterexamples and among these we remark 13.4 and 13.5 which show that under weaker conditions for the space S (quasi compact, T_0 but not T_1) the two Duality Theorems do not hold.

0) Background.

Graphs and their subsets. (See [2] § 1, [3] § 1).

Let G be a *finite directed graph*.

If v, w are two vertices of G , we use the symbol $v \rightarrow w$ (resp. $v \nrightarrow w$) to denote that vw is (resp. is not) a directed edge of G . If $v \rightarrow w$, we call v a *predecessor* of w and w a *successor* of v .

The graph G^* with the same vertices of G and such that $(u \rightarrow v \text{ in } G) \Leftrightarrow (v \rightarrow u \text{ in } G^*)$, is called the *dually directed graph* as regards G . (If $G \equiv G^*$, i.e. if for all $v, w \in G$ we have $(v \rightarrow w) \Leftrightarrow (w \rightarrow v)$, the graph is called *undirected*).

Let X be a non-empty subset of G . A vertex of X is called a *head* (resp. a *tail*) of X in G , if it is a predecessor (resp. a successor) of all the other vertices of X . We denote by $H_G(X)$ (resp. $T_G(X)$) or, simply, by $H(X)$ (resp. $T(X)$) the set of the heads (resp. tails) of X in G . If $H(X) \neq \emptyset$ (resp. $T(X) \neq \emptyset$), X is called *headed* (resp. *tailed*); otherwise, X is called *non-headed* (resp. *non-tailed*). Finally, X is called *totally headed* (resp. *totally tailed*), if all the non-empty subsets of X are headed (resp. tailed). If X is a singleton, we agree to say that X is headed.

Regular and completely regular functions. (See [2] § 1, [3] § 2).

Let S be a *topological space*.

Given a function $f: S \rightarrow G$ from S to G , we denote by capital letter V the set of all the f -counterimages of $v \in G$, and if we want to emphasize the function f , we write $V^f = f^{-1}(v)$.

A function $f: S \rightarrow G$ is called *o-regular* (resp. *o*-regular*), if for all $v, w \in G$ such that $v \neq w$ and $v \nrightarrow w$, it is $V \cap \bar{W} = \emptyset$ (resp. $\bar{V} \cap W = \emptyset$).

Let $I = [0,1]$ be the unit interval in R^1 . Two o-regular (resp. o*-regular) functions $f, g: S \rightarrow G$ are called *o-homotopic* (resp. *o*-homotopic*), if there exists an o-regular (resp. o*-regular) function $F: S \times I \rightarrow G$, such that $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$, for all $x \in S$. The o-regular (resp. o*-regular) function F is

called an *o-homotopy* (resp. *o*-homotopy*) between f and g . The *o-homotopy* (resp. *o*-homotopy*) is an equivalence relation and we denote by $Q(S,G)$ (resp. $Q^*(S,G)$) the set of *o-homotopy* (resp. *o*-homotopy*) classes. We note that $Q^*(S,G)$ coincides with $Q(S,G^*)$ and $Q^*(S,G^*)$ with $Q(S,G)$.

DUALITY PRINCIPLE. - *Every true proposition in which appear the concepts of headed set, tailed set, o-regularity, o*-regularity, o-homotopy, o*-homotopy, $Q(S,G)$, $Q^*(S,G)$, remains true if the concepts of headed set and tailed set, o-regularity and o*-regularity, o-homotopy and o*-homotopy, $Q(S,G)$ and $Q^*(S,G)$, are interchanged through the statement of the proposition.*

Given an *o-regular* (resp. *o*-regular*) function $f: S \rightarrow G$, a n -tuple $X = \{v_1, \dots, v_n\}$, ($n \geq 2$) is called a *singularity* of f if:

i) X is non-headed (resp. non-tailed);

ii) $\overline{V_1^f} \cap \dots \cap \overline{V_n^f} \neq \emptyset$.

An *o-regular* (resp. *o*-regular*) function $f: S \rightarrow G$ from S to G is called *completely o-regular* (resp. *completely o*-regular*), or simply *c.o-regular* (resp. *c.o*-regular*), if there are no singularities of f . (If the graph G is undirected, then all the singularities are couples and the *c.regular* functions are called *strongly regular* functions).

Functions between pairs. (See [2] §5, [3] §2).

Let S' be a *subspace* of S and G' a *subgraph* of G .

A function $f: S, S' \rightarrow G, G'$ is called *o-regular* (resp. *o*-regular*) if both $f: S \rightarrow G$ and its restriction $f' = f|_{S'}: S' \rightarrow G'$ are *o-regular* (resp. *o*-regular*) functions.

Two *o-regular* (resp. *o*-regular*) functions $f, g: S, S' \rightarrow G, G'$ are called *o-homotopic* (resp. *o*-homotopic*), if there exists an *o-regular* (resp. *o*-regular*) homotopy $F: S \times I, S' \times I \rightarrow G, G'$, between f and g . The *o-homotopy* (resp. *o*-homotopy*) is an equivalence relation and we denote by $Q(S, S'; G, G')$ (resp. $Q^*(S, S'; G, G')$) the

set of o -homotopy (resp. o^* -homotopy) classes. We note that $Q^*(S, S'; G, G')$ coincides with $Q(S, S', G^*, G'^*)$ and $Q(S, S'; G, G')$ with $Q^*(S, S'; G^*, G'^*)$.

A function $f: S, S' \rightarrow G, G'$ is called *c.o-regular* (resp. *c.o*-regular*) if both $f: S \rightarrow G$ and $f': S' \rightarrow G'$ are *c.o-regular* (resp. *c.o*-regular*) functions.

As before, the *Duality Principle* holds for functions between pairs.

Main results of [2], [3].

R_a : $X \subseteq G$ is totally headed, iff it is totally tailed. (See [3], Proposition 4).

If S is a normal topological space and S' is a closed subspace of S , we have:

R_b : (The first Normalization Theorem). Let $f: S \rightarrow G$ (resp. $f: S, S' \rightarrow G, G'$) be an o -regular function. Then there exists a *c.o-regular* function, o -homotopic to f . (See [3], Theorems 12, 15).

R_c : (Extension Theorem between pairs). Let $f: S, S' \rightarrow G, G'$ be an o -regular function. Then there exist a closed neighbourhood U of S' and an o -regular function $g: S, S' \rightarrow G, G'$, which is o -homotopic to f and such that the function $g: S, U \rightarrow G, G'$ is o -regular, i.e. $g(U) \subseteq G'$ and the restriction $\hat{g}: U \rightarrow G'$ of g to U is o -regular. (See [2], Theorem 20).

R_d : In the construction of R_c , if there exist n vertices $p_1, \dots, p_n \in G$ and m vertices $q_1, \dots, q_m \in G'$, such that $\overline{P_1^f} \cap \dots \cap \overline{P_n^f} \cap \overline{Q_1^{f'}}$ $\cap \dots \cap \overline{Q_m^{f'}} = \emptyset$, then also it follows $\overline{P_1^g} \cap \dots \cap \overline{P_n^g} \cap \overline{Q_1^{\hat{g}}} \cap \dots \cap \overline{Q_m^{\hat{g}}} = \emptyset$. Similarly, from $\overline{P_1^f} \cap \dots \cap \overline{P_n^f} \cap X = \emptyset$ it results $\overline{P_1^g} \cap \dots \cap \overline{P_n^g} \cap U = \emptyset$. (See [2], Corollary 21).

Moreover, if $S \times I$ is normal, then it results:

R_e : (The first Normalization Theorem for homotopies). Let $f, g: S \rightarrow G$ (resp. $f, g: S, S' \rightarrow G, G'$) be two o -homotopic *c.o-regular* functions. Then, between the functions f and g , there also exists an o -homotopy, which is a *c.o-regular* function. (See [3], Theorem 16).

By Duality Principle, the results dual to the previous ones are also true.