7 EXAMPLE.

Let $\eta \equiv (E,p,M;B)$ be a 0-Lie derivable bundle.

Then $$B : E \times_M TM \to TE$$

results into a horizontal section (see [7], §5).

Moreover, if η is a vector bundle and B is a linear morphism on TM, the 0-Lie-derivative coincide with the covariant derivative.

8 EXAMPLE

We get the usual Lie derivative of tensors $M \to T_{(p,q)}M$, taking into account the previous proposition and the 1-Lie-derivable bundles

$$\eta \equiv (TM, \Pi_M, M; s \circ c) \qquad \text{and} \qquad \eta \equiv (T^*M, \rho_M, M, C^*) \quad .$$

9 EXAMPLE.

Let $\beta \equiv (E,p,M;B)$ a bundle of geometric objects (see [7]) .

Let β be of "order K", i.e. such that the following condition holds: if $v \in J^kTM$, $x',x'' : M \to TM$ are two representative of v and f',f'' are the one parameter groups generated by x',x''

then $$\partial(Bf') = \partial(Bf'') \quad .$$

Then the map $$B : E \times J^kTM \to TE,$$

given by $$B(e,v) \equiv \partial(Bf)(e)$$

makes $(E,p,M;B)$ a k-Lie derivable bundle.

4 CONNECTION ON A BUNDLE.

Let $\eta \equiv (E,p,M)$ be a bundle.

1 DEFINITION.

A CONNECTION on η is an affine bundle morphism on E

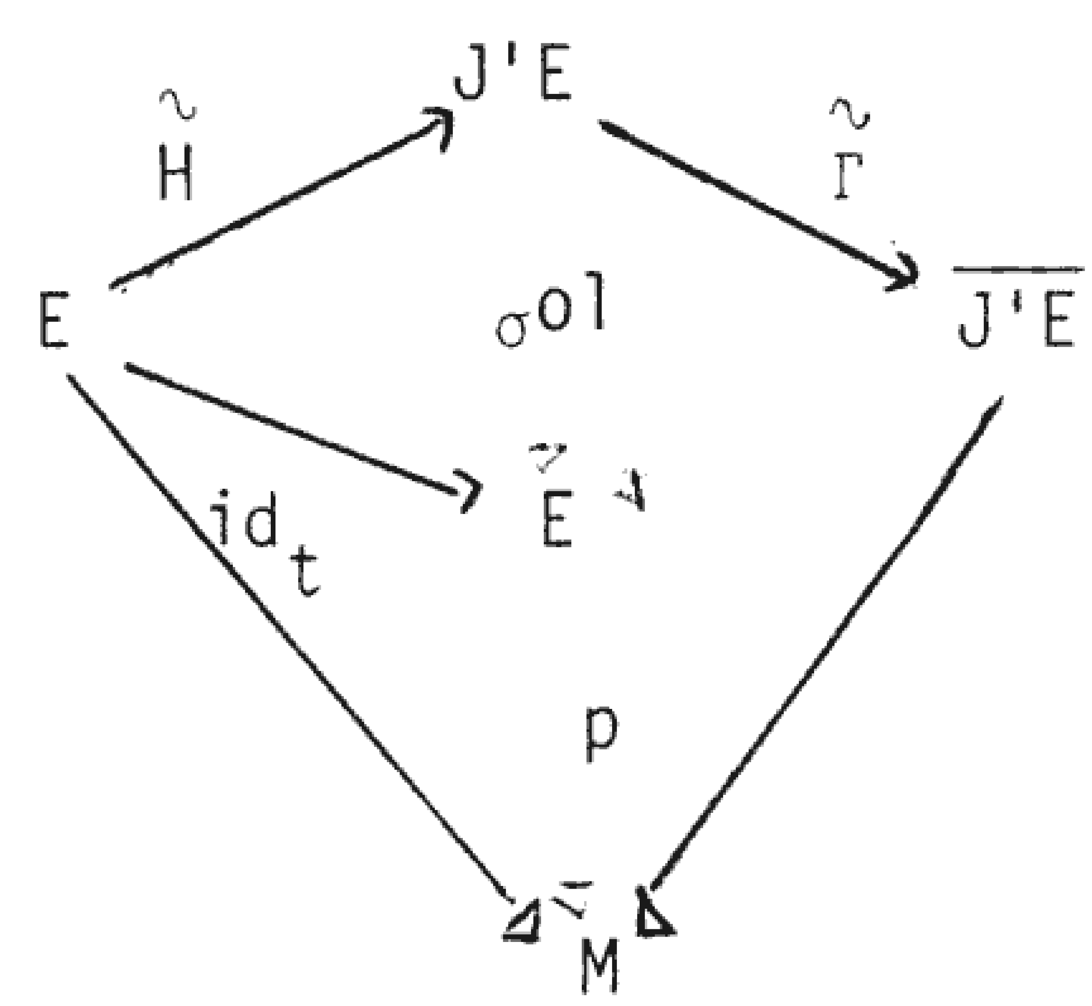
$$\tilde{\Gamma} : J'E \to \overline{J'E} = T^{*}M \otimes_{E} \nu\, TE$$

whose fiber derivatives are 1.

A HORIZONTAL SECTION is a section

$$\tilde{H} : E \to J'E$$

Hence the following diagram is commutative



2 PROPOSITION.

The maps α and β between the set of connections and the set of horizontal sections, given by

$$\alpha : \tilde{\Gamma} \to \tilde{H},$$

where $\check{H}$ is the unique horizontal section such that $\check{\Gamma} \circ \check{H} = 0$,

and $$\beta : \tilde{H} \to \tilde{\Gamma} \equiv id_{J'E} - \tilde{H} \circ \sigma^{01} ,$$

are inverse bijections.

Henceforth we will consider $\tilde{\Gamma}$ and $\tilde{H}$ as mutually related. Hence giving a connection is the choice of a point for each affine fiber of J'E, getting in this way an identification of the affine fibers with their vector spaces.

3 PROPOSITION.

The set $\tilde{\mathcal{J}}$ of all connections is the affine space of the sections of the affine bundle $\eta^{01}E$, whose vector space is the space of sections of the vector bundle $\overline{\eta^{01}E}$.

4 Let us remark that $\mathcal{J}$ [7] and $\tilde{\mathcal{J}}$ have the same vector space.

PROPOSITION.

Each one of the following commutative diagrams determine the same isomorphism, whose derivative is 1, between the two affine spaces $\mathcal{J}$ and $\tilde{\mathcal{J}}$:

a)

$$\begin{array}{ccc} TM \times_M J'E & \xrightarrow{\ id_{TM} \times \tilde{\Gamma}\ } & TM \times_M (T^*M \otimes_E \nu TE) \\ {\scriptstyle C}\big\downarrow & & \big\downarrow{\scriptstyle <,>} \\ T\,E & \xrightarrow{\quad \Gamma \quad} & \nu\, T\, E \end{array}$$

b)

$$\begin{array}{ccc} E \times_M T\,M & \xrightarrow{\ \tilde{H} \times id_{TM}\ } & J'E \times_M T\,M \\ & {\scriptstyle H}\searrow \quad \swarrow{\scriptstyle C} & \\ & T\,E & \end{array}$$

.

Henceforth we will write $\mathcal{J}$, Γ and H for $\tilde{\mathcal{J}}$, $\tilde{\Gamma}$ and $\tilde{H}$.

5 PROPOSITION.

Let $c : R \to E$ be a curve. The following conditions are equivalent.

a) $H \circ \sigma^{01} \circ j'c \equiv H \circ c = j'c$

a') $H \circ h \circ d\,c \equiv H \circ (c, d(p \circ c)) = d\,c$

b) $\Gamma \circ j'c = 0$

b') $\Gamma \circ d\,c = 0$.

Hence a curve $c : R \to E$ is HORIZONTAL if the previous conditions hold.

6 PROPOSITION.

Let η be a vector bundle. Let Γ be a connection.
The following conditions are equivalent.

a) $\Gamma: J'E \to \overline{J'E}$ is a vector bundle morphism on M.

a') $\Gamma: TE \to \bar{\nu}TE$ is a vector bundle morphism on TM.

b) $H : E \to J'E$ is a vector bundle morphism on TM.

b') $H : hTE \to TE$ is a vector bundle morphism on TM.

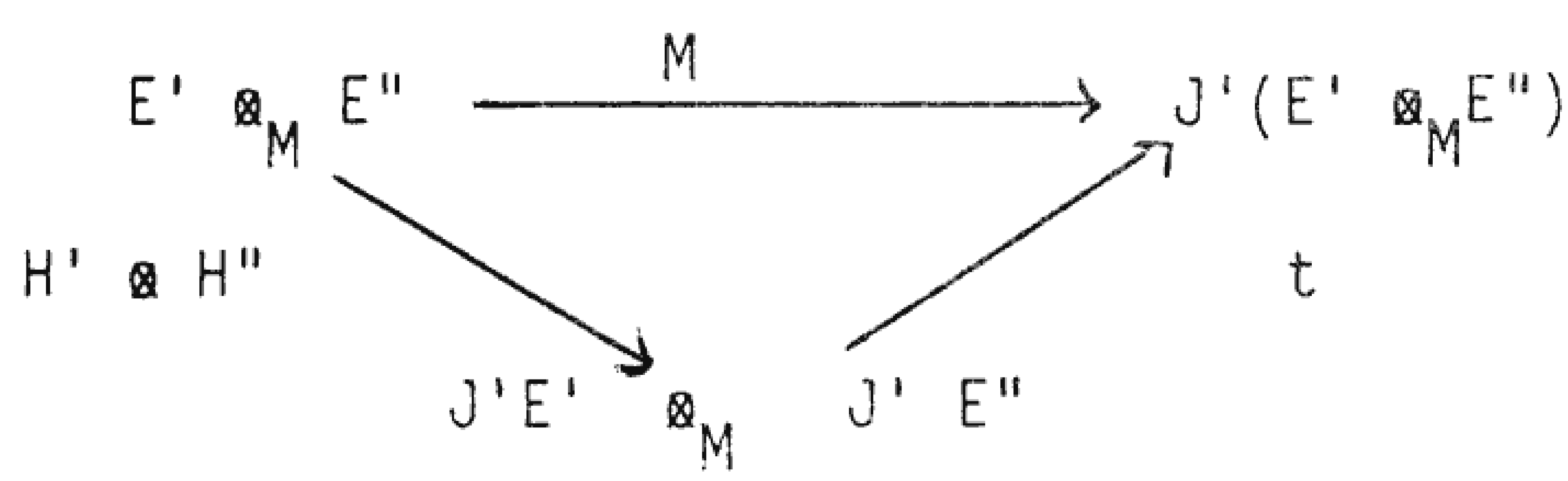
Hence a connection (horizontal section) is LINEAR if the previous conditions hold.

7 PROPOSITION.

Let Γ' and Γ'' be two linear connections of η' and η'', respectively
The tensor product of Γ' and Γ'' is the connection Γ on $\eta' \otimes \eta''$ associated with the horizontal section

$$H = t \circ (H' \otimes H'') .$$

Hence the following diagram is commutative



8 PROPOSITION.

Let Γ be a linear connection on η .

The dual connection of Γ is the linear connection Γ^* on η^* associated with the unique horizontal section H^*, which makes commutative the following diagram

$$\begin{array}{ccc} J'E \times_M J'E^* & \xrightarrow{\langle , \rangle} & T^*M \\ \uparrow (H,H^*) & & \uparrow 0 \\ E \times_M E^* & \longrightarrow & M \end{array}$$

$\dot{=}$

9 PROPOSITION.

Let Γ be a linear connection on η . Let $v : M \to E$ be a section .

We get $\nabla v = (id_{T^*M} \otimes \perp\!\!\perp_E) \circ \Gamma \circ j^1 v$ $\dot{=}$

Hence the following diagram is commutative

$$\begin{array}{ccc} J'E & \xrightarrow{\Gamma} & T^*M \otimes_E vTE \\ j^1 v \uparrow & & \downarrow id_{T^*M} \otimes \perp\!\!\perp_E \\ M & \xrightarrow[\nabla v]{} & T^*M \otimes_E E \end{array}$$

10 PROPOSITION.

Let $\eta \equiv \tau M$ and let $g : TM \times_M TM \to R$ be a non degenerate symmetrical bilinear map.

The Riemannian connection Γ induced by g is associated with the unique linear section

$$H : TM \to J'TM$$

such that

a) the following diagram is commutative

$$\begin{array}{ccc} J'TM \times_M J'TM & \xrightarrow{g} & T^*M \\ \uparrow (H,H) & & \uparrow 0 \\ TM & \longrightarrow & M \end{array}$$

b) the torsion $\Theta = 0$ $\dot{=}$