$$\begin{array}{ccc} T\,T^{*}M & \xrightarrow{\;s\;} & T^{*}T\,M \\ & {\scriptstyle h}\searrow \quad \swarrow{\scriptstyle \nu} & \\ & T\,M \times_{MT} {}^{*}M & \end{array}$$

b)' Reversing all the terms of b), we get the inverse isomorphism

$$s^{-1} : T^{*}T\,M \rightarrow T\,T^{*}M.$$

c) $(\,T\,T^{*}M, h, T^{*}M \times_M T\,M)$ is the pull-back bundle of
$(T^{*}T^{*}M, \nu,\; T^{*}M \times_M T\,M)$ with respect to the map

$$i \equiv (id_{T^{*}M} \times (-\,id_{TM})) : T^{*}M \times_M TM \rightarrow T^{*}M \times_M T\,M.$$

The induced map $\omega \equiv i^{*} : T\,T^{*}M \rightarrow T^{*}T^{*}M$

is an isomorphism (the SYMPLECTIC ISOMORPHISM) such that the following diagram is commutative

$$\begin{array}{ccc} T\,T^{*}M & \xrightarrow{\;\omega\;} & T^{*}T^{*}M \\ {\scriptstyle (\pi_{T^{*}M},\, T\rho_M)}\downarrow & & \downarrow{\scriptstyle \nu} \\ T^{*}M \times_M T\,M & \xrightarrow{\;i\;} & T^{*}M \times_M T\,M \end{array}$$

c)' In an analogous way we get the isomorphism

$$\omega \circ s^{-1} : T^{*}T\,M \rightarrow T^{*}T^{*}M \quad \underline{.}$$

5 DEFINITION.

The SYMMETRIC SUBMANIFOLD of $T\,T\,M$ is

$$s\,T\,T\,M \equiv \{\alpha \in T\,T\,M \mid s(\alpha) = \alpha\} \quad \underline{.}$$

4 - <u>Lie derivative of tensors</u>.

1 Let $\mathcal{M}$ be the category, whose objects are manifolds and whose morphisms are diffeomorphisms.

Let $T_{(r,s)} : \mathcal{M} \rightarrow \mathcal{M}$

be the covariant functor defined as follows:

a) $$T_{(r,s)} M \equiv \underset{r}{\otimes} T M \underset{s}{\otimes} T^* M$$

b) if $f : M \to N$ is a diffeomorphism,

then $$T_{(r,s)} f \equiv \underset{r}{\otimes} T f \underset{s}{\otimes} T^* f^{-1} : T_{(r,s)} M \to T_{(r,s)} N.$$

2 Let M and N be manifolds.

Let $$\phi : R \times M \to N$$

be a map (defined at least locally). Then

$$\partial\phi : M \to T N$$

is the map given by

$$\partial\phi(x) = T\phi_x(0,1) .$$

3 Let M be a manifold.

Let $u : M \to T M$ be a vector field and

let $C : R \times M \to M$ be the (locally defined)

group of local diffeomorphisms generated by u.

Namely we have $u = \partial\, c$.

Let $v : M \to T_{(r,s)} M$ be a tensor field .

Let $C\, v : R \times M \to T_{(r,s)} M$ be the (locally defined)

map, given, $\forall\, \lambda \in R$, by the tensor field

$$(C\, v)_\lambda \equiv T_{(r,s)}\, C_\lambda^{-1} \circ v \circ C_\lambda : M \to T_{(r,s)} M .$$

Let us remark that $\partial(Cv)$ takes its values in the subspace

$\nu\, T\, T_{(r,s)} M \hookrightarrow T\, T_{(r,s)} M$, since $(C\, v)_\lambda$ is a section of $T_{(r,s)} M$.

Then we can give the following definition.

4 DEFINITION.

The LIE DEFIVATIVE of $v : M \to T_{(r,s)} M$ with respect to $u : M \to TM$ is

the tensor field

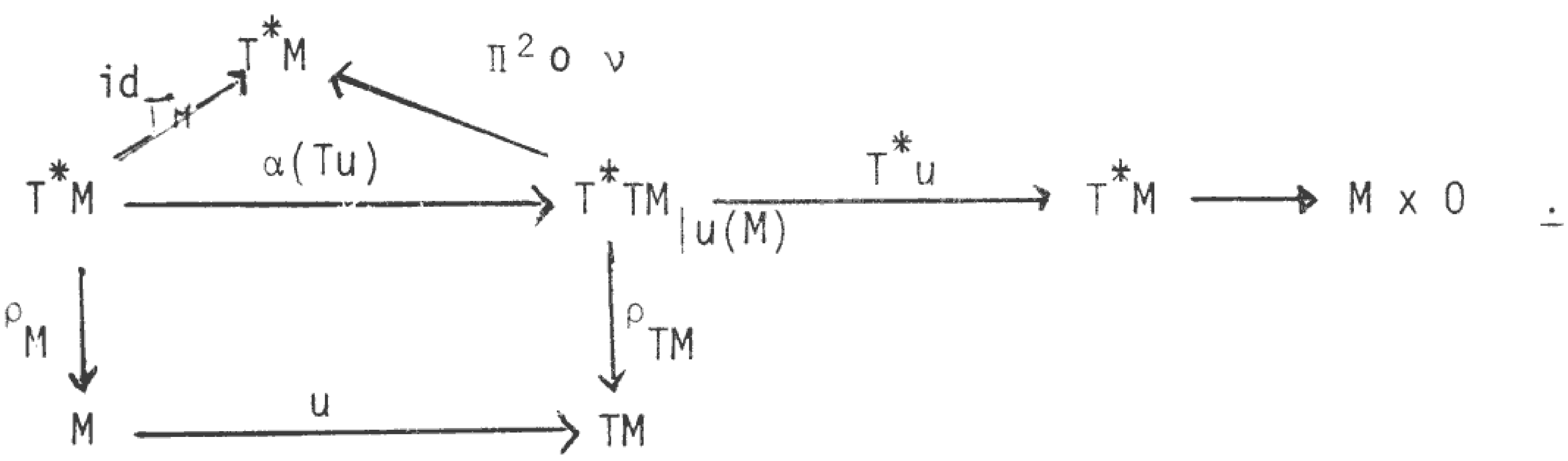
$$L_u v \equiv \perp\!\!\perp_{T_{(r,s)}M} \circ \partial(Cv) : M \to T_{(r,s)}M \qquad \dot{-}$$

5 LEMMA

Let $u : M \to T\,M$ be a section. There is a unique map

$$\alpha(Tu) : T^*M \to T^*T\,M$$

such that the following diagram is commutative and exact in T^*M



6 PROPOSITION.

We have

$$L_u v = \perp\!\!\perp_{T(r,s)M} \circ (Tv\circ u - t \circ (\underset{\iota}{\otimes}\, s \circ Tu \underset{s}{\otimes}\, s \circ (\alpha(Tu)) \circ v) \quad . \quad (*)$$

PROOF.

It suffices to give the proof for $v : M \to T_{(1,0)}M$ and $v : M \to T_{(0,1)}M$.

In the first case, we get

$$Cv = (T_2 C^{-1}) \circ (\Pi', v) \circ (\Pi^1, C) : R \times M \to TM,$$

hence

$$\partial(Cv) = (TT_2 C^{-1}) \circ (\Pi', Tv) \circ (\Pi', T_1 c)_{(0,1)} =$$

$$= ((\partial T_2 C^{-1}) \circ \Pi_{TM} + T_2 T_2 C_0^{-1}) \circ (Tv) \circ \partial C =$$

$$= (s \circ T\partial C^{-1}) \circ \Pi_{TM} + id_{TTM}) \circ (Tv) \circ u =$$

$$= - s \circ T\, u \circ v + T\, v \circ u \; .$$

In the second case, we get

$$Cv = (T_2^* C) \circ (\Pi^1, v) \circ (\Pi^1, c) : R \times M \to T^*M$$

hence $\partial(Cv) = (TT_2^{*}C) \circ (\pi^1, Tv) \circ (\pi^1, T_1 C)_{(0,1)} =$

$=((\partial T_2^{*}C) \circ \pi_{TM} + T_2 T_2^{*} C_o) \circ (TV) \circ \partial C =$

$=(- s \circ \alpha(T_2 \partial C) \circ \pi_{TM} + id_{TT^{*}M}) \circ (Tv) \circ u =$

$= - s \circ \alpha(Tu) \circ v + Tv \circ u$.

Let us remark that both tensors in (*) are on the same affine fiber on $h\ TT_{(r,s)}M$.

5 Connection on a bundle.

Let $\eta \equiv (E,p,M)$ be a bundle.

1 DEFINITION.

A PSEUDO-CONNECTION on η is an affine bundle morphism on $h\ T\ E$

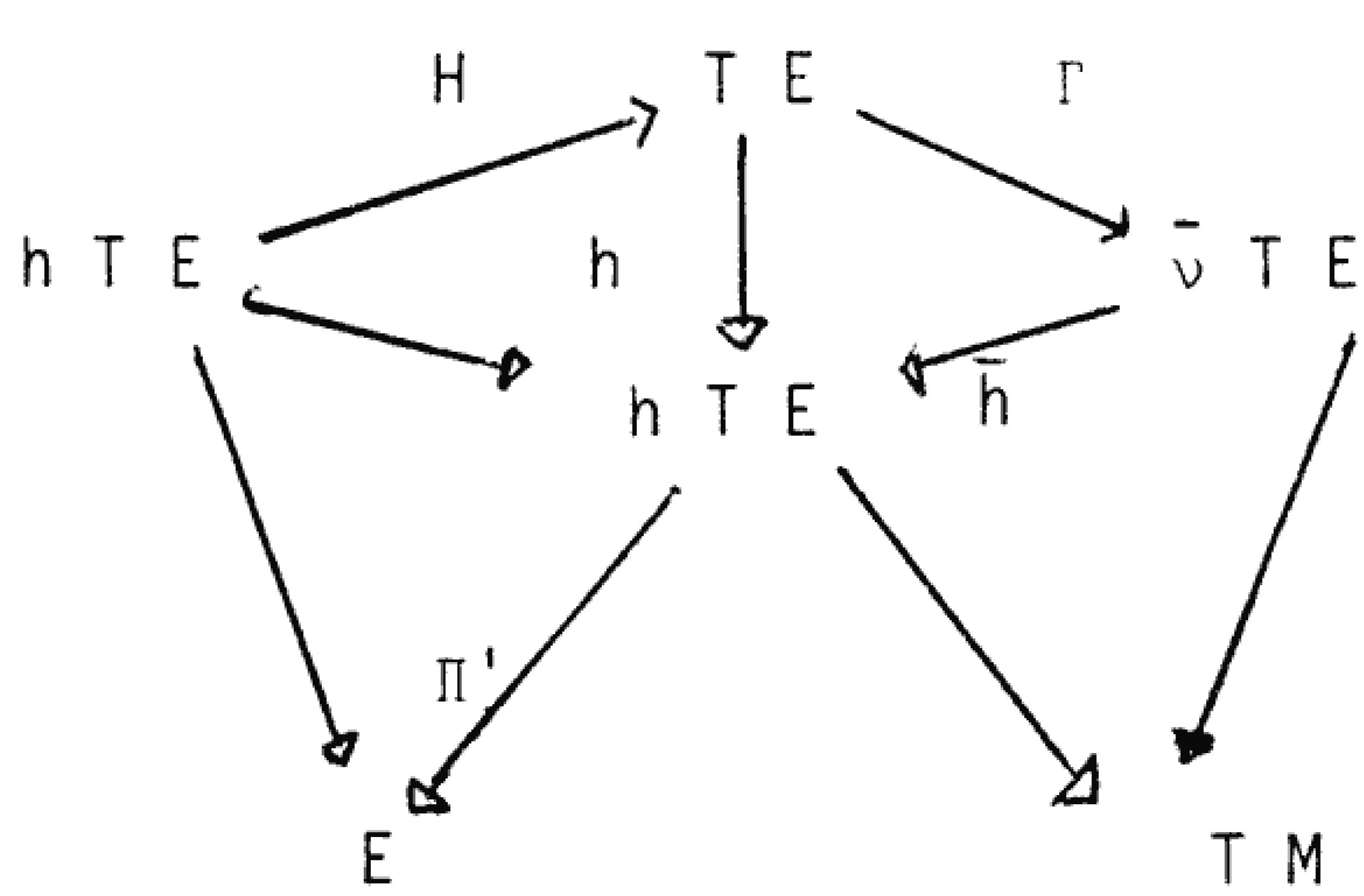
$$\Gamma : TE \to \bar{\nu}\ T\ E$$

whose fiber derivatives are 1.

A PSEUDO-HORIZONTAL SECTION is a section

$$H : h\ TE \to TE$$.

Hence the following diagram is commutative



Let us remark that $\Gamma : T\ E \to \bar{\nu}\ T\ E$ is characterized by the map $\Gamma' : TE \to \nu\ T\ E$ given by $TE \xrightarrow{\Gamma} \bar{\nu}\ TE \xrightarrow{\pi^2} \nu\ T\ E$.