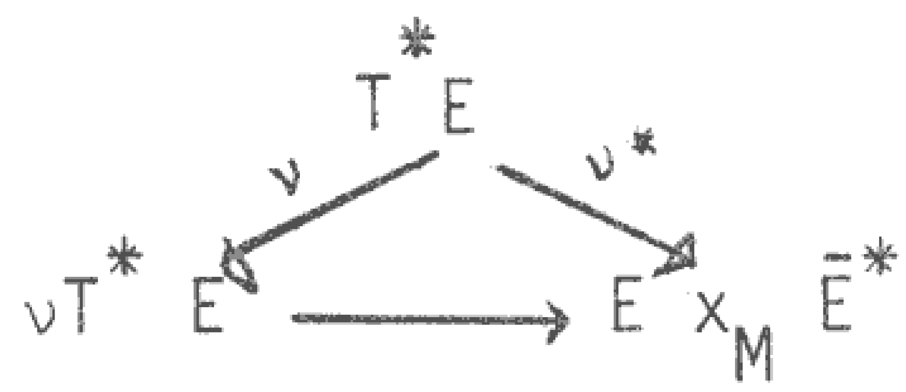
$$0 \to hT^*E \hookrightarrow T^*E \twoheadrightarrow E\times_M \bar{E}^* \to 0$$

Then there is a unique homomorphism over E

$$\nu T^*E \to E\times_M \bar{E}^*$$

such that the following diagram is commutative



Such a map is an isomorphism $\underline{\cdot}$

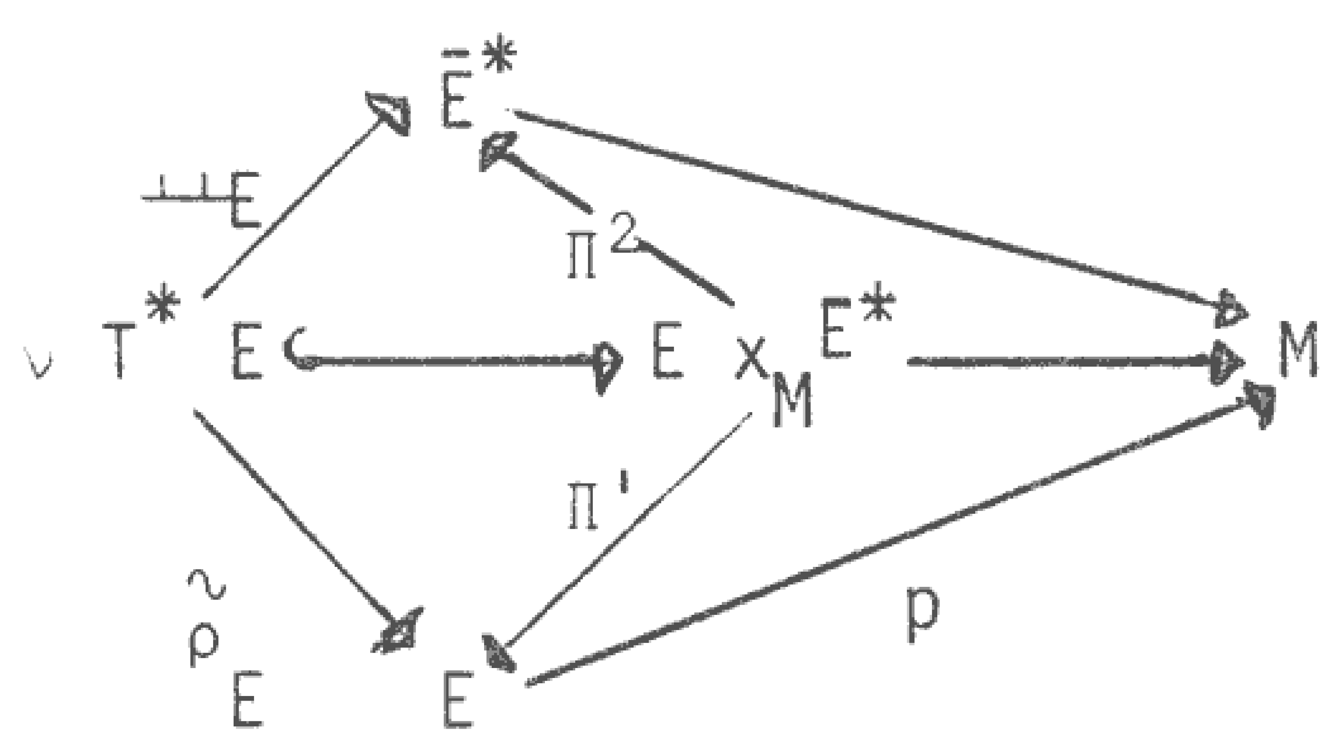
We will often make the identification

$$\nu T^*E \cong E \times_M \bar{E}^*$$

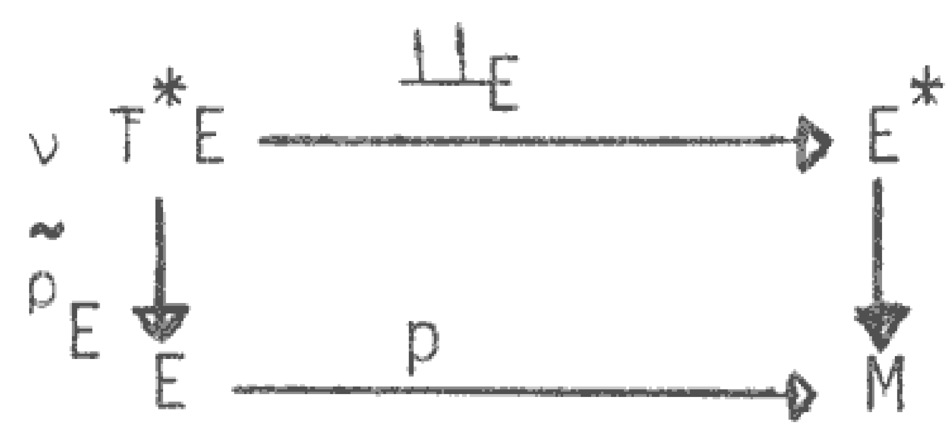
9 We define the map $\perp\!\!\perp_E : \nu T^*E \to \bar{E}^*$

by the composition $\nu T^*E \to E\times_M \bar{E}^* \overset{\Pi^2}{\to} \bar{E}^*$

Then we get the commutative diagram



and the homomorphism



is an isomorphism on fibers.

3 - THE SECOND TANGENT AND COTANGENT SPACES OF A MANIFOLD.

1 As a particular case of the previous results, let us consider

$$\eta \equiv (TM, \Pi_M, M) \quad \text{or} \quad \eta \equiv (T^*M, \rho_M, M) \quad .$$

Then we get the following spaces

$$h\,T\,T\,M = TM \times_M T\,M \qquad \nu\,T\,T\,M = T\,M \times_M T\,M$$

$$h\,T\,T^*M = T^*M \times_M T\,M \qquad \nu\,T\,T^*M = T^*M \times_M T^*M$$

$$h\,T^*T\,M = T\,M \times_M T^*M \qquad \nu\,T^*T\,M = T\,M \times_M T^*M$$

$$h\,T^*T^*M = T^*M \times_M T^*M \qquad \nu\,T^*T^*M = T^*M \times_M T\,M$$

and the following maps

$$(\pi_{TM}, T\pi_M) \equiv h : TT\,M \to h\,T\,T\,M \qquad \nu\,T\,T\,M \to T\,T\,M$$

$$(\pi_{T^*M}, T\rho_M) \equiv h : TT^*M \to h\,T\,T^*M \qquad \nu\,T\,T^*M \to T\,T^*M$$

$$h\,T^*T\,M \to T^*TM \qquad \nu : T^*T\,M \to \nu T^*T\,M$$

$$h\,T^*T^*M \to T^*T^*M \qquad \nu : T^*T^*M \to T^*T^*M$$

$$\perp\!\!\!\perp_{TM} : \nu T\,T\,M \to T\,M \qquad \perp\!\!\!\perp_{T^*M} : \nu TT^*M \to T^*M$$

$$\perp\!\!\!\perp_{T^*M} : \nu\,T^*T\,M \to T^*M \qquad \perp\!\!\!\perp_{T^*M} : \nu T^*T^*M \to T\,M$$

2 Taking into account that

$$h\,T\,T\,M = \nu T\,T\,M \quad \text{and} \quad \nu\,T^*T\,M = h\,T^*T\,M\,,$$

we define the following maps

$$\nu : T\,T\,M \to T\,T\,M$$

given by $\quad T\,T\,M \overset{h}{\to} h\,T\,T\,M = \nu T\,T\,M \to T\,T\,M$

and $\quad h : T^*T\,M \to T^*T\,M$

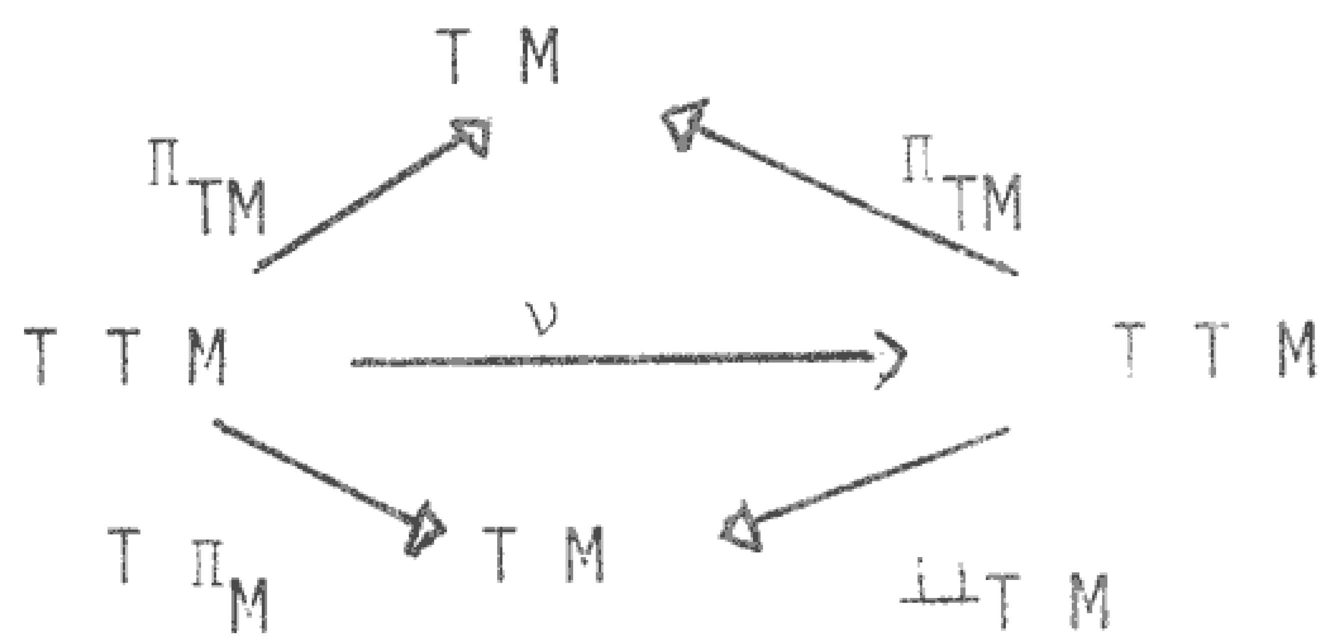
given by $\quad T^*T\,M \overset{\nu}{\to} \nu\,T^*T\,M = h\,T^*T\,M \to T^*T\,M\,.$

3 PROPOSITION.

a) The vertical endomorphism is the unique map

$$\nu : T\,T\,M \to T\,T\,M$$

which makes commutative the following diagram:



b) The horizontal endomorphism

$$h : T^*T\,M \rightarrow T^*T\,M$$

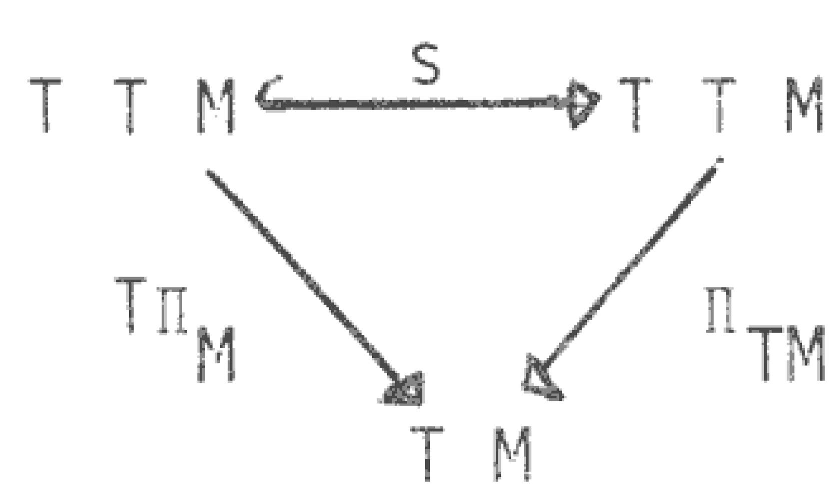
is the transpose of the vertical endomorphism $\nu : T\,T\,M \rightarrow T\,T\,M$, as

$(T^*T\,M,\ \rho_{TM},\ T\,M)$ is the dual of $(T\,T\,M\ ,\ \pi_{TM}\ \ T\,M)$ $\underline{\cdot}$

4 PROPOSITION.

a) $(T\,T\,M,\ h,\ T\,M\ x_M\ T\,M)$ is the pull-back bundle of $(T\,T\,M,\ h,\ T\,M\ x_M\ T\,M)$ with respect to the exchange endomorphism $ex : T\,M\ x_M\ T\,M \rightarrow TM\ x_M\ T\,M.$

The induced map $s \equiv (ex)^* : T\,T\,M \rightarrow T\,T\,M$ is an involutive automorphism such that the following diagram is commutative



b)$(\,T\,T^*M, h, T^*M\ x_M T\,M)$ is the pull-back bundle of $(\,T^*T\,M, \nu, T\,M\ x_M T^*M)$ with respect to the exange map

$$ex : T^*M\ x_M T\,M \rightarrow TM\ x_M\ T^*M\ .$$

The induced map $s \equiv (ex)^* : T\,T^*M \rightarrow T^*T\,M$

is an isomorphism such that the following diagram is commutative

$$\begin{array}{ccc} T\,T^*M & \xrightarrow{\ s\ } & T^*T\,M \\ & {\scriptstyle h}\searrow \qquad \swarrow{\scriptstyle \nu} & \\ & T\,M \times_{MT} {}^*M & \end{array}$$

b)' Reversing all the terms of b), we get the inverse isomorphism

$$s^{-1} : T^*T\,M \to T\,T^*M.$$

c) $(\,T\,T^*M, h, T^*M \times_M T\,M)$ is the pull-back bundle of $(T^*T^*M, \nu, T^*M \times_M T\,M)$ with respect to the map

$$i \equiv (id_{T^*M} \times (-\,id_{TM})) : T^*M \times_M TM \to T^*M \times_M T\,M.$$

The induced map $\omega \equiv i^* : T\,T^*M \to T^*T^*M$

is an isomorphism (the SYMPLECTIC ISOMORPHISM) such that the following diagram is commutative

$$\begin{array}{ccc} T\,T^*M & \xrightarrow{\ \omega\ } & T^*T^*M \\ {\scriptstyle (\pi_{T^*M}, T\rho_M)}\downarrow & & \downarrow{\scriptstyle \nu} \\ T^*M \times_M T\,M & \xrightarrow{\ i\ } & T^*M \times_M T\,M \end{array}$$

c)' In an analogous way we get the isomorphism

$$\omega \circ s^{-1} : T^*T\,M \to T^*T^*M \qquad \underline{.}$$

5 DEFINITION.

The SYMMETRIC SUBMANIFOLD of $T\,T\,M$ is

$$s\,T\,T\,M \equiv \{\alpha \in T\,T\,M \mid s(\alpha) = \alpha\} \qquad \underline{.}$$

4 - Lie derivative of tensors.

1 Let $\mathcal{M}$ be the category, whose objects are manifolds and whose morphisms are diffeomorphisms.

Let $T_{(r,s)} : \mathcal{M} \to \mathcal{M}$