$p \circ c_\alpha = p \circ c = p \circ c_\beta$ and $T\,p(\alpha) = T\,p \circ d\,c = T\,p(\beta)$.

Since $d\,\gamma(o)$ depends only on α and β, we can put

$$\alpha + \beta \equiv d\,c(o) \quad \underline{\cdot}$$

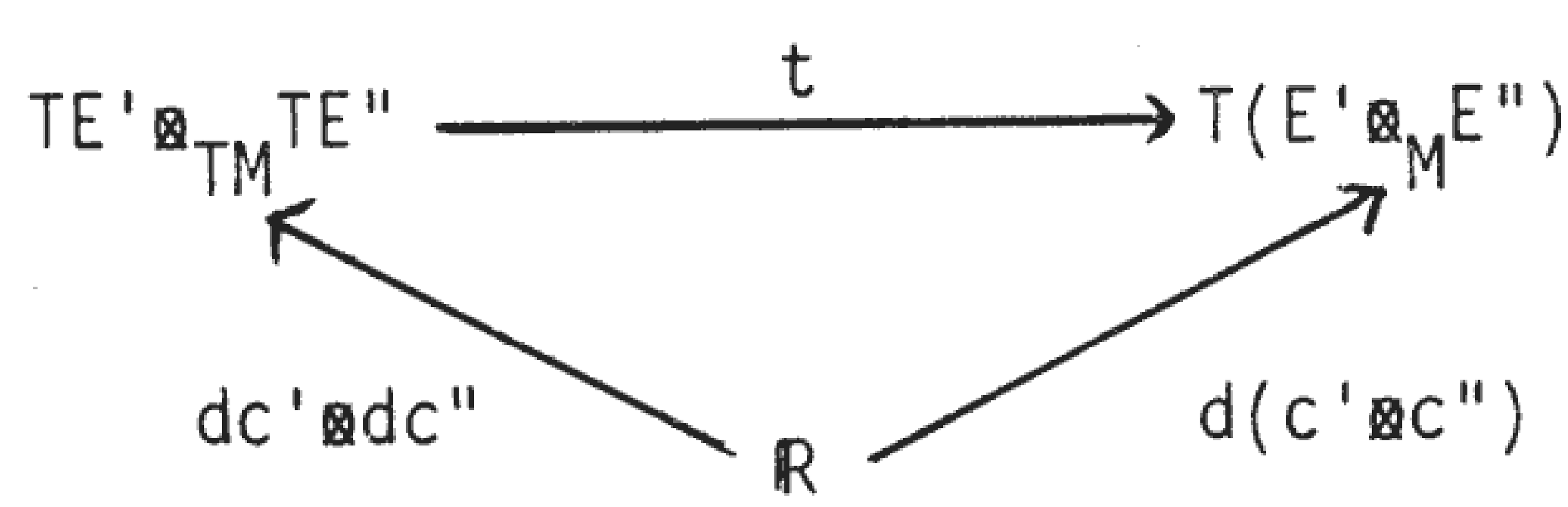
11 PROPOSITION.

Let $\eta' \equiv (E',p',M)$ and $\eta'' \equiv (E'',p'',M)$ be vector bundles.

There is a unique map

$$t : T\,E' \otimes_{TM} TE'' \longrightarrow T(E' \otimes_M E'')$$

such that the following diagram is commutative



for each $c' : \mathbb{R} \to E'$ and $c'' : \mathbb{R} \to E''$ such that

$$p' \circ c' = p'' \circ c'' .$$

This map is a surjective linear homomorphism over TM $\underline{\cdot}$

2. - THE COTANGENT SPACE OF A BUNDLE.

Let $\eta \equiv (E,p,M)$ be a C^∞ bundle.

1 DEFINITION.

The COTANGENT BUNDLE OF E is the vector bundle

$$\tau^* E \equiv (T^* E, \rho_E, E) \quad \underline{\cdot}$$

2 DEFINITION.

The HORIZONTAL BUNDLE OF $T^* E$ is the pull-back vector bundle

$$h\,\tau^*\,E \equiv (hT^* E, \pi^1, E),$$

where $$h\,T^*\,E \equiv E \times_M T^*\,M \quad \underline{\cdot}$$

Hence the following diagram is commutative

$$\begin{array}{ccc} h\,T^*E & \xrightarrow{\;\Pi^2\;} & T^*M \\ {\scriptstyle\Pi'}\downarrow & & \downarrow{\scriptstyle\rho_M} \\ E & \xrightarrow{\;p\;} & M \end{array}$$

3. PROPOSITION.

The transpose map of $h : T\,E \to h\,T\,E$ over E is an injective map

$$h\,T^*E \;\to\; T^*E \quad .$$

The following diagram is commutative

$$\begin{array}{ccc} h\,T^*E & \hookrightarrow & T^*E \\ & {\scriptstyle\Pi'}\searrow \quad \swarrow{\scriptstyle\rho_E} & \\ & E & \end{array}$$

PROOF.

In fact $h\,T^*E$ is the dual of $h\,T\,E$ and h is surjective $\underline{.}$

4 PROPOSITION.

The inclusion $h\,T^*E \to T^*E$ identifies $h\,T^*E$ with the orthogonal of νTE.

PROOF.

In fact νTE is the kernel of h $\underline{.}$

5 DEFINITION.

The VERTICAL BUNDLE OF T^*E is the quotient vector bundle

$$\nu\,{\scriptstyle T}^*E \equiv (\nu T^*E,\, \tilde{\rho}_E, E)$$

where $$\nu\,T^*E \equiv T^*E/hT^*E \quad \underline{.}$$

The following sequence is exact and the diagram commutative

$$\begin{array}{ccccccccc} 0 & \longrightarrow & h\,T^*E & \hookrightarrow & T^*E & \xrightarrow{\;\nu\;} & \nu T^*E & \longrightarrow & 0 \\ & & & {\scriptstyle\Pi'}\searrow & \downarrow{\scriptstyle\rho_E} & \swarrow{\scriptstyle\tilde{\rho}_E} & & & \\ & & & & E & & & & \end{array}$$

6 DEFINITION.

The COTANGENT BUNDLE OF E, ON νT^*E, is

$$\tau_\nu^* E \equiv (T^*E,\ \nu\ , \nu T^*E).$$

The HORIZONTAL BUNDLE OF T^*E, ON νT^*E, is the pull-back vector bundle

$$\bar{\tau}_\nu^* E \equiv (\bar{h}T^*E,\ \bar{\nu}, \nu T^*E)$$

where $\bar{h}T^*E \equiv \nu T^*E x_E h\, T^*E$ and $\bar{\nu} \equiv \Pi'$.

Hence the following diagram is commutative

$$\begin{array}{ccc} \bar{h}\, T^*\, E & \xrightarrow{\ \Pi^2\ } & h\, T^*E \\ \Pi' \downarrow & & \downarrow \Pi' \\ T^*E & \xrightarrow{\ \tilde{\rho}_E\ } & E \end{array}$$

7 PROPOSITION.

The bundle $\tau_\nu^* E \equiv (T^*E, \nu, \nu T^*E)$

is an affine bundle, whose vector bundle is

$$\bar{\tau}_\nu^* E \equiv (\bar{h}T^*E,\ \bar{\nu}, \nu T^*E)\ .$$

PROOF.

Let $[\beta] \in \nu T_e^* E$

We get $\nu^{-1}\ [\beta] = \{\alpha \in T_e^*E \mid \nu(\alpha) = [\beta]\}$.

Since $\nu_e : T_e^*E \rightarrow \nu T_e^*E$ is a linear map, then $\nu^{-1}\ [\beta]$ is an affine space, whose vector space is $\operatorname{Ker} \nu_e = h\ T_e^*E$.

Hence T^*E is an affine bundle on νT^*E and a vector bundle on E.

8 PROPOSITION.

Let $\eta \equiv (E,p,M)$ be an affine bundle, whose vector bundle is $\bar{\eta} \equiv (\bar{E},\bar{p},M)$.

Let $\nu^* : T^*E \rightarrow E x_M \bar{E}^*$

be the transpose map of the inclusion

$$E x_M \bar{E} \cong \nu TE \longrightarrow TE\ .$$

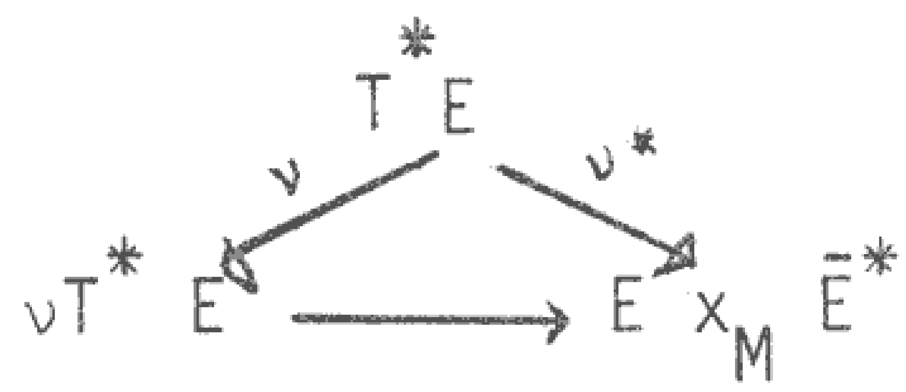
The following sequence is exact

$$0 \rightarrow hT^*E \hookrightarrow T^*E \twoheadrightarrow E\times_M\bar{E}^* \rightarrow 0$$

Then there is a unique homomorphism over E

$$\nu T^*E \rightarrow E\times_M\bar{E}^*$$

such that the following diagram is commutative



Such a map is an isomorphism $\underline{\cdot}$

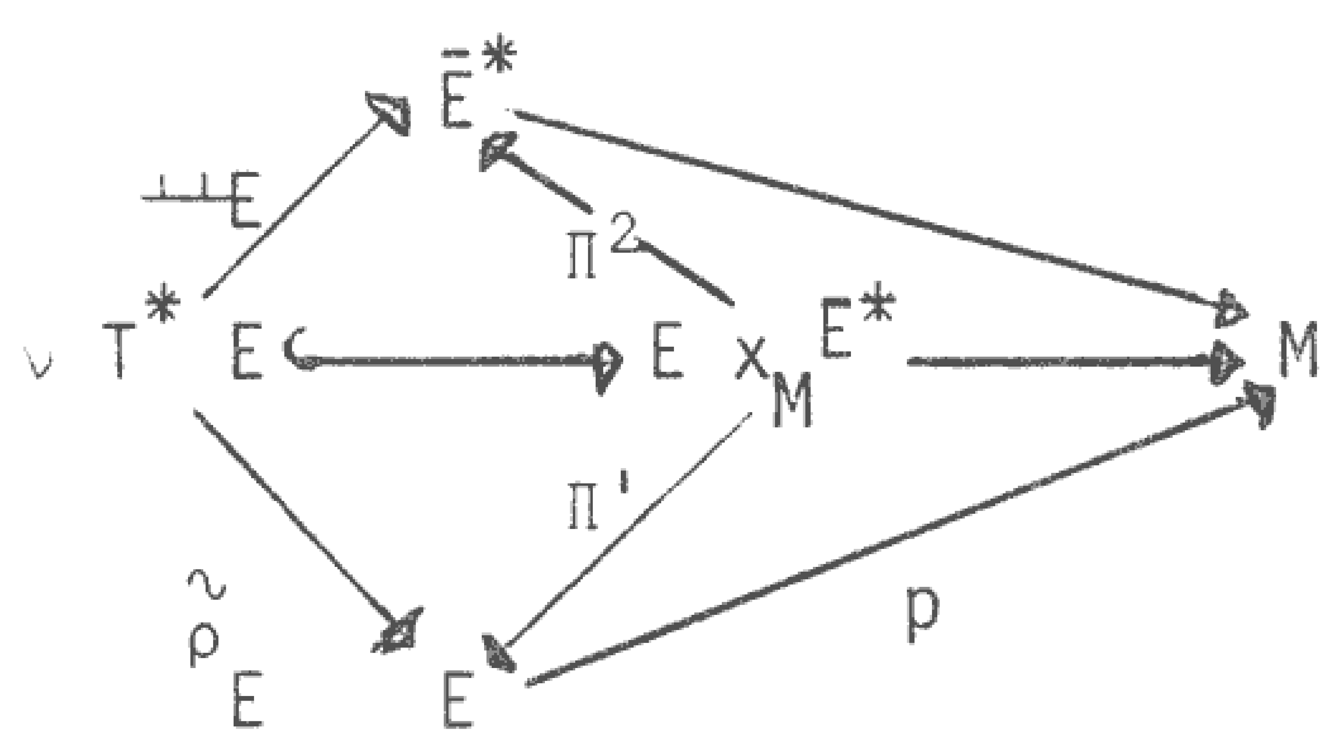
We will often make the identification

$$\nu T^*E \cong E\times_M\bar{E}^*$$

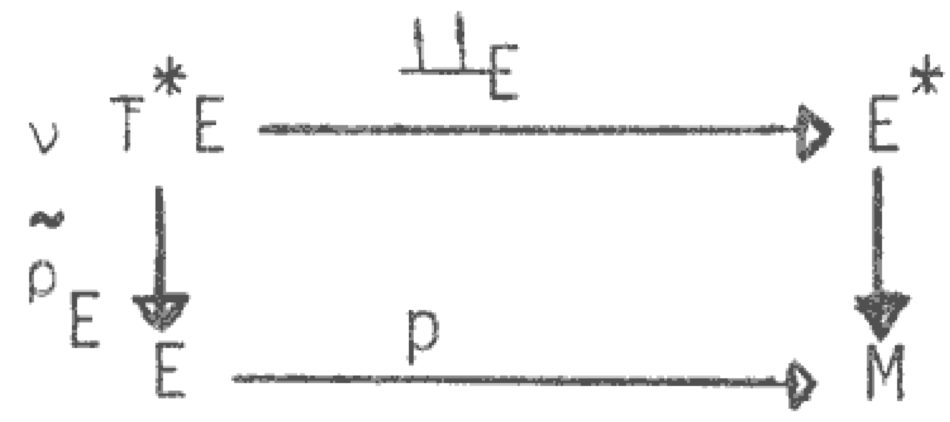
9 We define the map $\perp\!\!\perp_E : \nu T^*E \rightarrow \bar{E}^*$

by the composition $\nu T^*E \rightarrow E\times_M \bar{E}^* \xrightarrow{\Pi^2} \bar{E}^*$

Then we get the commutative diagram



and the homomorphism



is an isomorphism on fibers.

3 - THE SECOND TANGENT AND COTANGENT SPACES OF A MANIFOLD.

1 As a particular case of the previous results, let us consider

$$\eta \equiv (TM, \Pi_M, M) \quad \text{or} \quad \eta \equiv (T^*M, \rho_M, M) \quad .$$