1 - THE TANGENT SPACE OF A BUNDLE.

Let $\eta \equiv (E,p,M)$ be a C^{∞} bundle.

1 DEFINITION.

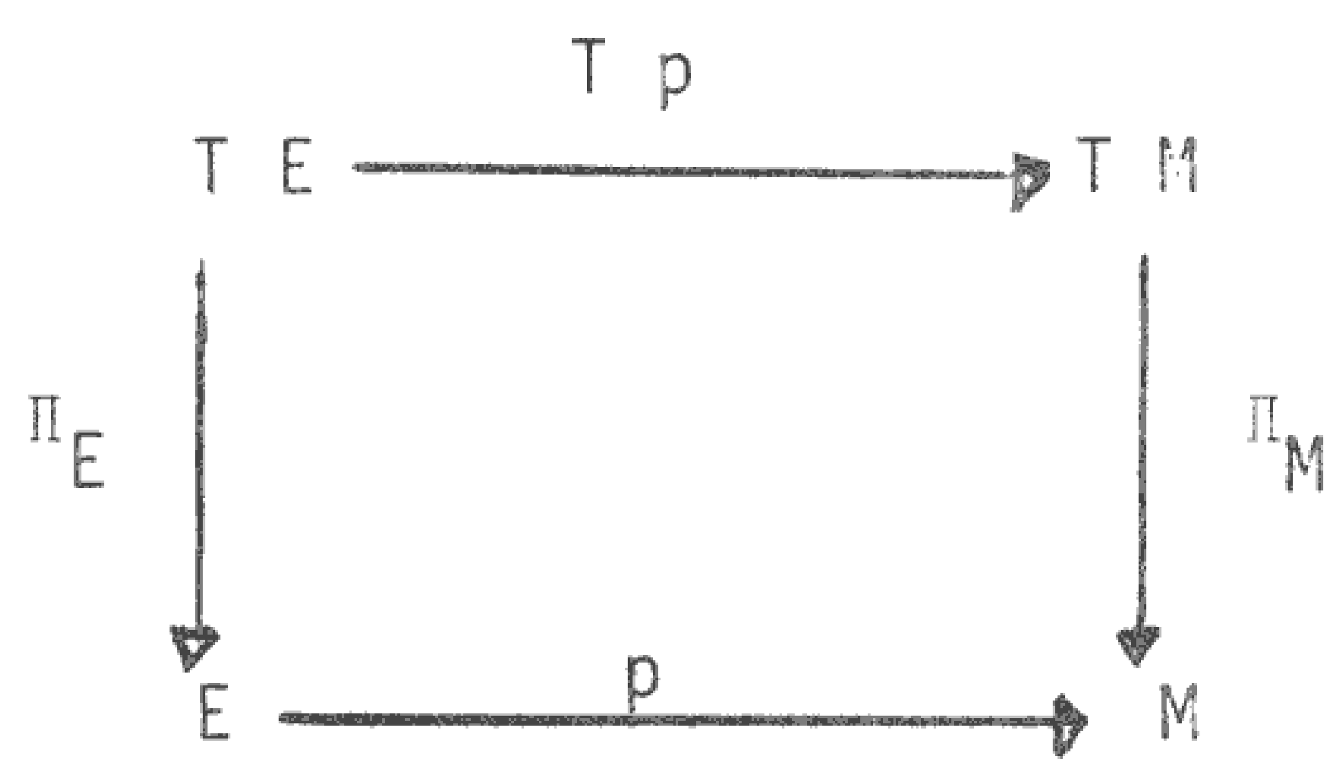
The TANGENT BUNDLE OF E is the vector bundle

$$\tau E \equiv (TE, \pi_E, E) \ .$$

The TANGENT BUNDLE OF η is the vector bundle

$$\tau \eta \equiv (TE, Tp, TM) \ .$$

The following diagram is commutative.



2 DEFINITION.

The HORIZONTAL BUNDLE OF TE is the pull-back vector bundle

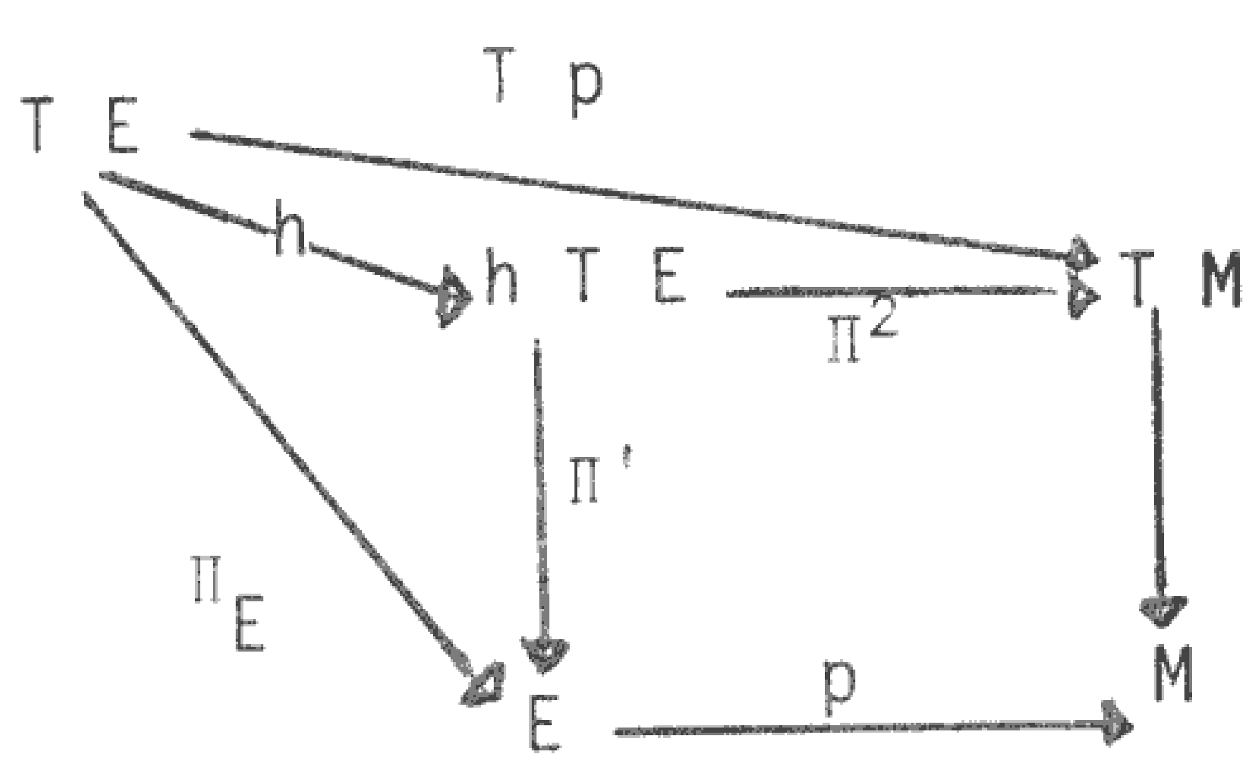
$$h \tau E \equiv h T E, \pi', E),$$

where

$$h T E \equiv E \times_M TM \ .$$

The map $\qquad h \equiv (\pi_E, T p) : T E \rightarrow h T E$

in the unique map which makes commutative the following diagram



3 DEFINITION.

The VERTICAL BUNDLE OF TE is the subbundle of τE, kernel of h on E

$$\nu\,\tau\,E \equiv (\nu\,T\,E,\ \Pi_E, E)\ \underline{.}$$

4 The following diagram is commutative

$$\begin{array}{ccc} \nu\,T\,E & \hookrightarrow & T\,E \\ \downarrow & & \downarrow \\ E & \hookrightarrow & h\,T\,E \end{array}$$

and the following sequence is exact

$$0 \to \nu\,T\,E \to T\,E \xrightarrow{h} h\,T\,E \to 0,$$

hence we have a canonical isomorphism

$$h\,T\,E \hookrightarrow T\,E/\nu\,T\,E.$$

5 PROPOSITION.

We get $$\nu\,T\,E = \bigsqcup_{e \in E} \{d\,c_e(0)\}$$

where $$\{c_e\} \equiv \{c : \mathbb{R} \to E \mid c(o) = e,\ p \circ c = p(c(o))\}\ \underline{.}$$

Such curves $c : \mathbb{R} \to E$ are called VERTICAL.

6 DEFINITION.

The TANGENT BUNDLE OF E, ON hTE, is the pull-back bundle

$$\tau_h E \equiv (TE,\ h,\ hTE)\ .$$

The VERTICAL BUNDLE OF TE, ON hTE, is the pull-back bundle

$$\bar{\tau}_h E \equiv (\bar{\nu}\,TE, \bar{h}, hT\,E),$$

where $$\bar{\nu}\,T\,E \equiv T\,M \times_M \nu T\,E \qquad \text{and} \qquad \bar{h} \equiv id_{TM} \times \Pi_E.$$

Hence the following diagram is commutative

$$\begin{array}{ccc} \bar{\nu}\,T\,E & \longrightarrow & \nu\,T\,E \\ \downarrow & & \downarrow \\ h\,T\,E & \longrightarrow & E \end{array}$$

7 PROPOSITION.

The bundle $$\bar{\tau}_h E \equiv (T\,E,\ h,\ h\,T\,E)$$

is an affine bundle, whose vector bundle is

$$\bar{\tau}_h E \equiv (\bar{\tau}\, T\, E,\ \bar{h},\ h\, T\, E)$$

PROOF.

Let $(e,u) \in E \times_M T\, M.$

We get $h^{-1}(e,u) = \{\alpha \in T_e E \mid T\, p(\alpha) = u\}$

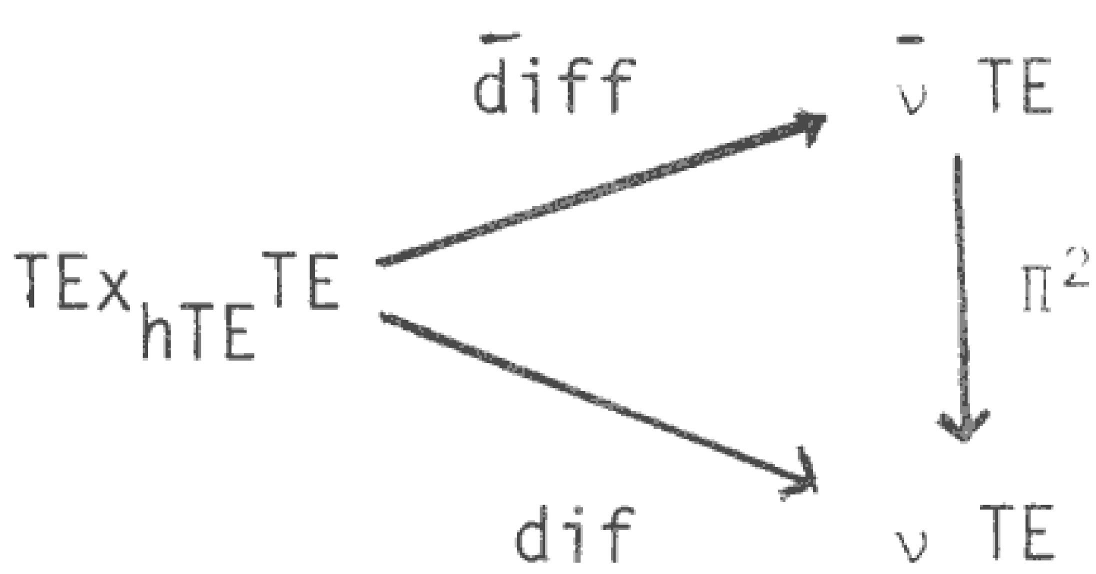
Since $T_e p : T_e E \to T_{p(e)} M$ is a linear map, then $h^{-1}(e,u)$ is an affine space, whose vector space is $\mathrm{Ker}\ T_e p = \bar{\nu}\, T_e\, E$ $\underline{\cdot}$

Hence $T\, E$ is an affine bundle on $h\, T\, E$ and a vector bundle on E.

Let us remark that we can consider the two difference maps, with respect to the two previous structures

$$\bar{\mathrm{diff}}: TE \times_{hTE} TE \to \bar{\nu}TE \qquad \text{and} \qquad \mathrm{diff}: TE \times_{hTE} TE \to \nu TE$$

and the following diagram is commutative

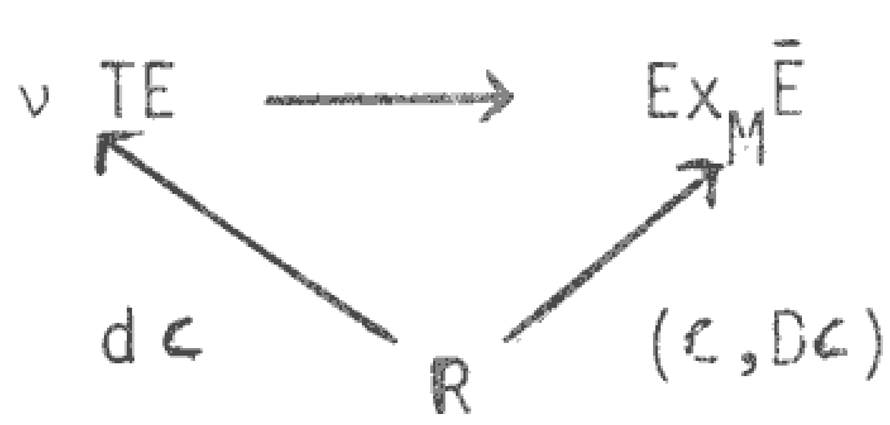


8 PROPOSITION.

Let $\eta \equiv (E,p,M)$ be an affine bundle, whose vector bundle is $\bar{\eta} \equiv (\bar{E},\bar{p},M)$. There is a unique diffeomorphism

$$\nu\, TE \hookrightarrow E \times_M \bar{E}$$

such that, for each vertical map $c : \mathbb{R} \to E$, the following diagram is commutative



Such a diffeomorphism is an isomorphism over E $\underline{\cdot}$

We will make often the identification

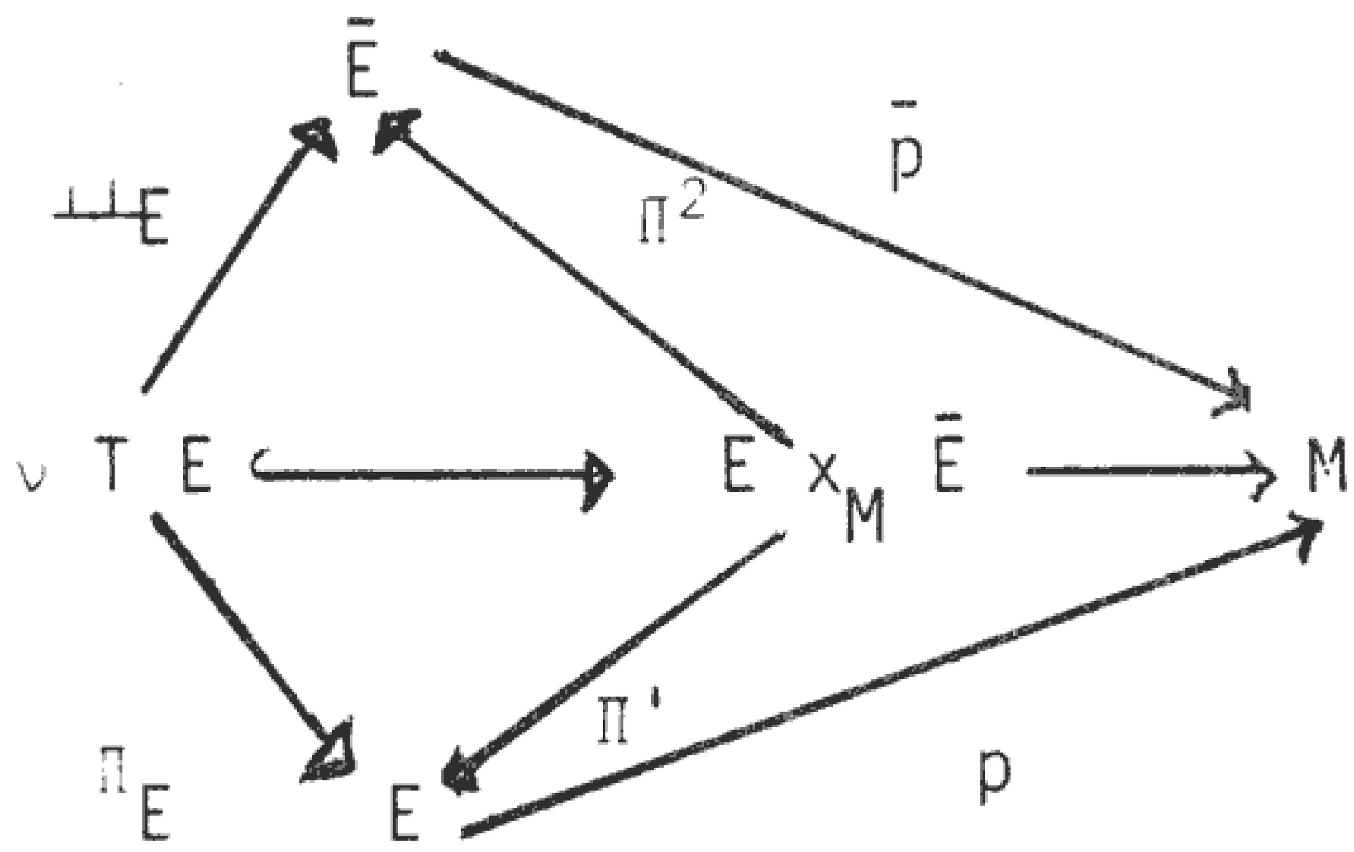
$$\nu T E \cong E \times_M \bar{E} .$$

9 We define the map

$$\perp\!\!\perp_E : \nu T E \to \bar{E}$$

by the composition $\nu T E \to E \times_M \bar{E} \xrightarrow{\pi^2} \bar{E}$.

Then we get the commutative diagram



and the homomorphism

$$\begin{array}{ccc} \nu T E & \xrightarrow{\perp\!\!\perp E} & \bar{E} \\ \downarrow{\scriptstyle \perp\!\!\perp_E} & & \downarrow{\scriptstyle \bar{p}} \\ E & \xrightarrow{p} & M \end{array}$$

is an isomorphism of fibers.

10 PROPOSITION.

Let $\eta \equiv (E,p,M)$ be a vector bundle. Then $T\eta \equiv (TE, Tp, TM)$ has a natural structure of vector bundle.

PROOF.

Let $\alpha,\beta \in T E$ be such that $T p(\alpha) = T p(\beta)$.

There exist $c_\alpha : \mathbb{R} \to E$ and $c_\beta : \mathbb{R} \to E$ such that

$$p \circ c_\alpha = p \circ c_\beta \quad \text{and} \quad d\, c_\alpha(o) = \alpha \ , \ d\, c_\beta(o) = \beta .$$

We can define $c \equiv c_\alpha + c_\beta : \mathbb{R} \to E$, for which we get

$p \circ c_\alpha = p \circ c = p \circ c_\beta$ and $T\, p(\alpha) = T\, p \circ d\, c = T\, p(\beta)$.

Since $d\, \gamma(o)$ depends only on α and β, we can put

$$\alpha + \beta \equiv d\, c(o) \quad \underline{.}$$

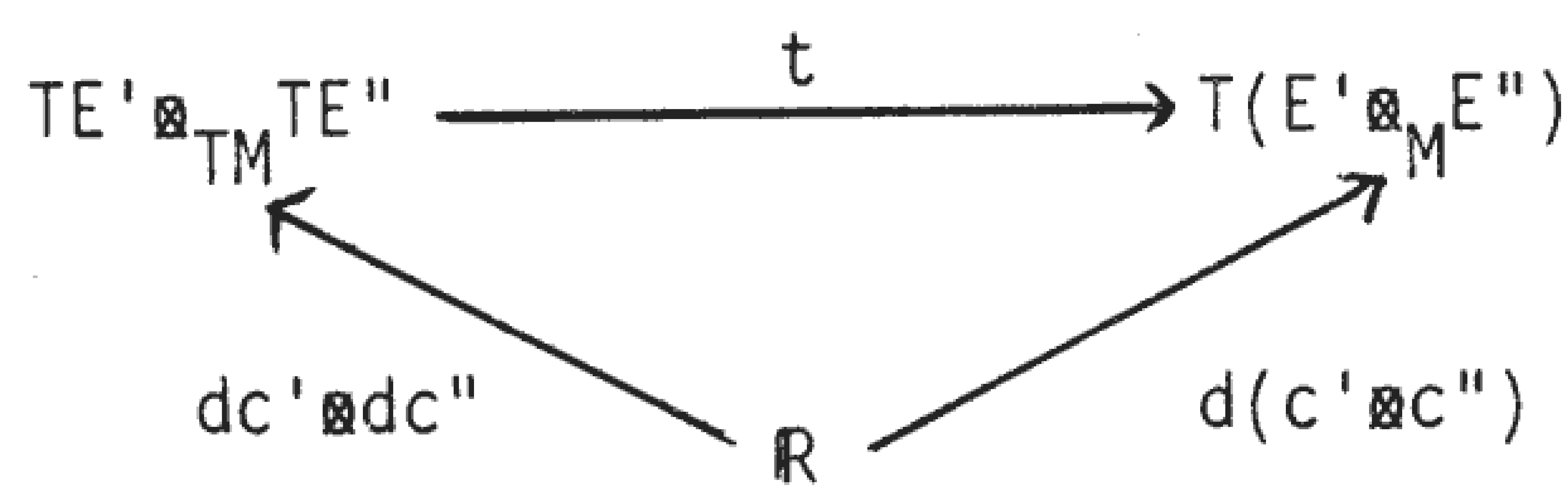
11 PROPOSITION.

Let $\eta' \equiv (E', p', M)$ and $\eta'' \equiv (E'', p'', M)$ be vector bundles.

There is a unique map

$$t : T\, E' \otimes_{TM} TE'' \longrightarrow T(E' \otimes_M E'')$$

such that the following diagram is commutative



for each $c' : \mathbb{R} \to E'$ and $c'' : \mathbb{R} \to E''$ such that

$$p' \circ c' = p'' \circ c'' .$$

This map is a surjective linear homomorphism over TM $\underline{.}$

2. - THE COTANGENT SPACE OF A BUNDLE.

Let $\eta \equiv (E, p, M)$ be a C^∞ bundle.

1 DEFINITION.

The COTANGENT BUNDLE OF E is the vector bundle

$$\tau^* E \equiv (T^* E, \rho_E, E) \quad \underline{.}$$

2 DEFINITION.

The HORIZONTAL BUNDLE OF $T^* E$ is the pull-back vector bundle

$$h\, \tau^*\, E \equiv (hT^* E, \pi^1, E),$$

where $$h\, T^*\, E \equiv E \times_M T^*\, M \quad \underline{.}$$