serious difficulties arise when we try to study this situation in the context of a general Fréchet space.

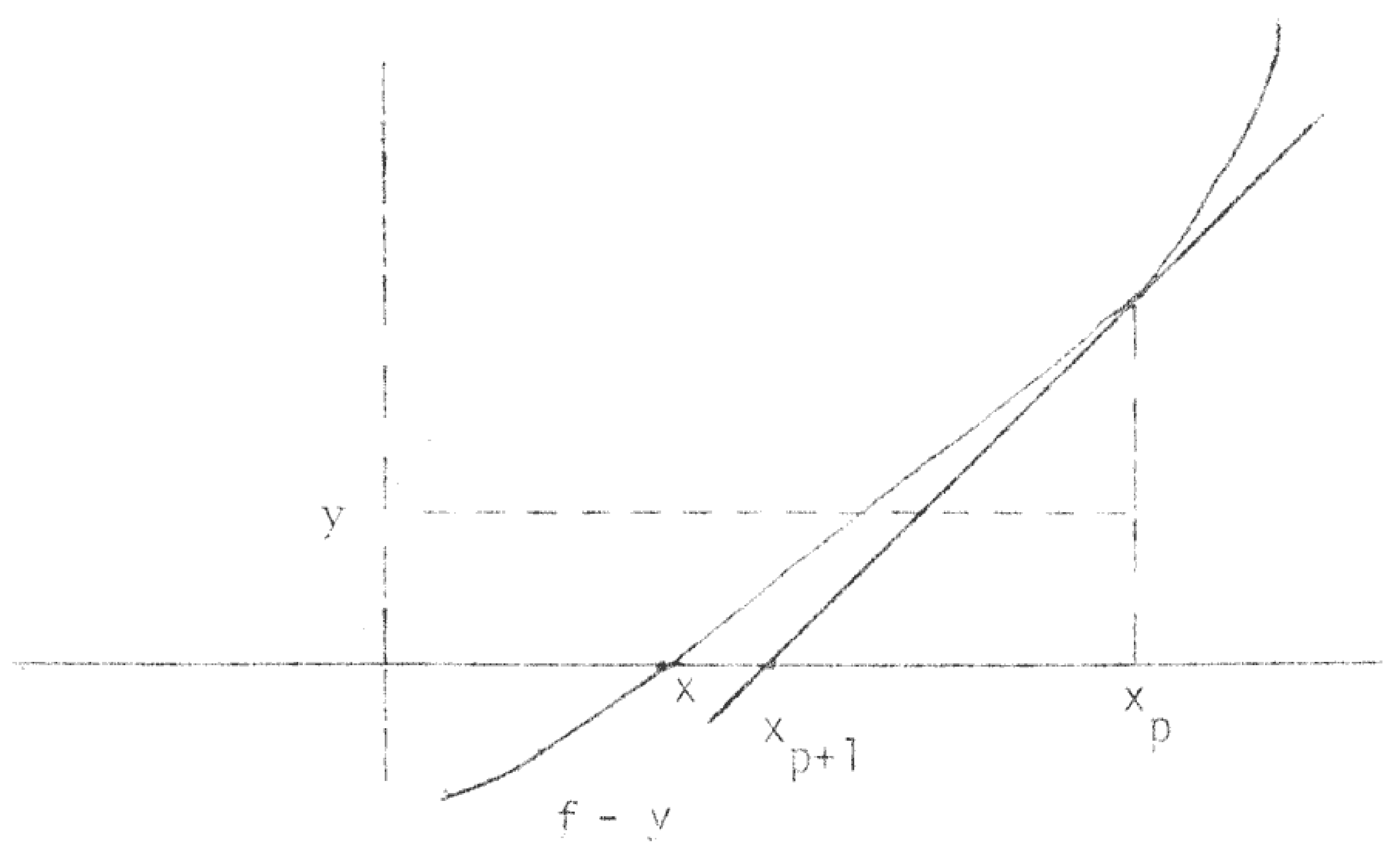
A totally different object of investigation is the structure of Fréchet spaces. There we consider a fixed space E and try to determine (up to isomorphism) all of its subspaces and quotient spaces. There are many other similar kinds of questions and this turns out to be rich area of study.

It is a little bit surprising that there are important connections between these two areas. These are being discovered in various current research activities and it is my main purpose in these lectures to describe some of them. Thus, the discussion will be divided into three parts: inverse function theorems, structure theory, and connections.

INVERSE FUNCTION THEOREMS.

We begin with $f : U \to E$ with $f(0) = 0$ and we want to solve $f(x) = y$ for small y. Of course, there are important related questions. Is the solution unique? Does it depend continuously on parameters? And so on. There are interesting things to say about such questions but, in these lectures, I will consider only the existence problem.

Our basic approach to solving $f(x) = y$ will be Newton's method. This works equally well when E is 1-dimensional, n-dimensional or even an infinite dimensional Banach space. The following picture describes the 1-dimensional situation but leads to formulas which work in the more general context:



The idea is to set us the recursion,

$$(1) \qquad x_{p+1} = x_p - ((f'(x_p))^{-1}(y-f(x_p)) \qquad (p\geq 0) \quad x_0 = 0$$

It is clear that if $\lim_p x_p = x$ and f is continuous, then $f(x) = y$.

Before we can use such a formula it is necessary to have a theory of differentiation which works in a Banach space. This is another vast subject and eventually any investigation into non-linear phenomena will have to deal with it extensively. For these lectures we take the short-cut of using the simplest definition and appealing to various regularity conditions (which we will not state explicitly) that imply, in our context, that all definitions are equivalent. This same definition can and will be used when E is a Fréchet space.

Thus we define the derivative of f at $x \in U$ to be the continuous linear function $f'(x) : E \to E$ which satifies :

$$f'(x)v = \lim_{t\to o} \frac{f(x+tv)-f(x)}{t} \qquad (x\in U \ , \ v\in E).$$

We then have the following result (see [8] for a proof).

THEOREM 1.

If E *is a Banach space and* $f'(0)$ *is invertible, then* $f(U)$ *is a neighborhood of* 0.

This is a very nice result and has important applications in partial differential equations. Unfortunately (and this has implications for the applications) nothing so broad is true in Fréchet spaces. It is useful to try to understand what goes wrong.

A first difficulty is that in Banach spaces it suffices to assume that $f'(0)$ is invertible because this implies that there is a whole neighborhood of 0, $W \subset U$, such that $f'(x)$ is invertible for every $x\in W$ which latter property is what is really needed. It is not hard to construct examples (I think there will be one in almost every non- Banach Fréchet space) that show that no such implication holds.

Actually, this is only a minor annoyance because no important examples are lost if we go the whole way and simply assume that $f'(x)$ is invertible for all x in some neighborhood of 0. Unfortunately, as the following example shows, this is still not enough.

Let E be the Fréchet space $C(\mathbb{R})$ of continuous real- valued functions on the real line $\mathbb{R}$ with the compact-open topology and let $f : E \to E$ be defined by taking $f(x)(t) = e^{x(t)} - 1$. Then $f(0) = 0$ and f is as regular as it could be. Moreover, it is easy to compute $f'(x)$ to obtain $f'(x)v = e^{x}v$ so that the inverse of $f'(x)$ is $f'(-x)$. On the other hand, any neighborhood of 0 will contain functions which take on values less than -1, but this is not possible for a function in the range of f.

We will try to analyse more closely what is going wrong, with the goal of getting some idea how to deal with this apparently chaotic situation. Let us see what in the proof of Theorem 1 does not work when we pass to Fréchet spaces.

Once the existence of an inverse of $f'(x)$, $x \in W$, is established there are two remaining issues in the proof of Theorem 1. First, we must guarantee that $x_p \in W$ so that the formula for x_{p+1} can be used and second, once the sequence (x_p) is defined, we must show that it converges or, at least, is Cauchy. Both concerns are dealt with using the same basic calculation:

$$f(x_p)-y = f(x_p)-f(x_{p-1})-f'(x_{p-1})(x_p-x_{p-1})$$

$$= \frac{1}{2}\int_0^1 f''(x_{p-1} + t(x_p-x_{p-1}))(x_p-x_{p-1})^2 dt.$$

With appropriate (and reasonable) regularity assumptions on f, this leads to the existence of positive constants C and δ with

$$\| f(x_p) - y\| \leq C\| x_p - x_{p-1}\|^2$$

and

$$\| x_{p+1} - x_p \| \leq C \| x_p - x_{p-1} \|^2 \leq \delta \| x_p - x_{p-1} \| .$$

This means that if $x_p - x_{p-1}$ is sufficiently small, then x_{p+1} will stay in W and $x_{p+1} - x_p$ will be even smaller. Thus it suffices to make $y-f(x_0) = y$ sufficiently small.

In a Fréchet space, however, the topology is defined by a sequence of norms $(\| \cdot \|_k)$ so that considerations of the above type lead only to relations of the form

$$\| x_{p+1} - x_p \|_k \leq \delta \| x_p - x_{p-1} \|_{\sigma(k)} , \tag{2}$$

where σ is a function determined by f and which is usually growing quite rapidly with k. Unfurtunately, if $\sigma(k)$ is much larger than k, the interation at each step leads to information about fewer norms and after finitely many steps, $\sigma(k) = 0$ and we have no information at all. Restrictions on the growth of σ are quite rare in the study of Fréchet spaces.

If $E = C^\infty(T)$ and f is a partial differential operator, then σ is related to the order of the operator and $\| \cdot \|_k$ is calculated in terms of the first k derivates. For this reason we call the function σ the *loss of derivatives function*. One of my major points in these lectures is that many phenomena occuring in the theory of Fréchet spaces can be related to this function, both conceptually and in the actual details of calculations.

In the case of the inverse function theorem, there is a method for dealing with the loss of derivatives. It is called the Nash-Moser method and it attacks the problem directly by using an additional structure with which a Fréchet space may be equipped.

Let $(S_\Theta)_{\Theta \geq 0}$ be a family of continuous linear operators, $S_\Theta : E \to E$, on a Fréchet space E, which satisfy the following conditions for $k \leq j$, $x \in E$, $\Theta > 0$ and an appropriate constant C which depends only on k,j but not on x,Θ:

$$\| S_\Theta x \|_j \leq C\Theta^{j-k} \| x \|_k$$

(3)

$$\| x - S_\Theta x \|_k \leq C\Theta^{k-j} \| x \|_j \quad .$$

Here $(\| \cdot \|_k)$ is an increasing sequence of seminorms which define the topology of E. (This will be discussed a little more in the next section). We refer to (S_Θ) as a *family of smoothing operators which satisfies* (3).

The recursion relation (1) is just changed to

(4) $$x_{p+1} = x_p - S_{\Theta_p}(f'(x_p))^{-1}(y - f(x_p)) \qquad (p \geq 0) \quad x_0 = 0,$$

where the sequence (Θ_p) must be chosen for the convenience of further calculations. For example, it no longer follows necessarily that, if (4) is used and (x_p) converges, then the limit is a solution of $f(x) = y$. This will be guaranteed by the second inequality in (3) provided $\lim_{p\to\infty} \Theta_p = \infty$. The calculations leading to (2) are then repeated by using the first relation in (3) to cancel to effect of σ. This turns out to be fairly delicate and requires that Θ_p does not grow too rapidly so that a balance must be struck. A more serious restriction is that nothing works unless there is a quite severe control on the growth of σ.

Nevertheless, it is possible to push through the calculations and we do get a theorem which, although very special, does have many important applications. The original idea is due to J. Nash [7], but J. Moser [6] was the first to realize how useful it could be. Subsequent refinements have been made by many authors, especially R. Hamilton, S. Łojasiewcz and E. Zehnder. The version given here is due jointly to the last two authors [4].

THEOREM 2.

Let E *be a Fréchet space which has a family of smoothing operators which satisfy* (3) *and let* $f : U \to E$ *be a continuous function on a neighborhood* U *of* 0 *in* E *which has a derivative at each point in* U. *Let* $f(0) = 0$ *and assume that there exist* $d > 0$ *and* $\lambda \in [1,2)$ *such that for all* $k \geq 0$ *we* *have constants* $C_k > 0$ *with*

$$\| f(x)\|_k \leq C_k \| x\|_{k+d} \qquad (x\in U)$$

$$\| f'(x)v\|_k \leq C_k(\| x\|_{k+d}\| v\|_o + \| v\|_{k+d}) \qquad (x\in U,\ v\in E)$$

$$\| f(x+v) - f(x) - f'(x)v\|_k \leq C_k(\| x\|_{k+d}\| v\|_o^2 + \| v\|_{k+d}\| v\|_o) \qquad (x, x+v\in U).$$

Suppose moreover that $f'(x)$ *has a right inverse* $L(x)$ *for each* $x\in U$ *and*

$$\| L(x)y\|_k \leq C_k(\| x\|_{k+d}\| y\|_d + \| y\|_{\lambda k+d}) \qquad (x\in U,\ y\in U)$$

(here $(\| \cdot \|_k)_{k \geq 0}$ *is again an increasing sequence of seminorms which defines the topology of* E*).*

Then $f(U)$ *is a neighborhood of* 0.

It is interesting to note that the requirement $\lambda<2$ in Theorem 2 is essential. In fact, in [4] there is given an example in which all of the hypotheses of the theorem hold except that $\lambda = 2$ and the conclusion of the theorem is false!

Of course, in order to even think of applying Theorem 2 it is necessary to consider how the smoothing operators might be constructed. In the original applications of Nash and Moser, E is always $C^\infty(T)$ and the smoothing operators are obtained either by convolution or the truncation of Fourier series.

In looking over the literature on this subject, it seemed curious to me that although many authors postulated wide classes of spaces for which Theorem 2 could be used, the concrete examples of Fréchet spaces which were actually written down were almost invariably (up to isomorphism) $C^\infty(T)$ or a space closely related to it.

This is in fact the case even when it did not appear so. For example, in [4] the authors use the Fréchet space $H(\mathbb{C})$ of entire functions in one variable. But this space is what we shall call later a "coordinate subspace" of $C^\infty(T)$ and in that case, analysis just carries over. It is like the situation in which you have an inverse function theorem valid for functions in three variables. It is then trivial to obtain (by holding one variable constant) a similar theorem for functions of two variables.

In any case, I tried to see if one could use a result like Theorem 2 for Fréchet spaces very different from $C^{\infty}(T)$. As we will see, the restriction to essentially this one space was no accident and it is necessary to change things quite a bit if we want to find an implicit function theorem valid in different kinds of Fréchet space.

Before we can get very far with such a program, it is necessary to say something about these spaces.

STRUCTURE OF FRÉCHET SPACES.

Recall that a Fréchet space is a complete, metrizable, locally convex space. Equivalently, it is a vector space E which is complete under a certain translation invariant metric and on which is defined an increasing sequence of semi-norms (sub-additive, positive scalar homogeneous real-valued functions) $(\|\cdot\|_k)_{k\geq 0}$ such that a sequence (x_n) in E converges to x in E iff $\lim_n \|x_n - x\|_k = 0$ $(k=0,1,2,\ldots)$.

In all of our applications we will take the seminorms $\|\cdot\|_k$ to be norms (that is, $\|x\|_k = 0$ iff $x = 0$). Somewhat more complicated is the fact that the Fréchet spaces we consider will all be nuclear. It is best to defer the definition of nuclear until we are in a more concrete situation.

The basic references for all of our discussion of the structure of nuclear Fréchet spaces will be [1], [5] and [10]. For us, the best starting point is to list some examples: