3 FRAMES AND THE REPRESENTATION OF $T^2\mathbb{E}$.

In this section we are dealing with the second order derivatives of the frame and tangent spaces.

Frame acceleration, second jacobians, strain and spin .

1 The acceleration of the frame is the vector field on $\mathbb{E}$ constituted by the accelerations of the world-lines of the frame. Hence it is the second derivative of the motion with respect to time. On the other hand, the second and mixed jacobians are the second derivatives with respect to event- event and time- event. We consider only free entities.

DEFINITION.

For simplicity of notations, leaving to the reader to write them in the complete from .

a) The (FREE) ACCELERATION-FUNDAMENTAL FORM - of $\mathcal{P}$ is the map

$$D_1^2\tilde{P} : \mathbb{T} \times \mathbb{E} \to \bar{\mathbb{E}} \quad .$$

The (FREE) ACCELERATION-EULERIAN FORM - of $\mathcal{P}$ is the map

$$\overset{=}{P} \equiv D_1^2 \; P \circ j : \mathbb{E} \to \bar{\mathbb{E}}$$

b) The (FREE) SECOND JACOBIAN-FUNDAMENTAL-EULERIAN FORM - of $\mathcal{P}$ is the map

$$D_2^2\tilde{P} : \mathbb{T} \times \mathbb{E} \to \bar{\mathbb{E}}^* \otimes \bar{\mathbb{E}}^* \otimes \bar{\mathbb{E}} \quad .$$

The (FREE) SECOND JACOBIAN-EULERIAN-EULERIAN FORM - of $\mathcal{P}$ is the map

$$\hat{\hat{P}} \equiv D_2^2 \; \tilde{P} \circ j : E \to \bar{\mathbb{E}}^* \otimes \bar{\mathbb{E}}^* \otimes \bar{\mathbb{E}} \quad .$$

The (FREE) SPATIAL SECOND JACOBIAN-FUNDAMENTAL-EULERIAN FORM - of $\mathcal{P}$ is the map

$$\check{\check{P}} \equiv \check{D}_2^2\tilde{P} : \mathbb{T} \times \mathbb{E} \to \bar{\mathbb{S}}^* \otimes \bar{\mathbb{S}}^* \otimes \bar{\mathbb{S}} \quad .$$

The (FREE) SPATIAL SECOND JACOBIAN-LAGRANGIAN - LAGRANGIAN FORM WITH RESPECT

TO THE INITIAL TIME $\tau \in T$ AND THE FINAL TIME $\tau' \in T$ - of $\mathcal{P}$ is the map

$$\check{\check{P}}_{(\tau',\tau)} \equiv D^2 P_{(\tau',\tau)} : \mathbb{S}_\tau \to \bar{\mathbb{S}}^* \otimes \bar{\mathbb{S}}^* \otimes \bar{\mathbb{S}} \quad .$$

c) The (FREE) MIXED SECOND JACOBIAN-FUNDAMENTAL-EULERIAN FORM of $\mathcal{P}$ is the map

$$D_2 D_1 \tilde{P} : \mathbb{T} \times E \to \mathbb{E}^* \otimes \bar{\mathbb{E}}$$

The (FREE) MIXED SECOND JACOBIAN-EULERIAN-EULERIAN FORM of $\mathcal{P}$ is the map

$$\hat{\bar{P}} \equiv D_2 D_1 \tilde{P} \circ j : \mathbb{E} \to \bar{\mathbb{E}}^* \otimes \bar{\mathbb{E}}$$

The (FREE) MIXED SPATIAL SECOND JACOBIAN-EULERIAN-EULERIAN FORM - of $\mathcal{P}$ is the map

$$\check{\bar{P}} \equiv \check{D}_2 D_1 P \circ j : \mathbb{E} \to \bar{\mathbb{S}}^* \otimes \bar{\mathbb{E}}$$

d) The (FREE) STRAIN-EULERIAN FORM - of $\mathcal{P}$ is the map

$$\epsilon_{\mathcal{P}} \equiv S \circ \check{\bar{P}} : \mathbb{E} \to \mathbb{S} \otimes \bar{\mathbb{S}} \, .$$

The (FREE) SPIN - EULERIAN FORM - of $\mathcal{P}$ is the map

$$\omega_{\mathcal{P}} \equiv \frac{A}{2} \circ \check{\bar{P}} : \mathbb{E} \to \bar{\mathbb{S}}^* \otimes \bar{\mathbb{S}}$$

The (FREE) ANGULAR VELOCITY-EULERIAN FORM - of $\mathcal{P}$ is the map

$$\Omega_{\mathcal{P}} \equiv \check{*} \frac{A}{2} \circ \check{\bar{P}} : \mathbb{E} \to \bar{\mathbb{S}} \, .$$

2 We get immediate important properties of these maps.

PROPOSITION.

We have

a) $$\underline{t} \circ D_1^2 \tilde{P} = 0$$

b) $$\underline{t} \circ D_2^2 \tilde{P} = 0 \ ,$$

c) $$\underline{t} \circ D_2 D_1 \tilde{P} = 0 \ ,$$

hence we can write

$$D_1^2 \tilde{P} : \mathbb{T} \times E \to \bar{\mathbb{S}} \qquad D_2^2 \tilde{P} : \mathbb{T} \times \mathbb{E} \to \bar{\mathbb{E}}^* \otimes \bar{\mathbb{E}}^* \times \bar{\mathbb{S}} \qquad D_2 D_1 \tilde{P} : \mathbb{T} \times E \to \bar{\mathbb{E}}^* \otimes \bar{\mathbb{S}}$$

$$\bar{\bar{P}} : \mathbb{E} \to \bar{\mathbb{S}} \qquad \hat{\hat{P}} : \mathbb{E} \to \mathbb{E}^* \otimes \bar{E}^* \otimes \bar{\mathbb{S}} \qquad \hat{\bar{P}} : \mathbb{E} \to \bar{\mathbb{E}}^* \otimes \bar{\mathbb{S}}$$

$$\check{\bar{P}} : \mathbb{E} \to \bar{\mathbb{S}}^* \otimes \bar{\mathbb{S}}$$

Moreover all the previous maps are expressible by $\tilde{P}, \bar{P}, DP$ and $\check{\check{P}}$:

d) $$D_1^2 \tilde{P} = \bar{\bar{P}} \circ \tilde{P}$$

e) $$\hat{\hat{P}} = - \bar{\bar{P}} \otimes \underline{t} \otimes \underline{t} - (\check{D}\bar{P} \circ \hat{P}) \otimes \underline{t} - \underline{t} \otimes (\check{D} \bar{P} \circ \hat{P})$$

f) $$\hat{\bar{P}} = \check{D} \bar{P} \circ \hat{P}$$

g) $$\bar{\bar{P}} = D \bar{P} (\bar{P})$$

h) $$\check{\bar{P}} = \check{D} \bar{P}$$

i) $$(D_2^2 \tilde{P})_{\tau'} |_{\mathbb{S}_\tau} = \check{\check{P}}_{(\tau', \tau)} \circ \hat{P} |_{\mathbb{S}_\tau}$$

l) $$(D_1 \check{P}) \circ j = \check{D} \bar{P} \ .$$

If $u \equiv u^\circ \bar{P} + u_P : \mathbb{E} \to \bar{\mathbb{E}}$, we can write

m) $$D \bar{P}(u) = u^\circ \bar{\bar{P}} + \frac{1}{2} \varepsilon_{\mathcal{P}}(\check{u}_P) + \Omega_{\mathcal{P}} \times \check{u}_P \ .$$

n) We have $$\underline{\varepsilon}_{\mathcal{P}} = L_{\bar{\mathbf{P}}} \, \mathbf{g} \ .$$

o) $$\bar{\bar{P}} = \Gamma^i_{oo} \, \delta x_i$$

$$\bar{\hat{P}} = \Gamma^k_{io}\, Dx^i \otimes \delta x_k$$

$$\hat{\hat{P}} = - \Gamma^k_{oo}\, Dx^\circ \otimes Dx^\circ - \Gamma^k_{io}(Dx^i \otimes Dx^\circ + Dx^\circ \otimes Dx^i) \otimes \delta x_k$$

$$\underline{\varepsilon}_{\mathcal{P}} = \Gamma_{j,o}{}^{i} + \Gamma^{i}{}_{oj})\, Dx^i \otimes Dx^j = \partial_o g_{ij}\, Dx^i \otimes Dx^j$$

$$\underline{\omega}_{\mathcal{P}} = \frac{1}{2}\,(\Gamma_{j,oi} - \Gamma_{i,oj})\, Dx^i \otimes Dx^j$$

$$\Omega_{\mathcal{P}} = \frac{1}{2}\, \sqrt{\det(g^{ij})}\quad \epsilon^{kij}\Gamma_{j,oi}\, \delta x_k$$

PROOF.

a),b) and c) follow from (II,1,10 a) by double derivation with respect to τ,τ; e.e. and τ,e.

d) follows from (II,1,10 c) by double derivation with respect to τ and taking $\sigma \equiv \tau$.

e) follows from (II,1,10 b) by double derivation with respect to e.

f) follows from (II,1,10 c) by double derivation with respect to τ and with respect to τ and e and taking $\sigma \equiv \tau$.

g) follows from (II,2,2c) by derivation with respect to e.

h) follows from f).

i) follows from (II,1,10 c) by double derivation with respect to e and taking $\tau \equiv t(e)$, $\sigma \equiv \tau$.

l) follows from $D_2 D_1 \tilde{P} = D_1 D_2 \tilde{P}$.

m) follows from g and f)

n) follows from $(L_{\bar{P}}\, g)_{ij} = \partial_o g_{ij} = \Gamma_{j,oi} + \Gamma_{i,oj}$ $\doteq$

Representation of $T^2\mathbf{P}$ and $\nu T^2\mathbf{P}$.

3 In order to get the space $T^2\mathbf{P}$ handy, it is useful to regard it as a quotient. In this way we could view $T^2\mathbf{P}$ as a quotient space $T^2\mathbf{E}_{/\mathbf{P}}$. But a

reduced representation by means of $T^2\mathbb{E}_{/\mathbb{P}}$ is more simple for the equivalence classes have a unique representative for each time $\tau \epsilon T$.

PROPOSITION.

Let $v \in T^2\mathbb{P}$. Then

$$\mathbb{C}_v \equiv T^2p^{-1}(v) = (T_2^2\, P)_v(\mathbb{T}) \to T^2\mathbb{E} \qquad \text{(a)}$$

is a C^∞ submanifold.

Then we get a partition of $T^2\mathbb{E}$, given by

$$T^2\mathbb{E} = \bigsqcup_{v \in T^2\mathbb{P}} \mathbb{C}_v \quad ,$$

and the quotient space $T^2\mathbb{E}_{/\mathbb{P}}$, which has a natural C^∞ structure and whose equivalence classes are characterized by

$$[e,u,v,w] = [e',u',v',w'] \Longleftrightarrow p(e)=p(e'),\ \check{P}(t(e'),e)(u) = u' \ ,$$
$$\check{P}(t(e'),e)(v) = v',\ \check{\check{P}}(t(e'),e)(u,v) + \check{P}(t(e'),e)(w) = w' \qquad \text{(b)}$$

We get a natural C^∞ diffeomorphism between $T^2\mathbb{P}$ and $\check{T}^2\mathbb{E}_{/\mathbb{P}}$ given by the unique maps

$$T^2\mathbb{P} \to \check{T}^2\mathbb{E}_{/\mathbb{P}} \qquad \text{and} \qquad \check{T}^2\mathbb{E}_{/\mathbb{P}} \to T^2\mathbb{P}$$

which make commutative the following diagrams, respectively,

$$\begin{tikzcd} \mathbb{T} \times T^2\,\mathbb{P} \arrow[r, "T_2^2 P"] \arrow[d, "\pi^2"'] & \check{T}^2\mathbb{E} \arrow[d] \\ T^2\,\mathbb{P} \arrow[r] & \check{T}^2\mathbb{E}_{/\mathbb{P}} \end{tikzcd} \qquad \begin{tikzcd} \check{T}^2\mathbb{E} \arrow[rr] \arrow[dr] & & T^2\mathbb{P} \\ & T^2\mathbb{E}_{/\mathbb{P}} \arrow[ur, hook] & \end{tikzcd}$$

PROOF.

Analogous to (II,2,3) $\underline{\cdot}$

4 Choicing a time $\tau \in T$ and taking, for each equivalence class, its representative at the time τ, we get a second interesting representation of $T^2\mathbb{P}$.

PROPOSITION.

The maps T^2P_τ and T^2p_τ are inverse C^∞ diffeomorphisms

$$T^2\tilde{P}_\tau : T^2P \to T^2\mathbb{S}_\tau \equiv \mathbb{S}_\tau \times \bar{\mathbb{S}} \times \bar{\mathbb{S}} \times \bar{\mathbb{S}} \, , \quad T^2p_\tau : T^2\mathbb{S}_\tau \equiv \mathbb{S}_\tau \times \bar{\mathbb{S}} \times \bar{\mathbb{S}} \times \bar{\mathbb{S}} \to T^2P \;\; \underline{.}$$

5 The relation among the different representations of $T^2\mathbb{P}$ is shown by the following commutative diagram

$$\begin{tikzcd}
& T^2\mathbb{P} \arrow[ld, "T^2\tilde{P}_\tau"'] \arrow[rr, hook] & & T^2\mathbb{E}_{/P} \\
T^2\mathbb{S}_\tau & & T^2\mathbb{S}_\tau \arrow[lu, "T^2\tilde{P}_\tau"'] \arrow[ll] \arrow[r, hook] & T\,\mathbb{E} \arrow[u]
\end{tikzcd}$$

6 The previous representations of $T^2\mathbb{P}$ reduce to analogous representations of $v\,T^2\mathbb{P}$.

COROLLARY.

The quotient space $(vT^2\mathbb{E})_{/\mathcal{P}}$ is a C^∞ submanifold of $T^2\mathbb{E}_{/\mathcal{P}}$ and its equivalence classes are characterized by

$[e,u,o,w] = [e',u',o,w] \iff p(e) = p(e')$, $P(t(e'),e)(u) = u'$, $P(t(e')e)(w) = w'$.

The diffeomorphism $T^2P \to T^2\mathbb{E}_{/\mathcal{P}}$ induces a diffeomorphism

$$vT^2\mathbb{P} \to (vT^2\mathbb{E})_{/\mathcal{P}}$$

and the diffeomorphism $T^2\mathbb{E}_{/\mathcal{P}} \to T^2\mathbb{P}$ induces the inverse diffeomorphism

$$(vT^2\mathbb{E})_{/\mathcal{P}} \to vT^2\mathbb{P} \; .$$

Moreover, the following diagrams are commutative

$$\begin{array}{ccc} \nu T^2\,\mathbb{E} & \xrightarrow{\coprod} & T\,\mathbb{E} \\ {\scriptstyle T^2 p}\downarrow & & \downarrow{\scriptstyle T p} \\ \nu T^2\mathbb{P} & \xrightarrow[\coprod_{\mathcal{P}}]{} & T\,\mathbb{P} \end{array} \qquad \begin{array}{ccc} \nu T^2\$_\tau & \xrightarrow{\coprod} & T\,\$_\tau \\ \downarrow & & \downarrow \\ \nu T^2\mathbb{P} & \xrightarrow[\coprod_{\mathcal{P}}]{} & T\,\mathbb{P} \end{array}$$

.

7 Taking into account the identification $T^2\mathbb{P} \overset{\sim}{=} \check{T}^2\mathbb{E}_{/\mathcal{P}}$, we get the following expression of T^2p and T^2P.

PROPOSITION.

a) $T^2p(e,u,v,w) = [e,P(e)(u), \hat{P}(e)(v), \hat{\hat{P}}(e)(u,v) + \hat{P}(e)(w)]$

b) $T^2P(\tau,\lambda,\mu,v;\ [e,u,v,w]) =$

$= (\tilde{P}(\tau,e),\ \lambda\bar{P}(\tilde{P}(\tau,e)) + P(\tau,e)(u),\ \mu\bar{P}(P^2(\tau,e)) + P(\tau,e)(v)\ ,$

$\lambda\mu\ \bar{\bar{P}}(P(\tau,e) + \lambda D\bar{P}(P(\tau,e))(P(\tau,e)(v)) + \lambda D\bar{P}(P(\tau,e)(P(\tau,e)(u)) + v\bar{P}(\theta^2(\tau,e) +$

$+ P(\tau,e)(u,v) + P(\tau,e)(w)$

PROOF.

Analogous to (II,2,6) .

<u>Frame connection and Cariolis map.</u>

8 For each $\tau\in T$, we can view $\mathbb{P}$ as an affine space ,depending on τ, taking into account the isomorphism $\mathbb{T} \times T\mathbb{P} \to T\mathbb{E}$. Hence we get a "time depending" affine connection on $\mathbb{P}$

$$\check{\Gamma}_P : \mathbb{T} \times_{s} T^2\,P \to \nu T^2\,\mathbb{P}.$$

THEOREM.

There is a unique map

$$\check{\Gamma}_{\mathcal{P}} : \mathbb{T} \times s\, T^2\, \mathbb{P} \to \nu T^2\, \mathbb{P} .$$

such that the following diagram is commutative

$$\begin{tikzcd}
s\, T^2\, \mathbb{E} \arrow[r, "\Gamma"] \arrow[d, "{(t,T^2p)}"'] & \nu T^2\, \mathbb{E} \arrow[d, "T^2p"] \\
\mathbb{T} \times s T^2 \mathbb{P} \arrow[r, "\check{\Gamma}_{\mathcal{P}}"] & \nu T^2 \mathbb{P}
\end{tikzcd} .$$

Such a map is given by the following commutative diagram

$$\begin{tikzcd}
s\, \check{T}^2\, \mathbb{E} \arrow[r, "\Gamma"] & \nu \check{T}^2\, \mathbb{E} \arrow[d, "T^2p"] \\
\mathbb{T} \times s\, T^2\, \mathbb{P} \arrow[u, "(T^2P)_{o,o,o}"] \arrow[r, "\check{\Gamma}_{\mathcal{P}}"] & \nu T^2\, \mathbb{P}
\end{tikzcd}$$

Namely we get

$$\check{\Gamma}_P(\tau,[e,u,u,w]) = [\tilde{P}(\tau,e), \check{P}(\tau,e)(u);\ 0, \check{P}(\tau,e)(u,u) + \check{P}(\tau,e)(w)] ,$$

hence, if $t(e) \equiv \tau$

$$\check{\Gamma}_P(\tau,[e,u,u,w]) = [e,u,o,w] .$$

PROOF.

(t,T^2p) is $T^2\mathbb{E} \to \mathbb{T} \times s\, T^2\mathbb{P}$ and $(T^2P)_{(o,o,o)} : \mathbb{T} \times s\, T^2\mathbb{P} \to s\, \check{T}^2\mathbb{E}$

are inverse C^∞ diffeomorphisms.

9 Then we can introduce the "following map", that will be used (III,1) to define the covariant derivative of maps $\mathbb{T} \to T\mathbb{P}$, hence the accelera tion of observed motion .

DEFINITION.

The FRAME TIME DEPENDING AFFINE CONNECTION is the map

$$\acute{\Gamma}_P : \mathbb{T} \times_s T^2\mathbb{P} \to \nu T^2\mathbb{P} ,$$

given by $\quad (\tau,[e,u,u,w]) \mapsto [\tilde{P}(\tau,e),\check{P}(\tau,e)(u);0,\acute{\check{P}}(\tau,e)(u,u)+\check{P}(\tau,e)(w)]$

10 The time depending affine connection $\check{\Gamma}_{\mathbb{P}}$ does not sufficies for Kinematics. Coriolis theorem, (III,1) which makes a comparison between the acceleration of an observed motion and the observed acceleration of a motion, requires a further map $\dot{\Gamma}_{\mathbb{P}} : \mathbb{T} \times_s T^2\mathbb{P} \to \nu T^2\mathbb{P}$, which is obteined taking into account the isomorphism $\mathbb{T} \times T\mathbb{P} \to \dot{T}\mathbb{E}$.

THEOREM.

There is a unique map

$$\dot{\Gamma}_{\mathbb{P}} : \mathbb{T} \times_s T^2\mathbb{P} \to \nu T^2\mathbb{P}$$

such that the following diagram is commutative

$$\begin{tikzcd}
\dot{T}^2\mathbb{E} \arrow[r, "\Gamma"] \arrow[d, "{(\ddot{t},T^2p)}"'] & \nu \dot{T}^2\mathbb{E} \arrow[d, "T^2p"] \\
\mathbb{T} \times_s T^2\mathbb{P} \arrow[r, "\dot{\Gamma}_{\mathbb{P}}"] & \nu T^2\mathbb{P}
\end{tikzcd}$$

Such a map is given by the following commutative diagram

$$\begin{tikzcd}
\dot{T}^2\mathbb{E} \arrow[r, "\dot{\Gamma}"] & \nu \dot{T}^2\mathbb{E} \arrow[d, "T^2p"] \\
\mathbb{T} \times_s T^2\mathbb{P} \arrow[u, hook, "(T^2\mathbb{P})_{(1,1,0)}"] \arrow[r, "\dot{\Gamma}_{\mathbb{P}}"] & \nu T^2\mathbb{P}
\end{tikzcd}$$

Namely we get

$$\dot{\Gamma}_{\mathbb{P}}(\tau, [e,u,u,w]) =$$

$$= [\tilde{P}(\tau,e),\check{P}(\tau,e)(u),0, \acute{P}(\tau,e)(w) + 2\acute{\bar{P}}(\tau,e)(P(\tau,e)(u)) + \bar{\bar{P}}(\grave{P}(\tau,e))]$$

hence, if $t(e) \equiv \tau$,

$$\acute{\Gamma}_{\mathcal{P}}(\tau,[e,u,u,w]) = [e,u,0,w + 2\,\check{\bar{P}}(e)(u) + \bar{\bar{P}}(e)] \;.$$

Thus we have

$$\acute{\Gamma}_{\mathcal{P}} = \check{\Gamma}_{\mathcal{P}} + \check{C}_{\mathcal{P}} + \check{D}_{\mathcal{P}} \;,$$

where

$C_{\mathcal{P}} : \mathbb{T} \times T\,\mathbb{P} \to T\,\mathbb{P}$ and $D_{\mathcal{P}} : \mathbb{T} \times \mathbb{P} \to T\,\mathbb{P}$ are given by

$$C_{\mathcal{P}}(\tau,[e,u]) \equiv [\tilde{P}(\tau,e)\;,\; 2\check{\bar{P}}(\tilde{P}(\tau,e))(u)]$$

$$D_{\mathcal{P}}(\tau,e) \equiv [\;\tilde{P}(\tau,e)\;,\; \bar{\bar{P}}\;(\tilde{P}(\tau,e))]$$

hence $\check{C}_{\mathcal{P}} : \mathbb{T} \times s\,T^2\mathbb{P} \to \nu T^2\mathbb{P}$ and $\check{D}_{\mathcal{P}} : \mathbb{T} \times s\,T^2\mathbb{P} \to \nu T^2\mathbb{P}$

are given by

$$\check{C}_{\mathcal{P}}(\tau,[e,u,u,w]) \equiv [\tilde{P}(\tau,e),\check{P}(\tau,e)(u),0,\; 2\,\check{\bar{P}}(\tilde{P}(\tau,e))(\check{P}(\tau,e)(u))\;]$$

$$\check{D}_{\mathcal{P}}(\tau,[e,u,u,w]) \equiv [\tilde{P}(\tau,e),\check{P}(\tau,e)(u),0,\; \bar{\bar{P}}(\tilde{P}(\tau,e)] \;.$$

PROOF.

$(\check{\dot{t}},T^2p) : \acute{T}^2\mathbb{E} \to \mathbb{T} \times sT^2\mathbb{P}$ and $(T^2\mathbb{P})_{(1,1,0)} : \mathbb{T} \times sT^2\mathbb{P} \to \acute{T}^2\mathbb{E}$

are inverse C^∞ diffeomorphisms.

11 Then we can give the following definition

DEFINITION.

The FRAME CORIOLIS MAP is the map

$$C_{\mathcal{P}} : \mathbb{T} \times T\mathbb{P} \to T\mathbb{P}$$

given by $(\tau,[e,u]) \mapsto [\tilde{P}(\tau,e),\; 2\,\check{\bar{P}}(P(\tau,e))(u)]$

The FRAME DRAGGING MAP is the map

$$D_{\mathcal{P}} : \mathbb{T} \times \mathbb{P} \to T\,\mathbb{P}$$

given by $$(\tau, [e]) \to [\tilde{P}(\tau,e), \bar{\bar{P}}(P(\tau,e))]$$.

Physical description.

$\bar{\bar{P}}$ is the field of acceleration of the field continuum. $\mathcal{E}_{\mathcal{P}}$ is the rate of change, during time, of the spatial metric; $\Omega_{\mathcal{P}}$ describes the rate of change, during time, of the spatial directions. This facts are implicitly proved in the next section.

It is not easy to describe by picture the fundamental ,but not straight forward,results of this section.