The COVARIANT DERIVATIVE of v with respect to u is

$$\nabla_u v \equiv \underline{||} \circ \Gamma \circ T\, v \circ u = (id_{\mathbb{E}}, D\tilde{u}(\tilde{v})) : E \to T\mathbb{E} \quad \underline{.}$$

3 - ASSOLUTE KINEMATICS.

Here we introduce the basic elements of one-body kinematics independent of any frame of reference.

Absolute world-line and motion.

1 The basic definition of kinematics is the following. Here we consider a C^∞ world-line extending along the whole T. We leave to the reader the easy generalization to the case when it is C^2 almost every where, or when it extends along an interval of T.

DEFINITION.

A WORLD-LINE is a connected C^∞ submanifold

$$\mathbb{M} \hookrightarrow \mathbb{E}$$

such that $\quad \mathbb{S}_\tau \cap \mathbb{M}$ is a singleton, $\quad \forall\, \tau \in \mathbb{T}$.

The MOTION, RELATIVE TO THE WORLD LINE $\mathbb{M}$, is the map

$$M : \mathbb{T} \to \mathbb{E}$$

given by $\quad \tau \mapsto$ the unique element $\in \mathbb{S}_\tau \cap \mathbb{M}$ $\underline{.}$

Henceforth in this section we suppose a world-line $\mathbb{M}$, or its motion M, to be given.

2 PROPOSITION.

$\mathbb{M}$ is an ambedded 1-dimensional submanifold of $\mathbb{E}$, diffeomorphic to $\mathbb{R}$.

M is a section of $(\mathbb{E}, t, \mathbb{T})$, namely it is a C^∞ embending, such that

$$t \circ M = id_{\mathbb{T}} \ ,$$

i.e. such that

$$M^\circ \equiv x^\circ \circ M = \hat{x}^\circ \ .$$

Hence the map

$$M : T \to \mathbb{M} \qquad \text{is a } C^\infty \text{ diffeomorphism .}$$

The world line $\mathbb{M}$ is characterized by its motion M .

3 The affine structures of $\mathbb{T}$ and $\mathbb{E}$ admit a Kind of privileged world-lines.

DEFINITION.

$\mathbb{M}$ is INERTIAL if it is an affine subspace of $\mathbb{E}$

4 PROPOSITION.

$\mathbb{M}$ is inertial if and only if M is an affine map, i.e.

$$M(\tau') = M(\tau) + DM(\tau'-\tau), \quad \text{with} \quad DM \in U \ .$$

<u>Absolute velocity and acceleration.</u>

5 Previously we introduce useful notations .

a) Let $\mathbb{F}$ be an affine space and let

$$\phi : \mathbb{T} \to \mathbb{F}$$

be a C^∞ map.

Then we put $\quad d\phi \equiv (\phi, D\phi) : \mathbb{T} \to T\mathbb{F}$.

In particular, if $\quad \phi \equiv f : \mathbb{T} \to \mathbb{E}$,

we get $\quad df \equiv (f, Df) : \mathbb{T} \to T\mathbb{E}$

and $\quad d^2 f \equiv (f, Df, Df, D^2 f) : \mathbb{T} \to T^2\mathbb{E}$.

b) We put $\quad \nabla df \equiv \underline{\sqcup\!\sqcup} \circ \Gamma \circ d^2 f = (f, D^2 f) : \mathbb{T} \to T\mathbb{E}$.

The coordinate expressions are

$$df = Df^\alpha (\partial x_\alpha \circ f)$$

$$d^2 f = Df^\alpha (\partial x_\alpha \circ df) + D^2 f^\alpha (\partial \dot{x}_\alpha \circ df)$$

$$\nabla df = (D^2 f^\alpha + (\Gamma^\alpha_{\beta\gamma} \circ f) Df^\beta Df^\gamma)(\partial x_\alpha \circ f) .$$

6 We can view the absolute velocity in terms of free or of applied vectors, equivalently.

DEFINITION.

The FREE VELOCITY of M is the map

$$DM : \mathbb{T} \to \bar{\mathbb{E}} .$$

The VELOCITY of M is the map

$$d\ M \equiv (M,\ D\ M) : \mathbb{T} \to T\mathbb{E} \quad \underline{.}$$

7 PROPOSITION.

We have $$< \underline{t}\ ,\ D\ M > = 1\ . \qquad (*)$$

Hence, we can write

$$D\ M : T \to \mathbb{U}$$

and $$d\ M : \mathbb{T} \to \mathbb{V} \equiv \overset{\prime}{T}\mathbb{E}$$

and we get $$D\ M^{\circ} = 1$$

$$D\ M = \delta x_0 \circ M + D\ M^k(\delta x_k \circ M)$$

$$d\ M = \partial x_0 \circ M + D\ M^k(\partial x_k \circ M)\ .$$

PROOF. $(*)$ follows from $t \circ M = id_{\mathbb{T}}$ $\underline{.}$

8 We can view the absolute acceleration in terms of free or of applied vectors, equivalently and second order tangent space may intervene esplicitly or not .

DEFINITION.

The FREE ACCELERATION of M is the map

$$D^2M : \mathbb{T} \to \bar{\mathbb{E}}\ .$$

The LIFTED ACCELERATION of M is the map

$$\Gamma \circ d^2M = (M, DM\ ;\ 0,\ D^2M) : \mathbb{T} \to \mathbf{v}\, T^2\mathbb{E}\ .$$

The ACCELERATION of M is the map

$$\nabla\ d\ M \equiv \underline{||} \circ \Gamma \circ d^2M = (M, D^2M) : \mathbb{T} \to T\ \mathbb{E} \quad \underline{.}$$

9 PROPOSITION.

We have $$<\underline{t},\ D^2M> = 0 \qquad (*)$$

Hence, we can write

$$D^2M : \mathbf{T} \to \bar{\mathbb{S}},$$

$$\Gamma \circ d^2M : \mathbf{T} \to \nu \overset{\prime}{T}{}^2\mathbb{E}$$

and $$\nabla\, d\, M : \mathbf{T} \to \mathbb{A} \equiv \check{T}\,\mathbb{E}$$

and we get $$D^2M^\circ = 0$$

$$D^2M = (D^2M^k + (\Gamma^k_{ij} \circ M)DM^iDM^j + (\Gamma^k_{oj} \circ M)DM^j + \Gamma^k_{oo} \circ M)\ \delta x_k$$

$$\Gamma \circ d^2M = (D^2M^k + (\Gamma^k_{ij} \circ M)DM^iDM^j + (\Gamma^k_{oj} \circ M)DM^j + \Gamma^k_{oo} \circ M)\ \partial \dot{x}_k$$

$$\nabla\, d\, M = (D^2M^k + (\Gamma^k_{ij} \circ M)DM^iDM^j + (\Gamma^k_{oj} \circ M)Dm^j + \Gamma^k_{oo} \circ M)\ \partial x_k$$

Geometrical analysis.

Here we give some further element of analysis of $\mathbb{M}$, not essential from a kinematical point of view.

10 $\mathbb{M}$ has two structures: the C^∞ structure induced by $\mathbb{E}$ and the oriented euclidean affine structure induced by $\mathbb{T}$ (but, in general, $\mathbb{M}$ is not an affine subspace of $\mathbb{E}$) .

The embedding $$TM : T\,\mathbb{T} \to T\mathbb{M} \hookrightarrow T\mathbb{E}$$

is given by $$(\tau,\lambda) \mapsto (M(\tau)),\ \lambda DM(\tau))\ .$$

The embedding $$T^2M : T^2\mathbf{T} \to T^2\mathbb{M} \hookrightarrow T^2\mathbb{E}$$

is given by $$(\tau,\lambda;\mu,v) \to (M(\tau),\lambda DM(\tau);\mu DM(\tau), vDM(\tau)+\lambda\mu D^2M(\tau)).$$

Now, let us consider the two fields

$$\bar{\mathbb{M}} \equiv d\, M \circ M^{-1} : \mathbb{M} \to T\, \mathbb{M}$$

and $$\bar{\bar{\mathbb{M}}} \equiv \nabla dM \circ M^{-1} : \mathbb{M} \to \check{T}\, \mathbb{E}_{|\mathbb{M}} \, .$$

11 PROPOSITION.

$\bar{\mathbb{M}}$ results into the unitary oriented constant field, with respect to the oriented euclidean affine structure of $\mathbb{M}$ induced by $\mathbb{T}$.

Moreover, each vector field $X : \mathbb{M} \to T\, \mathbb{M}$ can be written as

$$X = X^\circ\, \bar{\mathbb{M}} \, , \qquad \text{where} \qquad X^\circ \equiv \langle \underline{t}, x \rangle .$$

12 PROPOSITION.

Let $X : \mathbb{M} \to T\, \mathbb{M}$ and $y : \mathbb{M} \to T\, \mathbb{M}$ be two C^∞ fields.
Then the covariant derivative

$$\nabla_X Y \equiv \underline{\coprod} \circ \Gamma \circ T\, Y \circ X : \mathbb{M} \to T\, \mathbb{E}_{|\mathbb{M}}$$

is given by $$\nabla_X Y = p''_{\bar{M}} \circ \nabla_X Y + p^{\perp}_{M} \circ \nabla_X Y \, ,$$

where $$p''_{\bar{M}} \circ \nabla_X Y = X^\circ\, D\, Y^\circ\, \bar{\mathbb{M}}$$

results into the covariant derivative with respect to the affine structure of $\mathbb{M}$ and

$$p^{\perp}_{M} \circ \nabla_X Y = X^\circ\, Y^\circ\, \bar{\bar{\mathbb{M}}}$$

shows that the tensor

$$\bar{\bar{\mathbb{M}}} \otimes \underline{t} \otimes \underline{t} : \mathbb{M} \to T_{(1,2)}\mathbb{E}_{|\mathbb{M}}$$

can be considered as the second fundamental form of $\mathbb{M}$.

Physical description.

The world-line $\mathbb{M}$ of a particle represents the set of all the events "touched" by the particle and the motion M is the map that associates with each time the relative event. Of course the events being absolute, i.e. independent of any frame of reference, the same occurs for the world line and the motion. The affine structure of $\mathbb{E}$ allows a privileged type of motions, namely the inertial ones.

As we have the absolute motion M, we have the absolute velocity DM and acceleration D^2M. These contain all the information necessary to derive the velocity and acceleration observed by any frame of reference, when it is chosen. The fact that DM is a unitary vector and D^2M is a spatial vector will put in evidence how the observed velocity changes and that the observed acceleration does not change from an inertial frame of reference to an other.

We can describe the previous facts by pictures.