I CHAPTER
ABSOLUTE KINEMATICS

In this paper we study the general event framework constituted by the event space, its partition into the symultaneity spaces, which generate the time and the spetial metric.

We analyse some remarkable spaces and maps connected with the previous ones. Finally we study the one-body absolute motion, velocity and acceleration. All these elements are considered regardless of any frame of reference.

1 - THE EVENT SPACE

First we introduce the general framework for classical mechanics.
Event space, simultaneity, spatial metric, future orientation, time.

1 - Basic assumptions on primitive elements of our theory are given by the following definition, which constitutes the framework of classical mechanics.

DEFINITION.

The CLASSICAL EVENT FRAMEWORK is a 4-plet

$$\mathcal{E} \equiv \{E, \bar{\mathbb{S}}, \check{\mathbb{G}}, 0\}$$

where

$E \equiv \{\mathbb{E}, \bar{\mathbb{E}}, \sigma\}$ is an affine space, with dimension 4;

$\bar{\mathbb{S}} \hookrightarrow \bar{\mathbb{E}}$ is a subspace of $\bar{\mathbb{E}}$, with dimension 3;

$\check{\mathbb{G}}$ is a conformal euclidean metric on $\bar{\mathbb{S}}$;

0 is an orientation on the quotient space $\mathbb{E}/\bar{\mathbb{S}}$.

$\mathbb{E}$ is the EVENT SPACE; $\bar{\mathbb{E}}$ is the EVENT INTERVAL SPACE;
$\bar{\mathbb{S}}$ is the SIMULTANEOUS INTERVAL SPACE or the SPATIAL INTERVAL SPACE;
$\check{\mathbb{G}}$ is the SPATIAL CONFORMAL METRIC;
0 is the FUTURE ORIENTATION,
-0 is the PAST ORIENTATION.

Henceforth we assume a classical event framework $\mathcal{E}$ to be given.

2 - The previous definition contains implicitly the notion of absolute time, which we are now giving explicitly.

DEFINITION.

The TIME SPACE is the quotient space

$$T \equiv \mathbb{E} / \bar{\mathbb{S}} .$$

The TIME VECTOR SPACE is the quotient space

$$\bar{\mathbf{T}} \equiv \bar{\mathbf{E}} / \bar{\mathbf{S}} .$$

The TIME PROJECTION is the quotient map

$$t : \mathbf{E} \to \mathbf{T} .$$

The SPACE AT THE TIME $\tau \in \mathbf{T}$ is the subspace

$$\mathbf{S}_\tau \equiv t^{-1}(\tau) \hookrightarrow \mathbf{E}.$$

The TIME BUNDLE is the 3-plet

$$\eta \equiv (\mathbf{E}, t, \mathbf{T}) .$$

Hence, each equivalence class is of the type

$$\mathbf{T} \ni \tau \equiv [e] \equiv e + \bar{\mathbf{S}} \equiv \mathbf{S}_\tau \to \mathbf{E}$$

having $$t(e) \equiv \tau .$$

Thus τ and $\mathbf{S}_\tau$ coincide, but τ is viewed as a point of $\mathbf{T}$ and $\mathbf{S}_\tau$ as a subset of $\mathbf{E}$.

Moreover we will denote by j the injective map

$$j \equiv (t, id_{\mathbf{E}}) : \mathbf{E} \to \mathbf{T} \times \mathbf{E}.$$

3 - We get immediate properties for the previous spaces.

PROPOSITION.

a) $(\mathbf{T}, \bar{\mathbf{T}})$ results naturally into an affine 1-dimensional oriented space.

b) t is an affine surjective map. We get

$$\bar{\mathbf{S}} = (D\, t)^{-1}(0).$$

c) For each $\tau \in \mathbf{T}$, $(\mathbf{S}_\tau, \bar{\mathbf{S}}, \sigma)$ is an affine 3-dimensional subspace of $\mathbf{E}$;

hence $\{\mathbf{S}_\tau\}_{\tau \in \mathbf{T}}$ is a family of parallel,(not canonically) isomorphic affine subspace of $\mathbf{E}$ and we have

$$\mathbf{E} = \bigsqcup_{\tau \in \mathbf{T}} \mathbf{S}_\tau .$$

d) η is an affine, (not canonically) trivial bundle $\underline{\cdot}$

4 - We have the absolute time component of an event interval.

DEFINITION.

The TIME COMPONENT of the vector $u \in \bar{\mathbf{E}}$ is

$$u^\circ \equiv \langle Dt, u \rangle \in \mathbf{T}.$$

u is FUTURE ORIENTED or PAST ORIENTED, according as

$$u^\circ \in \bar{\mathbf{T}}^+ \qquad \text{or} \qquad u \in \bar{\mathbf{T}}^- \underline{\cdot}$$

Moreover u is spatial if and only if $u^\circ = 0$.

5 - Thus, the equence

$$0 \rightarrow \bar{\mathbf{S}} \longrightarrow \bar{\mathbf{E}} \twoheadrightarrow \bar{\mathbf{T}} \rightarrow 0$$

is exact, but we have not a canonical splitting of $\bar{\mathbf{E}}$, as we have not a canonical projection $\bar{\mathbf{E}} \twoheadrightarrow \bar{\mathbf{S}}$, or a canonical inclusion $\mathbf{T} \hookrightarrow \bar{\mathbf{E}}$. However, each vector $v \in \mathbf{E}$, such that $\langle Dt, v \rangle \neq 0$, determines a splitting of $\bar{\mathbf{E}}$.

Namely we get the inclusion

$$\bar{\mathbf{T}} \hookrightarrow \bar{\mathbf{E}},$$

given by

$$\lambda \mapsto \frac{\lambda}{v^\circ} v,$$

and the projection

$$\overset{\perp}{p}_v : \bar{\mathbf{E}} \rightarrow \bar{\mathbf{S}} ,$$

given by

$$u \mapsto u - \frac{u^\circ}{v^\circ} v,$$

which determine the decomposition in the direct sum

$$\bar{\mathbb{E}} = \bar{\mathbb{T}} \oplus \bar{\mathbb{S}} \ ,$$

given by $$u = \frac{u^\circ}{v^\circ} v + \left(u - \frac{u^\circ}{v^\circ} v\right) \equiv p_v^{\shortparallel}(u) + p_v^{\perp}(u) \ .$$

6 - According to the bundle structure of $\mathbb{E}$ on $\mathbb{T}$, we can define the vertical derivative of maps, i.e. the derivative along the fibers. Generally we will denote by "v" the quantities connected with η .

DEFINITION.

Let $\mathbb{F}$ be an affine space and let $f : \mathbb{E} \to \mathbb{F}$ be a C^∞ map.

The VERTICAL DERIVATIVE of f is the map

$$\check{D}f \equiv Df_{|\bar{\mathbb{S}}} : \mathbb{E} \to \bar{\mathbb{S}}^* \otimes \bar{\mathbb{F}} \ .$$

Poincaré's and Galilei's maps.

7 - A Poincaré's map is a map $\mathbb{E} \to \mathbb{E}$ which preserves the structure of $\mathcal{E}$ and the associated Galilei's map is its derivative.

DEFINITION.

A POINCARÉ'S MAP is an affine map

$$G : \mathbb{E} \to \mathbb{E},$$

such that
a) $DG(\bar{\mathbb{S}}) = \bar{\mathbb{S}}$
b) $DG \in U(\bar{\mathbb{S}})$,

c) if $G^\circ : \mathbb{T} \to \mathbb{T}$ is the induced map on the quotient space $\mathbb{T} \equiv \mathbb{E}/\bar{\mathbb{S}}$,

then $$DG^\circ = id_{\bar{\mathbb{T}}} \ .$$

$DG : \bar{\mathbb{E}} \to \bar{\mathbb{E}}$ is the GALILEI'S MAP associated with G.

G is SPECIAL if it preserves the orientations of $\bar{\mathbb{E}}$ and $\bar{\mathbb{S}}$ (hence of $\bar{\mathbb{T}}$).

8 - PROPOSITION.

Each Poincaré's map G in bijective .

PROOF.

It follows from $DG \in U(\bar{\mathbb{S}})$, $DG^\circ = id_{\bar{\mathbb{T}}}$.

<u>Space and time measure unity</u>.

9 - We have assumed a 1-parameter family $\check{\mathbb{G}}$ of euclidean metrics on $\bar{\mathbb{S}}$. A 1-parameter family $\mathbb{G}^\circ$ of euclidean metrics on $\bar{\mathbb{T}}$ is given a priori, for dim $\bar{\mathbb{T}} = 1$.

An arbitrary choice of one among these makes important simplications in the following.

DEFINITION.

A SPATIAL MEASURE UNITY is a metric $\check{g} \in \check{\mathbb{G}}$.

A TIME MEASURE UNITY is a metric $g^\circ \in \mathbb{G}^\circ$.

The choice of a spatial measure unity $\check{g}$ is equivalent to the choice of the sphere (in the family determined by $\check{\mathbb{G}}$) of $\bar{\mathbb{S}}$, with radius 1 as measured by $\check{g}$.

The choice of a time measure unity g° is equivalent to the choice of the vector

$$\lambda^\circ \in \mathbb{T}^+$$

such that

$$g^\circ(\lambda^\circ,\lambda^\circ) = 1 .$$

Then λ° determines the isomorphism

$$\bar{\mathbb{T}} \to \mathbb{R}$$

given by

$$\lambda \mapsto \frac{\lambda}{\lambda^\circ} .$$

Henceforth we assume a spatial and a time measure unity to be given. Hence we get the identification

$$\bar{\mathbb{T}} \simeq \mathbb{R}$$

and the consequent identifications

$$L(\bar{\mathbb{T}}, \bar{\mathbb{E}}) \cong \bar{E}, \; L(\mathbb{T}, \bar{\mathbb{S}}) \cong \bar{\mathbb{S}}, \; L(\bar{\mathbb{E}},\bar{\mathbb{T}}) \cong \bar{\mathbb{E}}^*, \; L(\bar{\mathbb{S}},\bar{\mathbb{T}}) \cong \mathbb{S}^*, \; \ldots$$

In this way, the map $Dt \in L(\bar{\mathbb{E}},\bar{\mathbb{T}})$ is identified with the form

$$\underline{t} \cong Dt \in \bar{\mathbb{E}}^* .$$

10 - Besides the subspace $\bar{\mathbb{S}} \hookrightarrow \bar{\mathbb{E}}$, which results into $\bar{\mathbb{S}} = \underline{t}^{-1}(0)$,an interesting will be played by the subspace of normalized vectors $\underline{t}^{-1}(1)$.

DEFINITION.

The FREE VELOCITY SPACE is

$$\mathbb{U} \equiv \underline{t}^{-1}(1) \hookrightarrow \mathbb{E}.$$

11 - PROPOSITION.

$(\mathbb{U}, \bar{\mathbb{S}})$ results naturally into an affine (not vector) 3-dimensional subspace of $\bar{\mathbb{E}}$.

Of course $\mathbb{U}$ and $\bar{\mathbb{S}}$ are isomorphic as affine spaces, but we have not a canonical affine isomorphism between $\mathbb{U}$ and $\bar{\mathbb{S}}$.

Special charts.

12 - In calculations can be usefull a numerical representation of $\mathbb{E}$, which takes into account its time structure. For simplicity of notations, we consider only diffeomorphisms $\mathbb{E} \to \mathbb{R}^4$, leaving to the reader the obvious generalization to local charts, our considerations being essentially local.

DEFINITION.

A SPECIAL CHART is a C^∞ chart

$$x \equiv \{x^\circ, x^i\} : \mathbb{E} \to \mathbb{R} \times \mathbb{R}^3 ,$$

such that x° factorizes as follows

$$\begin{tikzcd} \mathbb{T} \arrow[rr, "\hat{x}^\circ"] & & \mathbb{R} \\ & \mathbb{E} \arrow[ul, "t"] \arrow[ur, "x^\circ"'] & \end{tikzcd}$$

where $\hat{x}^\circ : \mathbf{T} \to \mathbb{R}$ is a normal oriented cartesian map $\underline{\cdot}$

Naturally $\hat{x}^\circ$(hence x°) is determined up an initial time.

We make the usual convention

$$\alpha,\beta,\lambda,\mu,\ldots = 0,1,2,3 \qquad \text{and} \qquad i,j,h,k,\ \ldots = 1,2,3 \ .$$

We assume in the following a special chart x to be given.

13 - Let us give the coordinate expression of some important quantities.

PROPOSITION.

We have

$$Dx^\circ = \underline{t} \ ;$$

$$\delta x_i : \mathbf{E} \to \bar{\mathbf{S}} \ ;$$

if $u \in \bar{\mathbf{E}}$, then $u = u^\circ\, \delta x_o + u^i\, \delta x_i$, where $u^\circ \equiv \langle \underline{t}, u \rangle$;

$$\check{g} = g_{ij}\, \check{D}x^i \otimes \check{D}x^j \ ;$$

$$\Gamma^\circ_{\alpha\beta} \equiv D\delta x_\alpha(\delta x_\beta, Dx^\circ) = - D^2x^\circ(\delta x_\alpha, \delta x_\beta) = 0,$$

$$\Gamma^k_{ij} \equiv D\delta x_i(\delta x_j, Dx^k) = - D^2x^k(\delta x_i, \delta x_j) =$$

$$= \frac{1}{2}\, g^{kh}(\partial_i g_{hj} + \partial_j g_{hi} - \partial_h g_{ij}),$$

$$\Gamma_{i,oj} + \Gamma_{j,oi} = \partial_o g_{ij} \ , \qquad \text{where} \quad \Gamma_{h,\alpha\beta} \equiv g_{hi}\Gamma^i{}_{\alpha\beta} \quad \underline{\cdot}$$

Moreover $\Gamma^k_{oj} \equiv D\delta x_o(\delta x_j, Dx^k) = - D^2x^k(\delta x_o, \delta x_j)$

and $\Gamma^k_{oo} \equiv D\delta x_o(\delta x_o, Dx^k) = - D^2x^k(\delta x_o, \delta x_o)$

can be different from zero, if δx_o is not constant.

Notice that $Dx^{\circ} = \underline{t}$ is fixed a priori and that the unique conditions imposed a priori on δx_{α} are

$$<\underline{t},\delta x_{0}> = 1 \qquad\qquad <\underline{t},\delta x_{i}> = 0 .$$

Physical description.

The event space $\mathbb{E}$ represents the set of all the possible events considered from the point of view of their mutual space-time collocation and without reference to any particular frame of reference. This space $\mathbb{E}$ must be viewed exactl in the same sence as the event space of Special and General Theory of Relativity.

The event space $\mathbb{E}$ is the disjoint union of a family $\{\mathbb{S}_{\tau}\}_{\tau\in\mathbb{T}}$ of three dimensional affine euclidean, mutually diffeomorphic, spaces. This partition represents the equivalence relation of absolute simultaneity among events. The structure of each space $\mathbb{S}_{\tau}$ permits all the physical operations considered in the classical time-independent Euclidean Geometry, as stright lines, parallelism, intervals,sum of intervals, by the parallelogram rule, circles, etc. We have not selected a priory a spatial measure unity, for it is not physically significant: by means of rigid rods we can only find ratios between lenghts is all directions and the choice of a particular interval of a rigid rod is a useful but not necessary convenction

The symultaneity spaces $\mathbb{S}_{\tau}$ are mutually ,but not canonically,isomorphic, for a particular family of bijections among these leads to a determination of positions, i.e. to a frame of reference, which we have excluded in the general context. Notice that in $\mathbb{S}_{\tau}$ we have not privileged points or axes.

The required four dimensional affine structure of $\mathbb{E}$ leads to the affine structures of the subspaces $\mathbb{S}_{\tau}$ and to the one dimensional affine structure of the set $\mathbb{T}$, whose points are the equivalence classes $\mathbb{S}_{\tau}$.

This space represents the classical absolute time. Its affine structure admits the time intervals, independent of an initial time, and their sum. The one dimensional affine structure of $\mathbb{T}$ leads also to the measure of time intervals with respect to an arbitrary chosen unity. Hence the affine structure of $\mathbb{E}$ contains implicitly the idea of "good clocks". The dimension one describes also the total ordinability of times and the assumed orientation o describes the future orientation. Notice that in $\mathbb{T}$ we have not a privileged initial time.

To make more evident the described properties of event framework, we can make some pictures using the affine euclidean structure of the paper. We must take care essentially in two things: we must neglect two (or one) dimension of $\mathbb{E}$ and we must partially neglect the euclidean structure of the paper, for we have not a measure of angles between spatial and time vectors. So a time vector orthogonal to a spatial vector is nonsense.