Corollary 1 - Let β be an ultrafilter mass and λ a continuous mass on $\mathcal{A} \subseteq \mathcal{P}(\Omega)$, with $\beta \ll \lambda$. Then $\mu = \beta + \lambda$ is non-atomic and non-continuous.

Corollary 2 - Let β be an ultrafilter mass on $\mathcal{A} \subseteq \mathcal{P}(\Omega)$ such that $\beta \ll \lambda$, where λ is a continuous measure on $\mathcal{A}$. Then β cannot be a measure on $\mathcal{A}$.

Remark - It is interesting also to look at Theorem 5 as another counterexample to known results for measures: in [7] it is shown that, given two measures λ and ν, with $\nu \ll \lambda$ and λ non atomic (i.e. continuous), then ν also is non-atomic. Actually, this need not be true if ν is only a mass (and not a measure), for example if it is an ultrafilter mass β, as that of Corollary 2. The existence of such a mass (given λ) can be proved (cfr. [1]) taking an $\mathcal{A}$-ultrafilter containing the filter

$$\mathcal{F} = \{E \in \mathcal{A} : \lambda(E) = \lambda(\Omega)\} .$$

4. Atomic masses and measurable cardinals.

Since the mass μ occurring in Theorem 5 is non-atomic (and non-continuous), Theorem 4 can be suitably applied to it, giving easily a countably additive sequence of sets also for the atomic mass ν.

Theorem 6 - Let ν be an atomic mass on a σ-algebra $\mathcal{A} \subseteq \mathcal{P}(\Omega)$, such that $\nu \ll \lambda$, where λ is a continuous measure on $\mathcal{A}$. Then there exists a sequence (A_n) of mutually disjoint measurable sets, such that

$$\nu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \nu(A_n) .$$

Proof - Use Theorem 4 for $\mu = \nu + \lambda$, taking into account the countable additivity of λ. ∎

Now, in order to deal with the so-called "Ulam's measure problem", we recall some known facts about ultrafilter over a set Ω; we limit owrselves to free ultrafilters(cfr. the remark following Proposition 3).

Definition 6 - An ultrafilter $\mathcal{U}$ over Ω is δ-complete if, given any sequence of sets $A_n \in \mathcal{U}$, one has $\bigcap_{n=1}^{\infty} A_n \in \mathcal{U}$.

Proposition 4 - Let β be an ultrafilter mass on $\mathcal{A} = \mathcal{P}(\Omega)$, and let $\mathcal{U}$ be the corresponding (free) ultrafilter. Then β is a measure if and only if $\mathcal{U}$ is δ-complete.

Proof - Countable additivity of β implies that, given any sequence of sets $A_n \in \mathcal{U}$, for $A'_n = \Omega - A_n \notin \mathcal{U}$ we must have $\bigcup_{n=1}^{\infty} A'_n \notin \mathcal{U}$.

Therefore $\Omega - \bigcup_{n=1}^{\infty} A'_n = \bigcap_{n=1}^{\infty} A_n \in \mathcal{U}$, i.e. $\mathcal{U}$ is δ-complete. The converse is also easily seen, since β is two-valued.

Definition 7 - Let Ω be a set: card Ω is said measurable when there exists a δ-complete free ultrafilter over Ω.

Corollary 3 - An ultrafilter measure exists on $\mathcal{A} = \mathcal{P}(\Omega)$ if and only if card Ω is measurable.

(Notice that the latter measure is finite,defined for all subsets of Ω, and zero on singletons).

The question concerning the existence of measurable cardinals (known also under the name of Ulam's measure problem) cannot be settled in ZFC (Zermelo-Fraenkel set theory with the Axiom of Choice).

It was shown that a measurable cardinal (assuming its existence) must be very large and, in fact, must be an inaccessible cardinal: really, if k is a measurable cardinal, then there are k inaccessible cardinals preceding it (cfr., e.g., [11], p. 26 and [14], p. 26).

Moreover, the existence of a measurable cardinal settles many mathematical problems: see [8].

On the other hand, if we assume that "all" sets are constructible (the so-called "axiom of constructibility" V = L), no measurable cardinal exists: in fact, if there is a measurable cardinal, then V = L " is as false as it possibly can be" (cfr. [14], p. 31).

Notice that, by Corollary 3, the existence of an ultrafilter measure on $\mathcal{P}(\Omega)$ is equivalent to the statement that card Ω is measurable, while an ultrafilter mass always exists, by a classical result due to Tarski [15].

Proposition 5 - Let card $\Omega = \mathfrak{c}$ and assume the continuum hypothesis (CH). Then no ultrafilter measure exists on $\mathcal{P}(\Omega)$ (i.e., under CH, $\mathfrak{c}$ is not a measurable cardinal).

Proof - See [17] or [11]. ∎

We point out that Corollary 2 (cfr. Section 3) gives non-existence of a particular class of ultrafilter measures, without any assumption on the cardinality of Ω.

We end this Section with a necessary condition for a cardinal to be measurable, which gives an interesting remark to Ulam's measure problem; we state first the following obvious

Lemma - Let Ω be a set such that card Ω is measurable, and let β be the corresponding ultrafilter measure. Then $\beta(E)>0$ implies card $E > \aleph_0$.

Theorem 7 - Let β be an ultrafilter measure on $\mathcal{P}(\Omega)$. Then, given any continuous measure λ on $\mathcal{A} \subseteq \mathcal{P}(\Omega)$, necessarily $\beta \perp \lambda$ (i.e., β is singular with respect to λ) and there are sets $E \subset \Omega$, with card $E>\aleph_0$, such that $\lambda(E) = 0$.

Proof - It is essentially a reformulation of Corollary 2, taking into account the preceding Lemma. ∎

Remark - Theorem 7 can be looked at to give some grounds for the acceptance or not of the axiom concerning the existence of measurable cardinals: for example, if we assume that, given a set Ω, there exists at least a continuous measure on a σ-algebra $\mathcal{A} \subseteq \mathcal{P}(\Omega)$, vanishing only on countable (*) sets, then card Ω is not measurable.

This result is also a partial converse to a theorem given by Ulam (cfr. Satz 2, p. 147) in [17]: he proved that, if card Ω is not measurable and there exists a measure on Ω, then this measure is necessarily continuous.

(*) Here it would be possible to replace "countable sets" by "sets of cardinality less than card Ω", just using a suitable definition of measure, in which countable additivity is replaced by the "natural" stronger requirement (cfr. [14], p. 20).