difficult to see that such subset must necessarily be \( A \cap B \cap E_0 \). So the mass

\[
\beta_1(E) = \begin{cases} 
0 & \text{if } E \notin U_1 \\
\alpha_1 & \text{if } E \in U_1
\end{cases}
\]

is atomic. Put \( \nu_1 = \mu - \beta_1 \); if the mass \( \nu_1 \) is non-continuous, then

\[
\alpha_2 = \inf_{p \in \mathcal{P}} \nu_1(E(p)) > 0,
\]

and so it is possible to go on in the same fashion.

After \( n \) steps, we get

\[
\nu_n = \mu - \sum_{k=1}^{n} \beta_k
\]

and, if \( \nu_n \) is continuous, eq.(4) holds with \( \nu_0 = \nu_n \) and with each \( \beta_k \) null for \( k > n \). If \( \nu_n \) is non-continuous for any \( n \), we get a sequence \( (\beta_n) \) such that the corresponding series \( \sum_{n=1}^{\infty} \beta_n(E) \) converges for every \( E \in \mathcal{A} \) (since \( \nu(E) < +\infty \)). Then \( \lim_{k \to \infty} \alpha_k = 0 \), and it follows that

\[
\nu_0 = \lim_{n \to \infty} \nu_n \text{ is continuous.} \]

3. Non atomic masses.

In the classical case of a measure, non-atomicity is equivalent to
continuity. This can be seen, for example, as an easy consequence of the following

**Theorem 2 (Saks)** - Let \( \mu \) be a measure on a \( \sigma \)-algebra \( \mathcal{A} \subseteq \mathcal{P}(\Omega) \). Given any \( \varepsilon > 0 \), there exists \( p \in \mathcal{B} \) such that each \( E_k \in p \) is either an atom or \( \mu(E_k) < \varepsilon \).

*Proof*: see [2], p. 308.

This equivalence cannot be carried over to the general case of a mass: in [1] it is shown (cfr. also the following Theorem 5) that, if \( \mu = \nu + \lambda \), where \( \nu \) is atomic and \( \lambda \) is continuous, with \( \nu < \lambda \) (i.e. \( \nu \) is absolutely continuous with respect to \( \lambda \)), then \( \mu \) is both non-atomic and non-continuous.

In other words, while atomic masses are necessarily non-continuous (see Proposition 2), non-atomic ones can be either continuous or not.

For continuous masses, the following result has been established in [13]:

**Theorem 3** - Let \( \mu \) be a continuous mass on a \( \sigma \)-algebra \( \mathcal{A} \subseteq \mathcal{P}(\Omega) \). Then there exists a sequence \( (F_n) \) of mutually disjoint measurable sets, with \( \mu(F_n) > 0 \), such that \( \sum_{n=1}^{\infty} \mu(F_n) \) equals any preassigned \( \alpha \), with \( 0 < \alpha < \mu(\Omega) \), and

\[
(5) \quad \alpha = \sum_{n=1}^{\infty} \mu(F_n) = \mu(\bigcup_{n=1}^{\infty} F_n).
\]

So to say, the mass \( \mu \) "behaves" like a measure on each collection \( (F_n) \): then it would seem interesting a deeper investigation of the family (or of some suitable subfamily) of all such sets obtained when \( \alpha \) ranges...
in the open interval from 0 to $\mu(\Omega)$.

A simple corollary of Theorem 3 is the following: the range of a continuous $\mu$ is the whole interval $[0, \mu(\Omega)]$. The latter statement is no longer true if $\mu$ is non-continuous: a counterexample is given in [1]; moreover, there it is shown that the range of $\mu$ need not even be a closed subset of $\mathbb{R}$, contrary to the classical case of a measure.

As far as continuous masses are concerned, let us quote also a recent result obtained through non-standard methods: a necessary and sufficient condition for the existence of a continuous mass, which is invariant for a transformation of $\Omega$ into itself, is given in [16].

We want now to extend Theorem 3 to the more general case of a non-atomic $\mu$: the previous remarks show that we can hope, at most, in countable additivity on a suitable sequence of sets (and not also, as in eq. (5), in a beforehand given value of $\alpha$).

**Theorem 4** - Let $\mu$ be a non-atomic mass on a $\sigma$-algebra $A \subseteq P(\Omega)$. Then there exists a sequence $(A_n)$ of mutually disjoint measurable sets, with $\mu(A_n) > 0$, such that

$$
\mu(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n).
$$

**Proof** - Let $B \in A$, with $0 < \mu(B) < \mu(\Omega)$: then also $B' = \Omega - B$ satisfies $0 < \mu(B') < \mu(\Omega)$. At least one of them (call it $B_1$) is such that $\mu(B_1) \leq \frac{1}{2} \mu(\Omega)$. Now, let $B_2 \in A$, $B_2 \subset B_1$, be such that $0 < \mu(B_2) < \mu(B_1)$ and $\mu(B_2) \leq \frac{1}{2} \mu(B_1)$. In general, we define $B_n \subset B_{n-1}$, with $0 < \mu(B_n) < \mu(B_{n-1})$ and $\mu(B_n) \leq \frac{1}{2} \mu(B_{n-1}) \leq \frac{1}{2^n} \mu(\Omega)$.
Put $A_n = B_n - B_{n+1}$; we have $\mu(A_n) = \mu(B_n) - \mu(B_{n+1}) > 0$ and $A_i \cap A_j = \emptyset$ for $i \neq j$. Moreover

$$
\mu\left( U_{n=1}^{\infty} A_n \right) = \sum_{n=1}^{\infty} \mu(A_n) + \mu\left( U_{n=k+1}^{\infty} A_n \right) =
$$

$$
= \sum_{n=1}^{k} \mu(A_n) + \mu(B_{k+1}) \leq \sum_{n=1}^{k} \mu(A_n) + \frac{1}{2^{k+1}} \mu(\Omega).
$$

As $k \to \infty$, we get

$$
\mu\left( U_{n=1}^{\infty} A_n \right) \leq \sum_{n=1}^{\infty} \mu(A_n),
$$

which, taking into account Proposition 1, gives (6). 

The next theorem will enable us to extend (in Section 4) the previous result also to a particular class of atomic masses.

**Theorem 5** - Let $\mathcal{A} \subseteq P(\Omega)$ be a $\sigma$-algebra, $\nu$ an atomic mass on $\mathcal{A}$, and $\nu$ a continuous mass on $\mathcal{A}$ such that $\lambda(A) > 0$ for any atom $A$ of $\nu$ (e.g., such that $\nu \ll \lambda$). Then $\mu = \nu + \lambda$ is non atomic and non-continuous.

**Proof**: see [1].

**Remark**: Put $\nu = \sum \beta_n$ in Theorem 1, eq.(4): $\nu$ need not be atomic (an example is given in [10], p. 47), and so we may have masses which are at the same time non-atomic and non-continuous, but not of the form given by Theorem 5.
Corollary 1 - Let $\mathcal{B}$ be an ultrafilter mass and $\lambda$ a continuous mass on $\mathcal{C}_c(\Omega)$, with $\beta \ll \lambda$. Then $\mu = \beta + \lambda$ is non-atomic and non-continuous.

Corollary 2 - Let $\mathcal{B}$ be an ultrafilter mass on $\mathcal{C}_c(\Omega)$ such that $\beta \ll \lambda$, where $\lambda$ is a continuous measure on $\mathcal{A}$. Then $\beta$ cannot be a measure on $\mathcal{A}$.

Remark - It is interesting also to look at Theorem 5 as another counterexample to known results for measures: in [7] it is shown that, given two measures $\lambda$ and $\nu$, with $\nu \ll \lambda$ and $\lambda$ non-atomic (i.e. continuous), then $\nu$ also is non-atomic. Actually, this need not be true if $\nu$ is only a mass (and not a measure), for example if it is an ultrafilter mass $\beta$, as that of Corollary 2. The existence of such a mass (given $\lambda$) can be proved (cfr. [1]) taking an $\mathcal{A}$-ultrafilter containing the filter

\[ \mathcal{F} = \{ E \in \mathcal{A} : \lambda(E) = \lambda(\Omega) \} . \]

4. Atomic masses and measurable cardinals.

Since the mass $\mu$ occurring in Theorem 5 is non-atomic (and non-continuous), Theorem 4 can be suitably applied to it, giving easily a countably additive sequence of sets also for the atomic mass $\nu$.

Theorem 6 - Let $\nu$ be an atomic mass on a $\sigma$-algebra $\mathcal{C}_c(\Omega)$, such that $\nu \ll \lambda$, where $\lambda$ is a continuous measure on $\mathcal{A}$. Then there exists a sequence $(A_n)$ of mutually disjoint measurable sets, such that

\[ \nu( \bigcup_{n=1}^{\infty} A_n ) = \sum_{n=1}^{\infty} \nu(A_n) . \]