linear. Let

$$f(s) = As + B$$

$$K_k(\underline{x}_k) = \sum_{i=1}^{i_k} a_{ki} x_{ki} + b_k ,$$

then the solution of the optimization problem (27) is a piece-wise linear function φ_k. In this case the reduced game can be solved easily as it is shown in [10], pp. 43-44.

5. Multiproduct oligopoly game

In this paragraph we will consider the game having the sets of strategies

$$X_k = \left[0, L_k^{(1)}\right] x \dots x \left[0, L_k^{(M)}\right] \qquad (30)$$

and pay-off functions

$$\varphi_k(\underline{x}_1, \dots, \underline{x}_n) = \sum_{m=1}^{M} x_k^{(m)} f_m\left(\sum_{\ell=1}^{n} x_\ell^{(1)}, \dots, \sum_{\ell=1}^{n} x_\ell^{(M)}\right) - K_k(\underline{x}_k), \qquad (31)$$

where $\underline{x}_k = \left(x_k^{(1)}, \dots, x_k^{(M)}\right)$, $\mathcal{D}(K_k) = X_k$, $\mathcal{R}(K_k) \subset R^1$,

$\mathcal{D}(f_m) = \left[0, \sum_{\ell=1}^{n} L_\ell^{(1)}\right] x \dots x \left[0, \sum_{\ell=1}^{n} L_\ell^{(M)}\right]$, $\mathcal{R}(f_m) \subset R^1$ for

k=1,2,...,n and m=1,2,..,M. This game can come up if the factories manufacture different products and sell them on the same market. Let M be the number of products, and let $x_k^{(m)}$, $L_k^{(m)}$ be the production level and capacity limit of factory k from product m. If f_m donotes the unit price of product m, than it is assumed that f_m is a function of the total production levels of the different products. The function K_k is the

production cost, and using the above terminology the income of factory k is given by function (31).
Similar interpretation can be given to the other applications shown in the section dealing with the classical oligopoly game but different qualities of water and waste-water have to be introduced.

The following result is basic in the theory of multiproduct economies, and it is a generalization of part c/ in the proof of Theorem 4.

Lemma 11. Let $\underline{g}$ be a vector-vector function such that $\mathcal{D}(\underline{g})$ is a convex set in the nonnegative orthant of R^M, $\mathcal{R}(\underline{g}) \subset R^M$. Assume that the components of $\underline{g}$ are concave and continuously differentiable. Let $\underline{\underline{J}}$ be the Jacobian matrix of $\underline{g}$. If $\underline{\underline{J}}(\underline{x}) + \underline{\underline{J}}(\underline{x})^T$ is nonnegative semidefinite for arbitrary $\underline{x} \in \mathcal{D}(\underline{g})$, then the function

$$h(\underline{x}) = \underline{x}^T \underline{g}(\underline{x})$$

is concave.

Proof. Let ∇ denote the gradient operation. Then simple calculations show that

$$\nabla h(\underline{x}) = \underline{g}(\underline{x})^T + \underline{x}^T \underline{\underline{J}}(\underline{x}). \qquad (32)$$

Since the components of $\underline{g}$ are concave, we have

$$\underline{g}(\underline{y}) - \underline{g}(\underline{x}) \leqq \underline{\underline{J}}(\underline{x})(\underline{y} - \underline{x}) \qquad (\underline{x}, \underline{y} \in \mathcal{D}(\underline{g})), \qquad (33)$$

and the condition given for the Jacobian $\underline{\underline{J}}$ implies

$$0 \geq \frac{1}{2} (\underline{y} - \underline{x})^T \left\{ \underline{\underline{J}}(\underline{x}) + \underline{\underline{J}}(\underline{x})^T \right\} (\underline{y} - \underline{x}) =$$

$$= (\underline{y} - \underline{x})^T \underline{\underline{J}}(\underline{x})(\underline{y} - \underline{x}). \tag{34}$$

The inequalities (33), (34) and $\underline{y} \geq \underline{0}$ imply

$$\underline{y}^T \left\{ \underline{g}(\underline{y}) - \underline{g}(\underline{x}) \right\} \leq \underline{y}^T \underline{\underline{J}}(\underline{x})(\underline{y} - \underline{x}) \leq \underline{x}^T \underline{\underline{J}}(\underline{x})(\underline{y} - \underline{x}),$$

consequently

$$\underline{y}^T \underline{g}(\underline{y}) - \underline{x}^T \underline{g}(\underline{x}) \leq \left[\underline{g}(\underline{x})^T + \underline{x}^T \underline{\underline{J}}(\underline{x}) \right] (\underline{y} - \underline{x}),$$

which and equation (32) give the inequality

$$h(\underline{y}) - h(\underline{x}) \leq \nabla h(\underline{x})(\underline{y} - \underline{x}),$$

Thus function h is concave. ■

As a corollary to this general result we can prove the main result of this section.

<u>Theorem 7.</u> Let $\underline{f} = (f_1, \ldots, f_M)$, and let $\underline{\underline{J}}$ be the Jacobian of $\underline{f}$. Assume that functions $\underline{f}$ and K_k $(1 \leq k \leq n)$ are continuous, the components of $\underline{f}$ are continuously differentiable and concave, K_k is convex and for arbitrary $\underline{s} \in \mathcal{S}(\underline{f})$ the matrix $\underline{\underline{J}}(\underline{s}) + \underline{\underline{J}}(\underline{s})^T$ is nonnegative semidefinite. Then the game has at least one equilibrium point.

<u>Proof.</u> Since X_k is a closed, convex, bounded subset of R^M, φ_k is continuous and Lemma 11. implies that φ_k is concave in $\underline{x}_k$, the game satisfies all conditions of the Nikaido-Isoda theorem. Thus the game has at least one equilibrium point. ■

Remark. The theorem does not give numerical methods for the determination of the equilibrium point. But in the linear case a very efficient algorithm can be constructed which is a generalization of the method of M. Maňas given for the one--product case.

Let us assume that

$$K_k\left(x_k^{(1)},\dots,x_k^{(M)}\right)=\sum_{m=1}^{M} A_k^{(m)}\ x_k^{(m)} + B_k \qquad (k=1,2,\dots,n),$$

$$f_\mu\left(s^{(1)},\dots,s^{(M)}\right) = \sum_{m=1}^{M} a_\mu^{(m)}\ s^{(m)} + b_\mu \qquad \left(1 \leqq \mu \leqq M\right),$$

where $s^{(m)} = \sum_{k=1}^{n} x_k^{(m)}$. Let us introduce the following notations: $L^{(m)} = \sum_{k=1}^{n} L_k^{(m)}$, $\underline{\underline{A}} = \left(a_\mu^{(m)}\right)_{\mu,m=1}^{M}$.

Finally let us assume that $\underline{\underline{A}} + \underline{\underline{A}}^T$ is nonnegative semidefinite. Under the above conditions the game has at least one equilibrium point, and since φ_k is concave in $\underline{x}_k$, a vector $\underline{x}^* = \left(\underline{x}_1^*,\dots,\underline{x}_n^*\right)$ is an equilibrium point of the game if and only if

$$\frac{\partial \varphi_k(\underline{x}^*)}{\partial x_k^{(m)}} \begin{cases} \leqq 0 & \text{for } x_k^{(m)*} = 0 \\ \geqq 0 & \text{for } x_k^{(m)*} = L_k^{(m)} \\ = 0 & \text{for } 0 < x_k^{(m)*} < L_k^{(m)}, \end{cases} \qquad (\forall\, k,m) \quad (35)$$

where $\underline{x}_k^* = \left(x_k^{(1)*},\dots,x_k^{(M)*}\right)$ $(k=1,2,\dots,n)$. Let

$$w_k^{(\mu)} = L_k^{(\mu)} - x_k^{(\mu)} \geqq 0$$

$$z_k^{(\mu)} \begin{cases} = 0 & \text{if} \quad x_k^{(\mu)} > 0 \\ \geqq 0 & \text{otherwise} \end{cases} \tag{36}$$

$$v_k^{(\mu)} \begin{cases} = 0 & \text{if} \quad x_k^{(\mu)} < L_k^{(\mu)} \\ \geqq 0 & \text{otherwise,} \end{cases}$$

then by calculating the partial derivatives of f_k we can easily verify that the conditions (35) are equivalent to the set of equations (see [10] pp. 46-47)

$$b_\mu + \sum_{m=1}^{M} a_\mu^{(m)} s^{(m)} + \sum_{m=1}^{M} a_m^{(\mu)} x_k^{(m)} - A_k^{(\mu)} - v_k^{(\mu)} + z_k^{(\mu)} = 0 \tag{37}$$

for $\mu=1,2,\ldots,M$; $k=1,2,\ldots,n$, where

$$s^{(m)} = \sum_{k=1}^{n} x_k^{(m)}. \tag{38}$$

The above system can be written in a simpler form if we introduce the following notations:

$$\underline{x} = \left(x_1^{(1)},\ldots,x_1^{(M)},\ldots,x_n^{(1)},\ldots,x_n^{(M)}\right)^T$$

$$\underline{a} = \left(A_1^{(1)},\ldots, A_1^{(M)},\ldots, A_n^{(1)},\ldots,A_n^{(M)}\right)^T$$

$$\underline{v} = \left(v_1^{(1)},\ldots, v_1^{(M)},\ldots, v_n^{(1)},\ldots, v_n^{(M)}\right)^T$$

$$\underline{z} = \left(z_1^{(1)},\ldots, z_1^{(M)},\ldots, z_n^{(1)},\ldots, z_n^{(M)}\right)^T$$

$$\underline{w} = \left(w_1^{(1)}, \ldots, w_1^{(M)}, \ldots, w_n^{(1)}, \ldots, w_n^{(M)} \right)^T$$

$$\underline{\ell} = \left(L_1^{(1)}, \ldots, L_1^{(M)}, \ldots, L_n^{(1)}, \ldots, L_n^{(M)} \right)^T$$

$$\underline{b} = \left(b_1, \ldots, b_M, \ldots, b_1, \ldots, b_M \right)^T$$

Furthermore let $\underline{\underline{P}}$ denote the $Mn \times Mn$ matrix

$$\underline{\underline{P}} = \begin{pmatrix} \underline{\underline{A}} & \cdots & \underline{\underline{A}} \\ \vdots & & \vdots \\ \underline{\underline{A}} & \cdots & \underline{\underline{A}} \end{pmatrix} + \begin{pmatrix} \underline{\underline{A}}^T & & \\ & \ddots & \\ & & \underline{\underline{A}}^T \end{pmatrix},$$

then the relations (36), (37), (38) have the form

$$\begin{aligned} \underline{\underline{P}}\,\underline{x} + \underline{b} - \underline{a} - \underline{v} + \underline{z} &= \underline{0} \\ \underline{z} + \underline{w} &= \underline{\ell} \\ \underline{x}^T \underline{z} = \underline{v}^T \underline{w} = \underline{v}^T \underline{z} &= 0 \\ \underline{x}, \underline{v}, \underline{z}, \underline{w} &\geqq \underline{0}\,. \end{aligned} \tag{39}$$

Thus we have proven the following result.

Lemma 12. A vector $\underline{x}^*$ is an equilibrium point of the linear oligopoly game with nonnegative definite matrix $\underline{\underline{A}} + \underline{\underline{A}}^T$ if and only if there exist vectors $\underline{v}^*$, $\underline{w}^*$, $\underline{z}^*$ such that conditions (39) are satisfied with $\underline{x} = \underline{x}^*$, $\underline{v} = \underline{v}^*$, $\underline{w} = \underline{w}^*$ and $\underline{z} = \underline{z}^*$.

In a further special case the uniqueness of the equilibrium point is assured, as it is shown in the following theorem.

Theorem 8. Assume that matrix $\underline{\underline{A}}$ is symmetric, negative definite. Then the game has a unique equilibrium point.

Proof. Let us consider the quadratic programming problem

$$\frac{\underline{0} \leq \underline{x} \leq \underline{c}}{\frac{1}{2}\,\underline{x}^T\,\underline{\underline{P}}\,\underline{x} + (\underline{b} - \underline{a})^T\,\underline{x} \longrightarrow \max .} \tag{40}$$

First we prove that problem (40) is a strictly convex programming problem. It is sufficient to prove that matrix $\underline{\underline{P}}$ is negative definite. Let $\underline{u} = (\underline{u}_1, \ldots, \underline{u}_n) \in R^{Mn}$, where $\underline{u}_k \in R^M$ for k=1,2,...,n. Then

$$\underline{u}^T\,\underline{\underline{P}}\,\underline{u} = \sum_{k=1}^{n} \underline{u}_k^T\,\underline{\underline{A}}\,\underline{u}_k + \sum_{i=1}^{n}\sum_{j=1}^{n} \underline{u}_i^T\,\underline{\underline{A}}\,\underline{u}_j =$$

$$= \sum_{k=1}^{n} \underline{u}_k^T\,\underline{\underline{A}}\,\underline{u}_k + \left(\sum_{i=1}^{n} \underline{u}_i\right)^T \underline{\underline{A}} \left(\sum_{j=1}^{n} \underline{u}_j\right) < 0$$

for $\underline{u} \neq \underline{0}$. If $\underline{\underline{A}}$ is symmetric, then obviously $\underline{\underline{P}}$ is also symmetric.

Next we observe that conditions (39) without the equation $\underline{v}^T\,\underline{z} = 0$ are the Kuhn-Tucker conditions of the quadratic programming problem (see G. Hadley [3]), and since it is convex, the Kuhn-Tucker conditions are necessary and sufficient conditions for the optimality. The fact that the matrix $\underline{\underline{P}}$ is negative definite implies that problem (40) has a unique solution, and since the game has an equilibrium point which must satisfy system (39) we conclude that the unique solution of (40) gives the unique solution of (39), which is the unique equilibrium point of the game. ∎

<u>Remark.</u> The numerical solution of problem (40) can be obtained by standard methods (see G. Hadley [3]).

Finally we remark that the statements of Lemma 12. and Theorem 8. can be extended for the multiproduct group equilibrium problem, but the details are not discussed here.