1. General results

A mathematical game is a set $\Gamma = (n; X_1, X_2, \ldots, X_n; \varphi_1, \varphi_2, \ldots, \varphi_n)$, where n is a positive integer, $X_1, X_2, \ldots, X_n$ are arbitrary sets and the functions φ_k $(1 \leq k \leq n)$ are such that $\mathcal{D}(\varphi_k) = X_1 \times X_2 \times \ldots \times X_n$, $\mathcal{R}(\varphi_k) \subset R^1$. Here n is called the number of players, the sets X_k are the strategy sets and the functions φ_k are the pay-off functions. Assuming that the first player chooses the strategy $x_1 \in X_1$, the second player chooses the strategy $x_2 \in X_2$, etc., than the value $\varphi_k(x_1, x_2, \ldots, x_n)$ is considered to be the income of player k $(k = 1,2,\ldots,n)$. In the special case of $\sum_{i=1}^{n} \varphi_i = 0$ the game is called a zero sum n-person game.

Definition 1. A vector $\underline{x}^* = (x_1^*, \ldots, x_n^*)$ is a Nash-equilibrium point of the game Γ, if

a/ $x_k^* \in X_k$ $(k = 1,2,\ldots,n)$;

b/ for $k = 1,2,\ldots,n$ and arbitrary $x_k \in X_k$,

$$\varphi_k(x_1^*, \ldots, x_k, \ldots x_n^*) \leq \varphi_k(x_1^*, \ldots, x_k^*, \ldots, x_n^*). \quad (1)$$

Remark. The equilibrium strategy x_k^* is optimal for the player k assuming that the other players choose the corresponding components of the equiliberium point.

Example 1. Let $n=2$,

$$X_1 = \{1,2,\ldots,m_1\}, \quad X_2 = \{1,2,\ldots,m_2\}.$$

In this special case the game Γ is called a <u>two-person finite game.</u> Let us introduce the following notations:

$$\varphi_1 (i,j) = a_{ij}$$

$$\varphi_2 (i,j) = b_{ij} \qquad (i=1,2,\dots,m_1;\ j=1,2,\dots,m_2),$$

$$\underline{\underline{A}} = (a_{ij}), \quad \underline{\underline{B}} = (b_{ij}).$$

Observe that $\underline{\underline{A}}$ and $\underline{\underline{B}}$ are $m_1 \times m_2$ matrices. The inequalities (1) imply that a pair (i_o, j_o) is an eyuilibrium point if and only if

$$b_{i_oj} \leqq b_{i_oj_o} \qquad (j=1,2,\dots,m_2)$$

$$a_{ij_o} \leqq a_{i_oj_o} \qquad (i=1,2,\dots,m_1).$$

In other words, the element $a_{i_oj_o}$ is maximal in its column (in matrix $\underline{\underline{A}}$), and the element $b_{i_oj_o}$ is maximal in its row (in matrix $\underline{\underline{B}}$). From this simple observation we can easily verify that the game determined by matrices

$$\underline{\underline{A}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \underline{\underline{B}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

has no equilibrium point; the game with matrices

$$\underline{\underline{A}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \underline{\underline{B}} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$$

has a unique equilibrium point $(1,1)$; and any pair (i,j) of the game given by matrices

$$\underline{\underline{A}} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad \underline{\underline{B}} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

is an equilibrium point.

The computation of the equilibrium points for finite games is an easy job since a finite number of inequalities has to be checked.

Example 2. Let $n=2$,

$$X_1 = \left\{ \underline{x}_1 \mid \underline{x}_1 \in R^{m_1} ,\ \underline{x}_1 \geqq \underline{0} ,\ \underline{1}^T \underline{x}_1 = 1 \right\},$$

$$X_2 = \left\{ \underline{x}_2 \mid \underline{x}_2 \in R^{m_2} ,\ \underline{x}_2 \geqq \underline{0} ,\ \underline{1}^T \underline{x}_2 = 1 \right\},$$

$$\varphi_1 (\underline{x}_1, \underline{x}_2) = \underline{x}_1^T \underline{\underline{A}}\, \underline{x}_2 , \quad \varphi_2 (\underline{x}_1, \underline{x}_2) = \underline{x}_1^T \underline{\underline{B}}\, \underline{x}_2 ,$$

where $\underline{0}$ is the zero vector, the vector $\underline{1}$ has unit components, $\underline{\underline{A}}$ and $\underline{\underline{B}}$ are $m_1 \times m_2$ real matrices. The game defined above is called a bimatrix game. In the special case of $\underline{\underline{B}} = - \underline{\underline{A}}$ the game is called a matrix game. It is known that the equilibrium problem of matrix games is equivalent to the solution of linear programming problems and the equilibrium problem of bimatrix games can be solved by the solution of quadratic programming problems. The details will be discussed later. Note that the bimatrix games are generalizations, extensions of finite two-person games, since the strategies of the players are the choices of distributions defined on the sets $\{1,2,\dots,m_1\}$ and $\{1,2,\dots,m_2\}$ instead of the choices of one-one element from each set. The pay-off of the generalized game is the expectation of the pay-off obtained in the finite game with respect to the distribution chosen by each player.

Example 3. Let us consider the following n-person game:

$$X_k = \{\underline{x}_k \mid \underline{\underline{A}}_k \underline{x}_k \leq \underline{b}_k\} \qquad (k=1,2,\dots,n)$$

$$\varphi_k(\underline{x}_1,\dots,\underline{x}_n) = \sum_{i_1=1}^{m_1} \dots \sum_{i_n=1}^{m_n} a^{(k)}_{i_1 i_2 \dots i_n} x^{(1)}_{i_1} x^{(2)}_{i_2} \dots x^{(n)}_{i_n} ,$$

where $\underline{\underline{A}}_k$ is an $\ell_k \times m_k$ real matrix, $\underline{b}_k \in R^{\ell_k}$ is a real vector, the numbers $a^{(k)}_{i_1 \dots i_n}$ are given real parameters, and for $k=1,2,\dots,n$, $\underline{x}_k = (x^{(k)}_1, \dots, x^{(k)}_{m_k})$. This game is called a <u>generalized polyhedral game</u>. To simplify our notations let

$$a^{(k)}_i(\underline{x}) = \sum_{i_1=1}^{m_1} \dots \sum_{i_{k-1}=1}^{m_{k-1}} \sum_{i_{k+1}=1}^{m_{k+1}} \dots \sum_{i_n=1}^{m_n} a^{(k)}_{i_1 \dots i_{k-1} i i_{k+1} \dots i_n} \cdot x^{(1)}_{i_1} \dots$$

$$\dots x^{(k-1)}_{i_{k-1}} x^{(k+1)}_{i_{k+1}} \dots x^{(n)}_{i_n} ,$$

and

$$\underline{a}_k(\underline{x}) = \left(a^{(k)}_1(\underline{x}), \dots, a^{(k)}_{m_k}(\underline{x})\right)^T ,$$

then

$$\varphi_k(\underline{x}) = \underline{a}_k(\underline{x})^T \underline{x}_k ,$$

where $\underline{a}_k(\underline{x})$ is independent of $\underline{x}_k$.

In the special case of

$$\underline{\underline{A}}_k = \begin{pmatrix} -\underline{\underline{I}} \\ \underline{1}^T \\ -\underline{1}^T \end{pmatrix} , \qquad \underline{b}_k = \begin{pmatrix} \underline{0} \\ 1 \\ -1 \end{pmatrix}$$

(where $\underline{\underline{I}}$ is the m_k dimensional unit matrix, $\underline{1}, \underline{0} \in R^{m_k}$, the vector $\underline{0}$ is the zero vector, the vector $\underline{1}$ has unit components) we have

$$X_k = \left\{ \underline{\underline{x}}_k \mid \underline{\underline{x}}_k \in R^{m_k}, \ \underline{\underline{x}}_k \geqq \underline{0}, \ \underline{1}^T \underline{\underline{x}}_k = 1 \right\},$$

and the game is called the <u>mixed extension</u> of finite n-person games. Observe that for n=2 we have the bimatrix games with

$$\underline{\underline{A}} = \left(a^{(1)}_{i_1 i_2} \right) \quad \text{and} \quad \underline{\underline{B}} = \left(a^{(2)}_{i_1 i_2} \right),$$

since

$$\varphi_1(\underline{\underline{x}}_1, \underline{\underline{x}}_2) = \sum_{i_1=1}^{m_1} \sum_{i_2=1}^{m_2} a^{(1)}_{i_1 i_2} \, x^{(1)}_{i_1} x^{(2)}_{i_2} = \underline{\underline{x}}_1^T \underline{\underline{A}} \, \underline{\underline{x}}_2$$

and

$$\varphi_2(\underline{\underline{x}}_1, \underline{\underline{x}}_2) = \sum_{i_1=1}^{m_1} \sum_{i_2=1}^{m_2} a^{(2)}_{i_1 i_2} \, x^{(1)}_{i_1} \, x^{(2)}_{i_2} = \underline{\underline{x}}_1^T \underline{\underline{B}} \, \underline{\underline{x}}_2 .$$

First we will show the connection between certain mathematical programming problems and two-person zero sum games.

Let us consider the mathematical programming problem

$$\begin{array}{c} \underline{\underline{x}} \in X \\ \underline{g}(\underline{\underline{x}}) \geqq \underline{0} \\ \hline f(\underline{\underline{x}}) \longrightarrow \max , \end{array} \tag{2}$$

where X is an arbitrary subset of R^n /it may be discrete/, $\mathcal{D}(\underline{g}) \subset R^n$, $\mathcal{R}(\underline{g}) \subset R^m$, $\mathcal{D}(f) \subset R^n$, $\mathcal{R}(f) \subset R^1$. Let

$$R^m_+ = \left\{ \underline{u} \mid \underline{u} \in R^m, \ \underline{u} \geqq \underline{0} \right\},$$

and let us consider the two-person zero sum game

$$\Gamma = \left(2 \ ; \ X \ , \ R^+_m \ ; \ F, \ - F \right), \tag{3}$$

where

$$F(\underline{x}, \underline{u}) = f(\underline{x}) + \underline{u}^T \underline{g}(\underline{x}).$$

Lemma 1. If $(\underline{x}^*, \underline{u}^*)$ is an equilibrium point of the game Γ, than $\underline{x}^*$ is an optimal solution to the programming problem (2).

Proof. The inequalities (1) imply that if $(\underline{x}^*, \underline{u}^*)$ is equilibrium point, than

$$f(\underline{x}^*) + \underline{u}^{*T} \underline{g}(\underline{x}^*) \geqq f(\underline{x}) + \underline{u}^{*T} \underline{g}(\underline{x}) \qquad (\forall\, \underline{x} \in X) \qquad (4)$$

$$f(\underline{x}^*) + \underline{u}^{*T} \underline{g}(\underline{x}^*) \leqq f(\underline{x}^*) + \underline{u}^{T} \underline{g}(\underline{x}^*) \qquad (\forall\, \underline{u} \in R^m_+). \qquad (5)$$

First we observe that $\underline{g}(\underline{x}^*) \geqq \underline{0}$. Let us assume that a component $g_i(\underline{x}^*) < 0$, then taking the ith component of $\underline{u}$ sufficiently large, the inequality (5) will not hold. Let $\underline{u} = \underline{0}$, then inequality (5) implies $\underline{u}^{*T} \underline{g}(\underline{x}^*) \leqq 0$. But $\underline{u}^* \geqq \underline{0}$, $\underline{g}(\underline{x}^*) \geqq \underline{0}$, consequently $\underline{u}^{*T} \underline{g}(\underline{x}^*) \geqq 0$. Thus $\underline{u}^{*T} \underline{g}(\underline{x}^*) = 0$.

Since $\underline{x}^* \in X$, $\underline{g}(\underline{x}^*) \geqq \underline{0}$, the vector $\underline{x}^*$ is a feasible solution of the problem (2). We can easily prove that $\underline{x}^*$ is an optimal solution. Let $\underline{x}$ be any feasible solution of the problem (2). Then inequality (4) implies

$$f(\underline{x}^*) = f(\underline{x}^*) + \underline{u}^{*T} \underline{g}(\underline{x}^*) \geqq f(\underline{x}) + \underline{u}^{*T} \underline{g}(\underline{x}) \geqq f(\underline{x}),$$

thus $\underline{x}^*$ is an optimal solution. ∎

Remark. The opposite statement is not true in general. The Kuhn-Tucker theory gives sufficent conditions, for an arbitrary optimal solution of the problem (2) to be obtainable from an equilibrium point of the game Γ.

Newt we will prove that the equilibrium problem of n-person general games is equivalent to the fixed-point problem of a certain point-to-set mapping. Let us consider the n-person game in a more generalized form: $\Gamma = (n\ ;\ X_1, X_2, \ldots, X_n, X\ ;\ \varphi_1, \varphi_2, \ldots, \varphi_n)$, where n is the number of players; $X_1, X_2, \ldots, X_n$ are the strategy sets of the players, $X \subset X_1 \times X_2 \times \ldots \times X_n$ is the simultaneous strategy set, the functions $\varphi_1, \varphi_2, \ldots, \varphi_n$ are the pay-off functions such that $\mathcal{D}(\varphi_k) = X$, $\mathcal{R}(\varphi_k) \subset R^1$ $(k=1,2,\ldots,n)$. Here we assume that the players can not choose their strategies independently of each other because of circumstances independent of the players /for instance in production games it is inpossible all players to have maximal production because of the bounded quantity of row materials/, and in the concrete realizations of the game only the elements of X can appear as strategy vectors.

Definition 2. A vector $\underline{x}^* = (x_1^*, \ldots, x_n^*)$ is an equilibrium point of the game Γ if

a/ $\underline{x}^* \in X$;

b/ for $k=1,2,\ldots,n$ for arbitrary $(x_1^*, \ldots, x_k, \ldots, x_n^*) \in X$,

$$\varphi_k(x_1^*, \ldots, x_k, \ldots, x_n^*) \leq \varphi_k(x_1^*, \ldots, x_k^*, \ldots, x_n^*). \qquad (6)$$

Let us consider the following function,

$$\Phi(\underline{x}, \underline{y}) = \sum_{k=1}^{n} \varphi_k(x_1, \ldots, y_k, \ldots, x_n),$$

where for $k=1,2,\ldots,n$, $(x_1,\ldots, y_k,\ldots,x_n) \in X$. Let us say that the pair $(\underline{x}, \underline{y})$ is feasible if $\underline{x} \in X$ and for $k=1,2,\ldots,n$, $(x_1,\ldots,y_k,\ldots,x_n) \in X$. Then function Φ is defined for arbitrary feasible pairs $(\underline{x}, \underline{y})$.

<u>Lemma 2.</u> The vector $\underline{x}^* = (x_1^*,\ldots,x_n^*)$ is an equilibrium point of the game Γ if and only if for arbitrary feasible pairs $(\underline{x}^*, \underline{y})$, $\Phi(\underline{x}^*, \underline{x}^*) \geqq \Phi(\underline{x}^*, \underline{y})$.

<u>Proof.</u> a/ Let us assume that $\underline{x}^*$ is an equilibrium point. Then for arbitrary k and $(x_1^*,\ldots,y_k,\ldots,x_n^*) \in X$ the inequality holds

$$\varphi_k(x_1^*,\ldots,x_k^*,\ldots,x_n^*) \geqq \varphi_k(x_1^*,\ldots,y_k,\ldots,x_n^*).$$

Let us add these inequalities for $k=1,2,\ldots,n$ and let $\underline{y} = (y_1,\ldots,y_n)$, then we have

$$\sum_{k=1}^{n} \varphi_k(x_1^*,\ldots,x_k^*,\ldots,x_n^*) = \Phi(\underline{x}^*, \underline{x}^*) \geqq \sum_{k=1}^{n} \varphi_k(x_1^*,\ldots,y_k,\ldots,x_n^*)$$

$$= \Phi(\underline{x}^*, \underline{y}).$$

b/ Let us now assume that $\underline{x}^* \in X$ and for an arbitrary feasible pair $(\underline{x}^*, \underline{y})$, $\Phi(\underline{x}^*, \underline{x}^*) \geqq \Phi(\underline{x}^*, \underline{y})$. Let k be fixed and let $\underline{y} = (x_1^*,\ldots,x_k,\ldots,x_n^*) \in X$. Then obviously the pair $(\underline{x}^*, \underline{y})$ is feasible and

$$\phi(\underline{x}^*, \underline{x}^*) \geqq \phi(\underline{x}^*, \underline{y}). \tag{7}$$

Since

$$\phi(\underline{x}^*, \underline{x}^*) = \sum_{i \neq k} \varphi_i(\underline{x}^*) + \varphi_k(x_1^*, \dots, x_k^*, \dots, x_n^*)$$

and

$$\phi(\underline{x}^*, \underline{y}) = \sum_{i \neq k} \varphi_i(\underline{x}^*) + \varphi_k(x_1^*, \dots, x_k, \dots, x_n^*)$$

the inequality (7) implies that

$$\varphi_k(x_1^*, \dots, x_k^*, \dots, x_n^*) \geqq \varphi_k(x_1^*, \dots, x_k, \dots, x_n^*),$$

thus $\underline{x}^*$ is an equilibrium point. ∎

By using the above notations let us introduce the following point-to-set mapping

$$\phi(\underline{x}) = \left\{ \underline{t} \mid (\underline{x}, \underline{t}) \text{ is feasible and } \phi(\underline{x}, \underline{t}) = \max \left\{ (\underline{x}, \underline{y}) ; (\underline{x}, \underline{y}) \text{ is feasible} \right\} \right\}.$$

As a simple consequence of Lemma 2. we have the following important result.

<u>Lemma 3.</u> A vector $\underline{x}^*$ is an equilibrium point of the game if and only if $\underline{x}^* \in \phi(\underline{x}^*)$ /i.e. $\underline{x}^*$ is a fixed point of the mapping ϕ /.

The most important existence theorem for n-person games can be proven by using the Kakutani fixed point theorem for showing that the mapping ϕ has at least one fixed point. This theorem is called Nikaido-Isoda theorem and it is the following:

<u>Theorem 1.</u> Assume that

a/ X is a bounded, closed and convex subset of a finite dimension Eucledian space;

b/ for $k=1,2,\dots,n$ the functions φ_k are continuous and for fixed $x_1,\dots,x_{k-1}, x_{k+1},\dots,x_n$ they are concave in x_k. Under these assumptions the game has at least one equilibrium point.

Proof. See J.B. Rosen [8] . ■

Remark. If we assume that the functions φ_k are strictly concave in x_k, then the uniqueness of the equilibrium point in general is not true /see Example 4./. For the uniqueness of the equilibrium point of n-person games J.B. Rosen [8] gave sufficient conditions, but the assumptions of the next paragraphs are independent of the conditions introduced by J.B. Rosen.

2. The solution of a special class of concave games

Let us assume that for $k=1,2,\dots,n$

$$X_k = \left\{ \underline{x}_k \mid \underline{x}_k \in R^{m_k} , \underline{h}_k(\underline{x}_k) \geqq \underline{0} \right\},$$

where

a/ $\mathcal{D}(\underline{h}_k) = R^{m_k}$, $\mathcal{R}(\underline{h}_k) \subset R^{\ell_k}$, the components of $\underline{h}_k$ are concave, continuously differentiable functions;

b/ X_k is bounded, and in each point of X_k the Kuhn-Tucker regularity condition holds /see G. Hadley [3]/ ;

c/ φ_k is continuous, concave in $\underline{x}_k$ for fixed $\underline{x}_1,\dots,\dots,\underline{x}_{k-1}, \underline{x}_{k+1},\dots,\underline{x}_n$ and continuously differentiable with respect to $\underline{x}_k$.

Lemma 4. The game $\Gamma = (n; X_1,\dots,X_n ; \varphi_1,\dots,\varphi_n)$ has at least one equilibrium point.

Proof. It is obvious that all conditions of the Nikaido--Isoda theorem are satisfied. ∎

Let $k=1,2,\dots,n$ and for fixed strategy vector $\underline{x}^* = (\underline{x}_1^*, \underline{x}_2^*, \dots, \underline{x}_n^*)$ consider the mathematical programming problem

$$\frac{\underline{h}_k(\underline{x}_k) \geqq \underline{0}}{\varphi_k(\underline{x}_1^*, \dots, \underline{x}_k, \dots, \underline{x}_n^*) \to \max .} \tag{8}$$

Lemma 5. A vector $\underline{x}^* = (\underline{x}_1^*, \dots, \underline{x}_n^*)$ is an equilibrium point if and only if for $k=1,2,\dots,n$ $\underline{x}_k^*$ is an optimal solution of the problem (8).

Proof. a/ If $\underline{x}_k^*$ is a feasible solution, then the constraint implies that $\underline{x}_k^* \in X_k$, thus $\underline{x}^* = (\underline{x}_1^*, \dots, \underline{x}_n^*)$ is a strategy vector. If $\underline{x}_k^*$ is an optimal solution, then for any feasible solution $\underline{x}_k \in X_k$, $\varphi_k(\underline{x}_1^*, \dots, \underline{x}_k^*, \dots, \underline{x}_n^*) \geqq \varphi_k(\underline{x}_1^*, \dots, \underline{x}_k, \dots, \underline{x}_n^*)$. Thus $\underline{x}^*$ is an equilibrium point.

b/ If $\underline{x}^*$ is an equilibrium point, then inequalities (1) imply that the components $\underline{x}_k^*$ are optimal solutions of the problems (8). ∎

Lemma 6. A vector $\underline{x}^* = (\underline{x}_1^*, \dots, \underline{x}_n^*)$ is an equilibrium point if and only if for $k=1,2,\dots,n$ there exists a vector $\underline{u}_k \in R^{\ell_k}$ such that

$$\begin{gathered}
\underline{u}_k \geqq \underline{0} \\
\nabla_k \varphi_k(\underline{x}^*) + \underline{u}_k^T \nabla_k \underline{h}_k(\underline{x}_k^*) = \underline{0}^T \\
\underline{h}_k(\underline{x}_k^*) \geqq \underline{0} \\
\underline{u}_k^T \underline{h}_k(\underline{x}_k^*) = 0
\end{gathered} \tag{9}$$

/where $\nabla_k \varphi_k$ is the gradient vector of φ_k with respect to $\underline{x}_k$ and $\nabla_k \underline{h}_k$ is the Jacobian matrix of $\underline{h}_k$/.

Proof. Under the assumptions given above, problem (8) is a concave programming problem. It is known that the Kuhn-Tucker equations and inequalities (9) are sufficient and necessary conditions for the optimality of a vector $\underline{x}_k^*$ $(k=1,2,\dots,n)$ /see Hadley [3] / . ∎

To the sake of simple notations let

$$\Psi_k(\underline{x}, \underline{u}_k) = \nabla_k f_k(\underline{x}) + \underline{u}_k^T \nabla_k \underline{h}_k(\underline{x}_k),$$

where $\underline{x} = (\underline{x}_1, \dots, \underline{x}_n)$.

Now we can prove our main theorem.

Theorem 2. A vector $\underline{x}^* = (\underline{x}_1^*, \dots, \underline{x}_n^*)$ is an equilibrium point if and only if there exists a vector $\underline{u}^* = (\underline{u}_1^*, \dots, \underline{u}_n^*)$ such that $(\underline{x}^*, \underline{u}^*)$ is an optimal solution of the mathematical programming problem

$$\left.\begin{aligned} \underline{u}_k &\geqq \underline{0} \\ \Psi_k(\underline{x}, \underline{u}_k) &= \underline{0}^T \\ \underline{h}_k(\underline{x}_k) &\geqq \underline{0} \end{aligned}\right\} \quad (k=1,2,\dots,n) \tag{10}$$

$$\overline{\sum_{k=1}^{n} \underline{u}_k^T \, \underline{h}_k(\underline{x}_k) \longrightarrow \min.}$$

Proof. a/ Let $\underline{x}^*$ be an equilibrium point /Lemma 4. implies that there exists at least one equilibrium point./ Then there exists a vector $\underline{u}^* = (\underline{u}_1^*, \dots, \underline{u}_n^*)$ such that the equations and inequalities are satisfied for $\underline{u}_k = \underline{u}_k^*$, thus the value of the objective function of the programming problem (10) is zero for $\underline{u}_k = \underline{u}_k^*$, $\underline{x}_k = \underline{x}_k^*$. For arbitrary feasible solution $(\underline{x}, \underline{u})$

of (1o) the objective function value is nonnegative, thus $(\underline{x}^*, \underline{u}^*)$ is an optimal solution.

b/ Let $(\underline{x}^*, \underline{u}^*)$ be an optimal solution of (1o). Since it is a feasible solution, each term of the objective function is nonnegative, conseqently the value of the objective function is nonnegative. But for the equilibrium point of the game /which exists/ the objective function has zero value, therefore the optimality of $(\underline{x}^*, \underline{u}^*)$ implies that the objective function at the point $(\underline{x}^*, \underline{u}^*)$ must have zero value. Since all terms are nonnegative in the objective function, all terms are equal to zero. Thus the equations and inequalities (9) are valid for $\underline{x} = \underline{x}^*$, $\underline{u} = \underline{u}^*$, consequently Lemma 6. implies that $\underline{x}^*$ is an equilibrium point. ■

<u>Remark 1.</u> Problem (1o) is a mathematical programming problem which can be solved by numerical methods /e.g. cutting plane or gradient type algorithms, see G. Hadley [3]/.

<u>Remark 2.</u> In the special case of n=2 and $\varphi_2 = -\varphi_1$ problem (1o) was discovered by M.D.Canon [2].

Finally we will shaw well-known algorithms can be derived from the above general method as special cases.

<u>General polyhedral games</u>

Using the notations of Example 3. we have

$$\nabla_k \varphi_k(\underline{x}^*) = \underline{a}_k(\underline{x})^T$$
$$\nabla_k \underline{h}_k(\underline{x}_k) = -\underline{\underline{A}}_k ,$$

since

$$\varphi_k(\underline{x}^*) = \underline{a}_k(\underline{x})^T \underline{x}_k$$

$$\underline{h}_k(\underline{x}_k) = \underline{b}_k - \underline{\underline{A}}_k \underline{x}_k .$$

Thus problem (10) has the form:

$$\left.\begin{array}{c} \underline{u}_k \geqq \underline{0} \\ \underline{a}_k(\underline{x})^T - \underline{u}_k^T \underline{\underline{A}}_k = \underline{0}^T \\ \underline{b}_k - \underline{\underline{A}}_k \underline{x}_k \geqq \underline{0} \end{array}\right\} \qquad (k=1,2,\dots,n) \tag{11}$$

$$\overline{\sum_{k=1}^{n} \underline{u}_k^T (\underline{b}_k - \underline{\underline{A}}_k \underline{x}_k) \longrightarrow \min.}$$

Let us observe that the second constraint implies that

$$\underline{a}_k(\underline{x})^T = \underline{u}_k^T \underline{\underline{A}}_k ,$$

and by using the fact that $\varphi_k(\underline{x}) = \underline{a}_k(\underline{x})^T \underline{x}_k$ we can write problem (11) in a more convenient form:

$$\left.\begin{array}{c} \underline{u}_k \geqq \underline{0} \\ \underline{a}_k(\underline{x})^T - \underline{u}_k^T \underline{\underline{A}}_k = \underline{0}^T \\ \underline{b}_k - \underline{\underline{A}}_k \underline{x}_k \geqq \underline{0} \end{array}\right\} \qquad (k=1,2,\dots,n)$$

$$\overline{\sum_{k=1}^{n} (\underline{u}_k^T \underline{b}_k - \varphi_k(\underline{x})) \longrightarrow \min.} \tag{12}$$

As a special case let $n=2$. Since

$$\underline{a}_1(\underline{x}) = \underline{\underline{A}}\, \underline{x}_2, \qquad \underline{a}_2(\underline{x}) = \underline{\underline{B}}^T \underline{x}_1 ,$$

where $\underline{\underline{A}} = \left(a^{(1)}_{i_1 i_2}\right)$ and $\underline{\underline{B}} = \left(a^{(2)}_{i_1 i_2}\right)$,

problem (12) can be rewritten as

$$
\begin{aligned}
\underline{u}_1 &\geqq \underline{0} \\
\underline{u}_2 &\geqq \underline{0} \\
\underline{x}_2^T \underline{\underline{A}} - \underline{u}_1^T \underline{\underline{A}}_1 &= \underline{0}^T \\
\underline{x}_1^T \underline{\underline{B}} - \underline{u}_2^T \underline{\underline{A}}_2 &= \underline{0}^T \\
\underline{b}_1 - \underline{\underline{A}}_1 \underline{x}_1 &\geqq \underline{0} \\
\underline{b}_2 - \underline{\underline{A}}_2 \underline{x}_2 &\geqq \underline{0}
\end{aligned} \tag{13}
$$

$$
\sum_{k=1}^{2} \left(\underline{u}_k^T \underline{b}_k\right) - \underline{x}_1^T \left(\underline{\underline{A}} + \underline{\underline{B}}\right) \underline{x}_2 \longrightarrow \min ,
$$

which is a quadratic programming problem with linear constraints. Let us observe that the unknown vector $(\underline{x}_1, \underline{x}_2, \underline{u}_1, \underline{u}_2)$ is $m_1 + m_2 + \ell_1 + \ell_2$ dimensional. In a further special case when $\underline{\underline{B}} = - \underline{\underline{A}}$ /zero-sum case/, problem (13) is a linear programming problem, which can be solved by the simplex method.

Mixed extension of finite games

As we have seen in Example 3. in our case

$$
\underline{\underline{A}}_k = \begin{pmatrix} -\underline{\underline{I}} \\ \underline{1}^T \\ -\underline{1}^T \end{pmatrix}, \qquad \underline{b}_k = \begin{pmatrix} \underline{0} \\ 1 \\ -1 \end{pmatrix}.
$$

Let us write the vectors $\underline{u}_k$ in block form corresponding to the special block form of $\underline{\underline{A}}_k$ and $\underline{b}_k$, then we have

$$
\underline{u}_k = \begin{pmatrix} \underline{v}_k \\ \alpha_k \\ \beta_k \end{pmatrix},
$$

where $\underline{v}_k \in R^{m_k}$, α_k and β_k are scalers. Using these special notations problem (12) can be written in the form

$$\left.\begin{array}{c} \underline{v}_k \geqq \underline{0} \\ \alpha_k \geqq 0 \\ \beta_k \geqq 0 \\ \underline{a}_k(\underline{x})^T + \underline{v}_k^T - \alpha_k \underline{1}^T + \beta_k \underline{1}^T = \underline{0}^T \\ \underline{x}_k \geqq \underline{0} \\ \underline{1}^T \underline{x}_k = 1 \end{array}\right\} (k=1,2,\dots,n) \tag{14}$$

$$\sum_{k=1}^{n} \left(\alpha_k - \beta_k - \varphi_k(\underline{x})\right) \longrightarrow \min.$$

Let us observe that the nonnegative vector $\underline{v}_k$ appears only in the fourth constraint and we can introduce the new variable $\gamma_k = \alpha_k - \beta_k$, which is not necessarily nonnegative. Then we get by multiplying the objective function by -1 the following problem:

$$\left.\begin{array}{c} \underline{a}_k(\underline{x})^T \leqq \gamma_k \underline{1}^T \\ \underline{x}_k \geqq \underline{0} \\ \underline{1}^T \underline{x}_k = 1 \end{array}\right\} (k=1,2,\dots,n) \tag{15}$$

$$\sum_{k=1}^{n} \left(\varphi_k(\underline{x}) - \gamma_k\right) \longrightarrow \max$$

which is the method of H. Mills [6].

Bimatrix games

From the previous case the bimatrix games can be obtained by choosing n=2. Simple calculations show that

$$\underline{a}_1(\underline{x}) = \underline{\underline{A}}\,\underline{x}_2 , \qquad \underline{a}_2(\underline{x}) = \underline{\underline{B}}^T \underline{x}_1 ,$$

thus problem (15) can be written as

$$\begin{aligned} \underline{\underline{A}}\,\underline{x}_2 &\leqq \gamma_1 \underline{1} \\ \underline{\underline{B}}^T \underline{x}_1 &\leqq \gamma_2 \underline{1} \\ \underline{x}_1 &\geqq \underline{0} \\ \underline{x}_2 &\geqq \underline{0} \\ \underline{1}^T \underline{x}_1 &= 1 \\ \underline{1}^T \underline{x}_2 &= 1 \\ \hline x_1^T (\underline{\underline{A}} + \underline{\underline{B}})\, \underline{x}_2 - \gamma_1 - \gamma_2 &\to \max , \end{aligned} \qquad (16)$$

which is a quadratic programming problem with linear constraints and it was discovered by O.L. Mangasarian and H. Stone [5]. For matrix games $\underline{\underline{B}} = -\underline{\underline{A}}$, thus problem (16) is a linear programming problem, which can be separated with respect to the variables $(\underline{x}_1, \gamma_2)$ and $(\underline{x}_2, \gamma_1)$, and so problem (16) can be reduced for two linear programming problems

$$\begin{aligned} \underline{\underline{A}}\,\underline{x}_2 &\leqq \gamma_1 \underline{1} \\ \underline{x}_2 &\geqq \underline{0} \\ \underline{1}^T \underline{x}_2 &= 1 \\ \hline \gamma_1 &\to \min \end{aligned} \qquad (17)$$

and

$$\begin{aligned} -\underline{\underline{A}}^T \underline{x}_1 &\leqq \gamma_2 \underline{1} \\ \underline{x}_1 &\geqq \underline{0} \\ \underline{1}^T \underline{x}_1 &= 1 \\ \hline \gamma_2 &\to \min , \end{aligned} \qquad (18)$$

where the problems have $m_2 + 1$ and $m_1 + 1$ variables, respectively.

3. The classical oligopoly game

In this section we will discuss a special economic game with sets of strategies

$$X_k = [0, L_k] \quad (L_k > 0,\ k=1,2,\dots,n) \qquad (19)$$

and pay-off functions

$$\varphi_k (x_1,\dots,x_n) = x_k\, f\left(\sum_{i=1}^{n} x_i\right) - K_k(x_k), \qquad (20)$$

where the functions f and K_k must have the properties:

$$\mathcal{D}(f) = [0, L], \quad \text{where} \quad L = \sum_{i=1}^{n} L_i \ ; \ \mathcal{D}(K_k) = [0, L_k];$$

$\mathcal{R}(f) \subset R^1$ and $\mathcal{R}(K_k) \subset R^1$. The game defined by the sets of strategies (19) and pay-off functions (20) is called the <u>classical oligopoly game</u>.

Before discussing the equilibrium problem of this game we show how the game appears in some applications.

<u>Application 1.</u> Assume that n factories manufacture the same product and they sell it on the same market. Let f be the unit price of the product being a function of the total production level, and let K_k be the cost function of the manufacturer k. Then L_k is the production bound for manufacturer k and $\varphi_k(x_1,\dots,x_n)$ is its netto income assuming that x_i is the production level of the manufacturer i for $i=1,2,\dots,n$.

<u>Application 2.</u> Assume that a multipurpose water supply system has to be designed. Let the water users denoted by k (k=1,2,...,n) and let the water quantity given to user k be

denoted by x_k. If the capacity bounds of the users are denoted by L_k, then obviously $x_k \in [0, L_k]$ for $k=1,2,\dots,n$. Let I be the investment cost being a function of $\sum_{k=1}^{n} x_k$, let $u_k(x_k)$, $v_k(x_k)$ and $w_k(x_k)$ be the production cost, income and the economic loss of the water shortage /penalty e.t.c./ of user k, respectively. Let us assume, that the total investment cost is devided by the users in the rate of the water quantity used by the water users. Thus the total income of user k can be determined by the function

$$-\frac{x_k}{\sum_{i=1}^{n} x_i} I\left(\sum_{i=1}^{n} x_i\right) - u_k(x_k) + v_k(x_k) - w_k(x_k). \tag{21}$$

By introduceing the notations

$$f\left(\sum_{i=1}^{n} x_i\right) = -\frac{1}{\sum_{i=1}^{n} x_i} I\left(\sum_{i=1}^{n} x_i\right)$$

$$K_k(x_k) = u_k(x_k) - v_k(x_k) + w_k(x_k)$$

function (21) has immediately form (2o).

<u>Application 3.</u> Let us now assume that n factories are on the bank of a river and they send a certain quantity of waste-water to the river. It is also assumed that the total penalty paid by the factories is a function of the total waste-water quantity sent to the river and it is devided among the factories proportionally to the waste-water quantity sent to the river by the different factories. Let L_k be the total waste-

-water quantity produced by factory k, let x_k be the waste-water quantity sent to the river by factory k. Then the total "income" of factory k can be given by the formula

$$\frac{-x_k}{\sum_{i=1}^{n} x_i} P\left(\sum_{i=1}^{n} x_i\right) - C_k\left(L_k - x_k\right), \qquad (22)$$

where P is the penalty function, C_k is the cleaning cost of factory k. Let

$$f\left(\sum_{i=1}^{n} x_i\right) = -\frac{1}{\sum_{i=1}^{n} x_i} P\left(\sum_{i=1}^{n} x_i\right), \quad K_k(x_k) = C_k\left(L_k - x_k\right),$$

then the function (22) immidiately has the form of (20).

First we show that the equilibrium problem of the classical oligopoly game is equivalent to a fixed point problem of a one dimension point-to-set mapping. It will be much more convenient than the application of the fixed point problem of Lemma 3., since the latter is an n-dimensional problem.

Let

$$\Psi_k(s, x_k, t_k) = t_k f(s - x_k + t_k) - K_k(t_k)$$

for $k=1,2,..,n$, $s \in [0, L]$, $x_k \in [0, L_k]$ and $t_k \in [0, \gamma_k]$, where $\gamma_k = \min\left\{L_k, L - s + x_k\right\}$. Since $\gamma_k \geqq 0$, the interval for t_k can not be empty. For $k=1,2,..,n$; $s \in [0, L]$; $x_k \in [0, L_k]$ let

$$T_k(s, x_k) = \left\{t_k \mid t_k \in [0, \gamma_k], \Psi_k(s, x_k, t_k) = \right.$$

$$\left. = \max_{0 \leq u_k \leq \gamma_k} \Psi_k(s, x_k, u_k)\right\}$$

and for $k=1,2,\dots,n$; $s \in [0, L]$ let

$$X_k(s) = \left\{x_k \mid x_k \in [0, L_k], \; x_k \in T_k(s, x_k)\right\}.$$

Lemma 7. A vector $\underline{x}^* = (x_1^*,\dots,x_n^*)$ is an equilibrium point of the classical oligopoly game if and only if $x_k^* \in X_k(s^*)$ $(k=1,2,\dots,n)$, where $s^* = \sum_{k=1}^{n} x_k^*$.

Proof. The definition of the equilibrium point implies that a strategy vector $\underline{x}^* = (x_1^*,\dots,x_n^*)$ is an equilibrium point if and only if

$$x_k^* f(s^* - x_k^* + x_k^*) - K_k(x_k^*) \geqq t_k f(s^* - x_k^* + t_k) - K_k(t_k) \quad (23)$$

for $k=1,2,\dots,n$ and $t_k \in [0, L_k]$. /It is easy to observe that for $s^* = \sum_{i=1}^{n} x_i^*$, $\gamma_k = L_k$/ Inequality (23) is equivalent to the fact that $x_k^* \in T_k(s^*, x_k^*)$, that is $x_k^* \in X_k(s^*)$. ∎

Let us finally introduce the following one dimensional point-to-set mapping:

$$X(s) = \left\{u \mid u = \sum_{i=1}^{n} x_i , \; x_i \in X_i(s)\right\} \quad (s \in [0, L]). \qquad (24)$$

Lemma 7. and definition (24) imply the following important result.

Theorem 3. A vector $\underline{x}^* = (x_1^*,\dots,x_n^*)$ is an equilibrium point of the classical oligopoly game if and only if for $s^* = \sum_{i=1}^{n} x_i^*$, $s^* \in X(s^*)$ and for $k=1,2,\dots,n$, $x_k^* \in X_k(s^*)$.

Remark. The solution of the game has two steps:

Step 1: the solution of the one dimensional fixed point problem $s^{*} \in X(s^{*})$;

Step 2: the determination of sets $X_k(s^{*})$ and the computation of the vectors $\underline{x}^{*} = (x_1^{*}, \ldots, x_n^{*})$ such that

$$x_k^{*} \in X_k(s^{*}) \quad (k=1,2,\ldots,n) \quad \text{and} \quad s^{*} = \sum_{k=1}^{n} x_k^{*} .$$

In the following parts of this section we will assume that the conditions given below are satisfied.

1. There exists a constant $\xi > 0$ such that

a/ $f(s) = 0$ for $s \geqq \xi$;

b/ f is continuous, concave and strictly decreasing in the interval $[0, \xi]$.

2. For $k=1,2,\ldots,n$ function K_k is continuous, convex and strictly increasing in the interval $[0, L_k]$.

Theorem 4. Under the above conditions the game has at least one equilibrium point.

Proof. The proof consists of several steps.

a/ First we prove that if $\underline{x}^{*} = (x_1^{*}, \ldots, x_n^{*})$ is an equilibrium point, then $\sum_{k=1}^{n} x_k^{*} \leqq \xi$. Let us suppose that $\sum_{k=1}^{n} x_k^{*} > \xi$. Then there are positive x_k^{*} and x_k such that $0 < x_k < x_k^{*}$ and $\sum_{i \neq k} x_i^{*} + x_k > \xi$. This implies

$$\varphi_k\left(x_1^*,\dots,x_k,\dots,x_n^*\right) = x_k\cdot 0 - K_k(x_k) > x_k^*\cdot 0 - K_k(x_k^*) =$$

$\varphi_k\left(x_1^*,\dots,x_k^*,\dots,x_n^*\right)$, which is a contradiction to inequality (1).

b/ Let

$$X = \left\{\underline{x} \mid \underline{x} = (x_1,\dots,x_n),\ \sum_{k=1}^{n} x_k \leqq \xi\ ,\ x_k \in [0,\ L_k],\ k=1,2,\dots,n\right\}.$$

Next we prove that any equilibrium point $\underline{x}^*$ of the generalized game $\Gamma = (n;\ X_1,\dots,X_n,\ X;\ \varphi_1,\dots,\varphi_n)$ gives an equilibrium point for the classical oligopoly game. Let $x_k \in [0,\ L_k]$. If $(x_1^*,\dots,x_{k-1}^*,\ x_k,\ x_{k+1}^*,\dots,x_n^*) \in X$, then the equilibrium property for game Γ gives

$$\varphi_k\left(x_1^*,\dots,x_k^*,\dots,x_n^*\right) \geqq \varphi_k\left(x_1^*,\dots,x_k,\dots,x_n^*\right),$$

and if $\left(x_1^*,\dots,x_k,\dots,x_n^*\right) \notin X$, then

$$\varphi_k\left(x_1^*,\dots,x_k,\dots,x_n^*\right) = x_k\cdot 0 - K_k(x_k) < -K_k(0) =$$

$$= 0.f\left(\sum_{i\neq k} x_i^*\right) - K_k(0) = \varphi_k\left(x_1^*,\dots,0,\dots,x_n^*\right) \leqq \varphi_k\left(x_1^*,\dots,x_k^*,\dots,x_n^*\right)$$

since $\left(x_1^*,\dots,0,\dots,x_n^*\right) \in X$.

c/ Next we prove that if function h is continuous, concave and strictly decreasing in a nonnegative interval $[A,\ B]$, then the function $xh(x)$ is concave in the same interval.

Let us first assume that h is twice continuously differentiable.

Then

$$\{xh(x)\}' = xh'(x) + h(x),$$

$$\{xh(x)\}'' = 2h'(x) + xh''(x) < 0,$$

which implies the assertion.

If h is continuous, then let h_m $(m=1,2,..)$ be twice continuously differenciable, concave, strictly decreasing functions such that $\lim_{m \to \infty} h_m = h$.

Let $A \leq x < y \leq B$; $\alpha, \beta \geq 0$; $\alpha + \beta = 1$, then for $m=1,2...$

$$(\alpha x + \beta y) h_m (\alpha x + \beta y) \geqq \alpha x h_m(x) + \beta y h_m(x).$$

By the limit relation $m \to \infty$ we obtain

$$(\alpha x + \beta y) h(\alpha x + \beta y) \geqq \alpha x h(x) + \beta y h(y),$$

thus $xh(x)$ is concave.

d/ The parts a/ and b/ imply that the classical oligopoly game and the generalized game $\Gamma = (n; X_1, \dots, X_n, X; \varphi_1, \dots, \varphi_n)$ have the same equilibrium points. Under the assumptions of the theorem X is a convex, closed, bounded subset of R^n, φ_k is continuous and part c/ implies that φ_k is concave in x_k. Thus the conditions of the Nikaido-Isoda theorem are satisfied, consequently the game has at least one equilibrium point. ∎

<u>Remark.</u> The uniqueness of the equilibrium point is not assured in general as the following example shows.

<u>Example 4.</u> Let $n=2$; $L_1 = L_2 = 1,2$;

$$f(s) = \begin{cases} 1,75 - 0,5s, & \text{if } 0 \leqq s \leqq 1,5 \\ 2,5 - s, & \text{if } 1,5 \leqq s \leqq 2,5 \\ 0, & \text{if } s > 2,5 \end{cases};$$

$$K_1(x) = K_2(x) = 0,5x \quad (x \geqq 0).$$

We will prove that an arbitrary point of the set

$$X^* = \left\{ (x_1, x_2) \,\middle|\, 0,5 \leqq x_1 \leqq 1,\ 0,5 \leqq x_2 \leqq 1,\ x_1 + x_2 = 1,5 \right\}$$

gives an equilibrium point of the game.

Lét $x^* \in [0,5 \,;\, 1]$ be fixed, and let

$$\Psi(x) = xf(1,5 - x + x^*) - K_k(x) \qquad (k=1,2,).$$

It is easy to verify that

$$\Psi'(x^* - 0) = x^*(-0,5) + 1 - 0,5 = 0,5(1 - x^*) \geqq 0,$$

and

$$\Psi'(x^* + 0) = x^*(-1) + 1 - 0,5 = 0,5 - x^* \leqq 0.$$

Part c/ implies that function Ψ is concave in x, consequently from the inequalities $\Psi'(x^* - 0) \geqq 0$ and $\Psi'(x^* + 0)$ we can conclude that x^* is a maximumpoint of the function Ψ. Thus arbitrary $x^* \in X^*$ is an equilibrium point.

Next we discuss a numerical algorithm for findig the equilibrium points of the classical oligopoly game. Under the assumptions of Theorem 4. the following statements are true.

Lemma 8.

a/ For $s \in [0, L]$, $X_k(s)$ is not empty and is a closed interval $[A_k(s), B_k(s)]$, $(k=1,2,\dots,n)$;

b/ for $0 \leq s < s' \leq L$ the inequality $B_k(s') \leqq A_k(s)$ holds for $k=1,2,\dots,n$;

c/ if f is differentiable at the point s, then $A_k(s) = B_k(s)$;

d/ if f is differentiable in the interval $[0, L]$, then $A_k(s)$ is a continuous function of s.

Proof. Parts a/ and b/ can be proven by simple modifications of parts C/a/ and C/b/ of the proof of Theorem 1. in paper [10]. The statements c/ and d/ are proven in the C/a,b,c part of the proof of Theorem 1. in paper [10]. ∎

Lemma 9. If $\underline{x}^* = \left(x_1^*,\dots,x_n^*\right)$ and $\underline{x}^{**} = \left(x_1^{**},\dots,x_n^{**}\right)$ are equilibrium points of the classical oligopoly game having the properties given in Theorem 4., then

$$\sum_{k=1}^{n} x_k^* = \sum_{k=1}^{n} x_k^{**} .$$

Proof. Assume that $s^* = \sum_{k=1}^{n} x_k^* < s^{**} = \sum_{k=1}^{n} x_k^{**}$. Then

$$s^* = \sum_{k=1}^{n} x_k^* \geqq \sum_{k=1}^{n} A_k(s^*) \geqq \sum_{k=1}^{n} B_k(s^{**}) \geqq \sum_{k=1}^{n} x_k^{**} = s^{**} ,$$

which is a contradiction. ∎

Corollary. The point-to-set mapping $X(s)$ has exactly one fixed point, which can be computed by the usual bisection method (see F. Szidarovszky, S.Yakowitz [12]).

Theorem 5. Assume that the conditions of Theorem 4. are satisfied. Let s^* be the unique fixed point of the mapping $X(s)$. Then all equilibrium points of the classical oligopoly game can be obtained by the solution of the system of linear equations and inequalities:

$$A_k(s^*) \leqq x_k \leqq B_k(s^*) \qquad (k=1,2,\ldots,n)$$

$$\sum_{k=1}^{n} x_k = s^*$$

Proof. The statement is a consequence of Lemma 8. and Lemma 9. ∎

Corollary. If in addition to the conditions of Theorem 4. function f is differentiable on the interval $[0, L]$, then the equilibrium point is unique.

Remark 1. It is interesting to observe that the game is not linear but the set of equilibrium points is a simplex.

Remark 2. The uniqueness of the equilibrium point depends on the differentiability of a function and not on strict concavity as it is usual in the theory of nonlinear programming.

Special cases.

1. In case of f and K_k $(1 \leqq k \leqq n)$ being twice differentiable the uniqueness was proved by O. Opitz [7] without giving any algorithm for finding it.

2. Under the assumptions of O. Opitz, F. Szidarovszky [9] proved the existence and uniqueness of the equilibrium point

and also gave an iterative algorithm for computing it.

3. If the cost functions K_k are identical and the conditions of O. Opitz are satisfied, then E. Burger [1] proved the existence and uniqueness of the equilibrium point and also gave an algorithm to compute it. We remark that the algorithm of Szidarovszky is a generalization of Burger's method.

4. If the functions f and K_k $(k=1,2,\dots,n)$ are linear, then the existence and uniqueness was proved by M. Maňas, [4], who gave an algorithm which is independent of the method of Szidarovszky. We remark that using the result of Theorem 5. the equilibrium point in this special case can be given in closed form (see pp. 37-39 of [10]).

4. The group equilibrium problem

In this paragraph we will discuss the generalized version of the classical oligopoly game Γ having the srategy sets

$$X_k = [0, L_{k1}] \times [0, L_{k2}] \times \dots \times [0, L_{ki_k}] \quad (1 \leq k \leq n) \tag{25}$$

and pay-off functions

$$\varphi_k(\underline{x}_1,\dots,\underline{x}_n) = \left(\sum_{i=1}^{i_k} x_{ki}\right) f\left(\sum_{\ell=1}^{n} \sum_{j=1}^{i_\ell} x_{\ell j}\right) - K_k(\underline{x}_k), \tag{26}$$

where for $k=1,2,\dots,n$, $\underline{x}_k = (x_{k1},\dots,x_{ki_k}) \in X_k$. This game can occur when the players of the classical oligopoly game form disjoint groups and they tend to the optimal income of the group. If the number of members in group k is equal to i_k, and the capacity limit of member i of group k is given by L_{ki}, then the strategy set of group k is the set X_k and the income of group k is the sum of the individual incomes of its members, given by the function (26).

For $k=1,2,\dots,n$ and $s_k \in \left[0, \sum_{i=1}^{i_k} L_{ki}\right]$ consider the problem

$$
\begin{aligned}
&0 \leq x_{ki} \leq L_{ki} \qquad (i=1,2,\dots,i_k)\\
&\sum_{i=1}^{i_k} x_{ki} = s_k \qquad\qquad (27)\\
\hline
&K(\underline{x}_k) \longrightarrow \min.
\end{aligned}
$$

If function K is continuous then problem (27) has an optimal solution. Let the optimal objective function value be denoted by $\varphi_k(s_k)$. Some properties of the functions φ_k are given in the following lemma.

<u>Lemma 10</u>. If K is continuous, convex and strictly increasing in the components of $\underline{x}_k$, then φ_k is continuous, convex and strictly increasing in s_k.

<u>Proof</u>. See Lemmas 2,3,4 of the paper [10]. ∎

<u>Remark</u>. Observe that the same properties were assumed in the main theorems of the previous section which are now stated in this lemma.

Let us now consider the classical oligopoly game $\tilde{\Gamma}$ with sets of strategies

$$\tilde{X}_k = \left[0, \sum_{i=1}^{i_k} L_{ki}\right] \quad (k=1,2,\dots,n) \tag{28}$$

and pay-off functions

$$\tilde{\varphi}_k(s_1,\dots,s_n) = s_k\, f\left(\sum_{\ell=1}^{n} s_\ell\right) - \varphi_k(s_k). \tag{29}$$

The connection between the generalized game (25), (26) and the classical oligopoly game (28), (29) is shown in the following theorem.

<u>Theorem 6.</u> Assume that K_k is continuous for $k=1,2,\dots,n$.

a/ Let $\underline{x}^* = (\underline{x}_1^*,\dots,\underline{x}_n^*)$ $\left(\underline{x}_k^* = (x_{k1}^*,\dots,x_{ki_k}^*)\right)$ be an equilibrium point of Γ, and let $s_k^* = \sum_{i=1}^{i_k} x_{ki}^*$. Then $(s_1^*,\dots,s_n^*)$ is an equilibrium point of $\tilde{\Gamma}$ and for $k=1,2,\dots,n$ $(x_{k1}^*,\dots,x_{ki_k}^*)$ is an optimal solution of problem (27) with $s_k = s_k^*$.

b/ Let $(s_1^*,\dots,s_n^*)$ be an equilibrium point of $\tilde{\Gamma}$ and let $\underline{x}_k^* = (x_{k1}^*,\dots,x_{ki_k}^*)$ be an optimal solution of problem (27) with $s_k = s_k^*$. Then $(\underline{x}_1^*,\dots,\underline{x}_n^*)$ gives an equilibrium point of game Γ.

<u>Proof.</u> See Lemma 1. of paper [10]. ∎

<u>Remark.</u> The group equilibrium problem is not a real generalization of the classical oligopoly game, since it can be reduced to the classical case.

Finally let as assume that the functions f and K_k are linear. Let

$$f(s) = As + B$$

$$K_k(\underline{x}_k) = \sum_{i=1}^{i_k} a_{ki} x_{ki} + b_k ,$$

then the solution of the optimization problem (27) is a piece-wise linear function φ_k. In this case the reduced game can be solved easily as it is shown in [10], pp. 43-44.

5. Multiproduct oligopoly game

In this paragraph we will consider the game having the sets of strategies

$$X_k = \left[0, L_k^{(1)}\right] \times \dots \times \left[0, L_k^{(M)}\right] \tag{30}$$

and pay-off functions

$$\varphi_k(\underline{x}_1,\dots,\underline{x}_n) = \sum_{m=1}^{M} x_k^{(m)} f_m\left(\sum_{\ell=1}^{n} x_\ell^{(1)},\dots,\sum_{\ell=1}^{n} x_\ell^{(M)}\right) - K_k(\underline{x}_k), \tag{31}$$

where $\underline{x}_k = \left(x_k^{(1)},\dots,x_k^{(M)}\right)$, $\mathcal{D}(K_k) = X_k$, $\mathcal{R}(K_k) \subset R^1$,

$$\mathcal{D}(f_m) = \left[0, \sum_{\ell=1}^{n} L_\ell^{(1)}\right] \times \dots \times \left[0, \sum_{\ell=1}^{n} L_\ell^{(M)}\right], \quad \mathcal{R}(f_m) \subset R^1 \quad \text{for}$$

$k=1,2,\dots,n$ and $m=1,2,\dots,M$. This game can come up if the factories manufacture different products and sell them on the same market. Let M be the number of products, and let $x_k^{(m)}$, $L_k^{(m)}$ be the production level and capacity limit of factory k from product m. If f_m donotes the unit price of product m, than it is assumed that f_m is a function of the total production levels of the different products. The function K_k is the

production cost, and using the above terminology the income of factory k is given by function (31).
Similar interpretation can be given to the other applications shown in the section dealing with the classical oligopoly game but different qualities of water and waste-water have to be introduced.

The following result is basic in the theory of multiproduct economies, and it is a generalization of part c/ in the proof of Theorem 4.

Lemma 11. Let $\underline{g}$ be a vector-vector function such that $\mathcal{D}(\underline{g})$ is a convex set in the nonnegative orthant of R^M, $\mathcal{R}(\underline{g}) \subset R^M$. Assume that the components of $\underline{g}$ are concave and continuously differentiable. Let $\underline{\underline{J}}$ be the Jacobian matrix of $\underline{g}$. If $\underline{\underline{J}}(\underline{x}) + \underline{\underline{J}}(\underline{x})^T$ is nonnegative semidefinite for arbitrary $\underline{x} \in \mathcal{D}(\underline{g})$, then the function

$$h(\underline{x}) = \underline{x}^T \underline{g}(\underline{x})$$

is concave.

Proof. Let ∇ denote the gradient operation. Then simple calculations show that

$$\nabla h(\underline{x}) = \underline{g}(\underline{x})^T + \underline{x}^T \underline{\underline{J}}(\underline{x}). \tag{32}$$

Since the components of $\underline{g}$ are concave, we have

$$\underline{g}(\underline{y}) - \underline{g}(\underline{x}) \leqq \underline{\underline{J}}(\underline{x})(\underline{y} - \underline{x}) \qquad (\underline{x}, \underline{y} \in \mathcal{D}(\underline{g})), \tag{33}$$

and the condition given for the Jacobian $\underline{\underline{J}}$ implies

$$0 \geqq \frac{1}{2}(\underline{y} - \underline{x})^T \left\{ \underline{\underline{J}}(\underline{x}) + \underline{\underline{J}}(\underline{x})^T \right\} (\underline{y} - \underline{x}) = \\ = (\underline{y} - \underline{x})^T \underline{\underline{J}}(\underline{x})(\underline{y} - \underline{x}). \tag{34}$$

The inequalities (33),(34) and $\underline{y} \geqq \underline{0}$ imply

$$\underline{y}^T \left\{ \underline{g}(\underline{y}) - \underline{g}(\underline{x}) \right\} \leqq \underline{y}^T \underline{\underline{J}}(\underline{x})(\underline{y} - \underline{x}) \leqq \underline{x}^T \underline{\underline{J}}(\underline{x})(\underline{y} - \underline{x}),$$

consequently

$$\underline{y}^T \underline{g}(\underline{y}) - \underline{x}^T \underline{g}(\underline{x}) \leqq \left[\underline{g}(\underline{x})^T + \underline{x}^T \underline{\underline{J}}(\underline{x}) \right] (\underline{y} - \underline{x}),$$

which and equation (32) give the inequality

$$h(\underline{y}) - h(\underline{x}) \leqq \nabla h(\underline{x})(\underline{y} - \underline{x}),$$

Thus function h is concave. ■

As a corollary to this general result we can prove the main result of this section.

<u>Theorem 7.</u> Let $\underline{f} = (f_1, \dots, f_M)$, and let $\underline{\underline{J}}$ be the Jacobian of $\underline{f}$. Assume that functions $\underline{f}$ and K_k $(1 \leqq k \leq n)$ are continuous, the components of $\underline{f}$ are continuously differentiable and concave, K_k is convex and for arbitrary $\underline{s} \in \mathcal{S}(\underline{f})$ the matrix $\underline{\underline{J}}(\underline{s}) + \underline{\underline{J}}(\underline{s})^T$ is nonnegative semidefinite. Then the game has at least one equilibrium point.

<u>Proof.</u> Since X_k is a closed, convex, bounded subset of R^M, φ_k is continuous and Lemma 11. implies that φ_k is concave in $\underline{x}_k$, the game satisfies all conditions of the Nikaido-Isoda theorem. Thus the game has at least one equilibrium point. ■

Remark. The theorem does not give numerical methods for the determination of the equilibrium point. But in the linear case a very efficient algorithm can be constructed which is a generalization of the method of M. Maňas given for the one-product case.

Let us assume that

$$K_k\left(x_k^{(1)},\dots,x_k^{(M)}\right)=\sum_{m=1}^{M} A_k^{(m)}\, x_k^{(m)} + B_k \qquad (k=1,2,\dots,n),$$

$$f_\mu\left(s^{(1)},\dots,s^{(M)}\right) = \sum_{m=1}^{M} a_\mu^{(m)}\, s^{(m)} + b_\mu \qquad \left(1 \leqq \mu \leqq M\right),$$

where $s^{(m)} = \sum_{k=1}^{n} x_k^{(m)}$. Let us introduce the following notations: $L^{(m)} = \sum_{k=1}^{n} L_k^{(m)}$, $\underline{\underline{A}} = \left(a_\mu^{(m)}\right)_{\mu,m=1}^{M}$.

Finally let us assume that $\underline{\underline{A}} + \underline{\underline{A}}^T$ is nonnegative semidefinite. Under the above conditions the game has at least one equilibrium point, and since φ_k is concave in $\underline{x}_k$, a vector $\underline{x}^* = \left(\underline{x}_1^*,\dots,\underline{x}_n^*\right)$ is an equilibrium point of the game if and only if

$$\frac{\partial \varphi_k(\underline{x}^*)}{\partial x_k^{(m)}} \begin{cases} \leqq 0 & \text{for } x_k^{(m)*} = 0 \\ \geqq 0 & \text{for } x_k^{(m)*} = L_k^{(m)} \\ = 0 & \text{for } 0 < x_k^{(m)*} < L_k^{(m)}, \end{cases} \qquad (\forall\, k,m) \quad (35)$$

where $\underline{x}_k^* = \left(x_k^{(1)*},\dots,x_k^{(M)*}\right)$ $(k=1,2,\dots,n)$. Let

$$w_k^{(\mu)} = L_k^{(\mu)} - x_k^{(\mu)} \geqq 0$$

$$z_k^{(\mu)} \begin{cases} = 0 & \text{if} \quad x_k^{(\mu)} > 0 \\ \geqq 0 & \text{otherwise} \end{cases} \tag{36}$$

$$v_k^{(\mu)} \begin{cases} = 0 & \text{if} \quad x_k^{(\mu)} < L_k^{(\mu)} \\ \geqq 0 & \text{otherwise,} \end{cases}$$

then by calculating the partial derivatives of f_k we can easily verify that the conditions (35) are equivalent to the set of equations (see [10] pp. 46-47)

$$b_\mu + \sum_{m=1}^{M} a_\mu^{(m)} s^{(m)} + \sum_{m=1}^{M} a_m^{(\mu)} x_k^{(m)} - A_k^{(\mu)} - v_k^{(\mu)} + z_k^{(\mu)} = 0 \tag{37}$$

for $\mu=1,2,\ldots,M$; $k=1,2,\ldots,n$, where

$$s^{(m)} = \sum_{k=1}^{n} x_k^{(m)}. \tag{38}$$

The above system can be written in a simpler form if we introduce the following notations:

$$\underline{x} = \left(x_1^{(1)}, \ldots, x_1^{(M)}, \ldots, x_n^{(1)}, \ldots, x_n^{(M)} \right)^T$$

$$\underline{a} = \left(A_1^{(1)}, \ldots, A_1^{(M)}, \ldots, A_n^{(1)}, \ldots, A_n^{(M)} \right)^T$$

$$\underline{v} = \left(v_1^{(1)}, \ldots, v_1^{(M)}, \ldots, v_n^{(1)}, \ldots, v_n^{(M)} \right)^T$$

$$\underline{z} = \left(z_1^{(1)}, \ldots, z_1^{(M)}, \ldots, z_n^{(1)}, \ldots, z_n^{(M)} \right)^T$$

$$\underline{w} = \left(w_1^{(1)}, \ldots, w_1^{(M)}, \ldots, w_n^{(1)}, \ldots, w_n^{(M)} \right)^T$$

$$\underline{\ell} = \left(L_1^{(1)}, \ldots, L_1^{(M)}, \ldots, L_n^{(1)}, \ldots, L_n^{(M)} \right)^T$$

$$\underline{b} = \left(b_1, \ldots, b_M, \ldots, b_1, \ldots, b_M \right)^T$$

Furthermore let $\underline{\underline{P}}$ denote the $Mn \times Mn$ matrix

$$\underline{\underline{P}} = \begin{pmatrix} \underline{\underline{A}} & \cdots & \underline{\underline{A}} \\ \vdots & & \vdots \\ \underline{\underline{A}} & \cdots & \underline{\underline{A}} \end{pmatrix} + \begin{pmatrix} \underline{\underline{A}}^T & & \\ & \ddots & \\ & & \underline{\underline{A}}^T \end{pmatrix},$$

then the relations (36), (37), (38) have the form

$$\begin{aligned} \underline{\underline{P}}\,\underline{x} + \underline{b} - \underline{a} - \underline{v} + \underline{z} &= \underline{0} \\ \underline{z} + \underline{w} &= \underline{\ell} \\ \underline{x}^T \underline{z} = \underline{v}^T \underline{w} = \underline{v}^T \underline{z} &= 0 \\ \underline{x}, \underline{v}, \underline{z}, \underline{w} &\geq \underline{0}\,. \end{aligned} \tag{39}$$

Thus we have proven the following result.

Lemma 12. A vector $\underline{x}^*$ is an equilibrium point of the linear oligopoly game with nonnegative definite matrix $\underline{\underline{A}} + \underline{\underline{A}}^T$ if and only if there exist vectors $\underline{v}^*, \underline{w}^*, \underline{z}^*$ such that conditions (39) are satisfied with $\underline{x} = \underline{x}^*$, $\underline{v} = \underline{v}^*$, $\underline{w} = \underline{w}^*$ and $\underline{z} = \underline{z}^*$.

In a further special case the uniqueness of the equilibrium point is assured, as it is shown in the following theorem.

Theorem 8. Assume that matrix $\underline{\underline{A}}$ is symmetric, negative definite. Then the game has a unique equilibrium point.

Proof. Let us consider the quadratic programming problem

$$\frac{\underline{0} \leqq \underline{x} \leqq \underline{c}}{\frac{1}{2}\, \underline{x}^T \underline{\underline{P}}\, \underline{x} + (\underline{b} - \underline{a})^T \underline{x} \longrightarrow \max .} \tag{40}$$

First we prove that problem (40) is a strictly convex programming problem. It is sufficient to prove that matrix $\underline{\underline{P}}$ is negative definite. Let $\underline{u} = (\underline{u}_1, \dots, \underline{u}_n) \in R^{Mn}$, where $\underline{u}_k \in R^M$ for $k=1,2,\dots,n$. Then

$$\underline{u}^T \underline{\underline{P}}\, \underline{u} = \sum_{k=1}^{n} \underline{u}_k^T \underline{\underline{A}}\, \underline{u}_k + \sum_{i=1}^{n} \sum_{j=1}^{n} \underline{u}_i^T \underline{\underline{A}}\, \underline{u}_j =$$

$$= \sum_{k=1}^{n} \underline{u}_k^T \underline{\underline{A}}\, \underline{u}_k + \left(\sum_{i=1}^{n} \underline{u}_i \right)^T \underline{\underline{A}} \left(\sum_{j=1}^{n} \underline{u}_j \right) < 0$$

for $\underline{u} \neq \underline{0}$. If $\underline{\underline{A}}$ is symmetric, then obviously $\underline{\underline{P}}$ is also symmetric.

Next we observe that conditions (39) without the equation $\underline{v}^T \underline{z} = 0$ are the Kuhn-Tucker conditions of the quadratic programming problem (see G. Hadley [3]), and since it is convex, the Kuhn-Tucker conditions are necessary and sufficient conditions for the optimality. The fact that the matrix $\underline{\underline{P}}$ is negative definite implies that problem (40) has a unique solution, and since the game has an equilibrium point which must satisfy system (39) we conclude that the unique solution of (40) gives the unique solution of (39), which is the unique equilibrium point of the game. ∎

<u>Remark.</u> The numerical solution of problem (40) can be obtained by standard methods (see G. Hadley [3]).

Finally we remark that the statements of Lemma 12. and Theorem 8. can be extended for the multiproduct group equilibrium problem, but the details are not discussed here.

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