Introduction - In the work (1) a general disjoint decomposition of semigroups was given, which can be applied for the case of regular semigroups. The aim of the present paper is to obtain a characteristic decomposition of regular semigroups based on the decomposition studied in (1). We shall investigate the components of this decomposition and the interrelations between them.

By making use of the work (2) we study the cases of regular semigroups with or without left or right identity element.

Finally we make some special remarks.

Notations: For two sets $A, B$ we write $A \triangleleft B$ if $A$ is a proper subset of $B$. By magnifying element we mean a left magnifying element.

§ 1.

Let $S$ be a semigroup without nonzero annihilator. This is not a proper restriction because every semigroup can be reduced to this case.

Then $S$ has the following disjoint decomposition:

\[ S = \bigcup_{i=0}^{5} S_i, \]
where

\[ S_0 = \{ a \in S \mid a \neq S \quad \text{and} \quad \exists x \in S, x \neq o \quad \text{and} \quad a \cdot x = 0 \}, \]

\[ S_1 = \{ a \in S \mid a = S \quad \text{and} \quad \exists y \in S, y \neq 0 \quad \text{and} \quad a \cdot y = 0 \}, \]

\[ S_2 = \{ a \in S \setminus (S_0 \cup S_1) \mid a \neq S \quad \text{and} \quad \exists x_1, x_2 \in S, x_1 \neq x_2 \quad \text{and} \quad a \cdot x_1 = a \cdot x_2 \}, \]

\[ S_3 = \{ a \in S \setminus (S_0 \cup S_1) \mid a = S \quad \text{and} \quad \exists y_1, y_2 \in S, y_1 \neq y_2 \quad \text{and} \quad a \cdot y_1 = a \cdot y_2 \}, \]

\[ S_4 = \{ a \in S \setminus (S_0 \cup S_1 \cup S_2 \cup S_3) \mid a \neq S \}, \]

\[ S_5 = \{ a \in S \setminus (S_0 \cup S_1 \cup S_2 \cup S_3) \mid a = S \}. \]

It is easy to see that the components \( S_i (i = 0, 1, \ldots, 5) \) are semigroups, \( S_i \cap S_j = \emptyset \) (\( i \neq j \)) and the following relations hold:

\[
\begin{align*}
S_5 S_1 & \leq S_i \quad , \quad S_1 S_5 \leq S_i \quad , \quad (0 \leq i \leq 5) \\
S_4 S_3 & \leq S_2 \quad , \quad S_4 S_2 \leq S_2 \quad , \quad S_4 S_1 \leq S_0 \quad , \quad S_4 S_0 \leq S_0 , \\
S_2 S_3 & \leq S_2 \quad , \quad S_0 S_1 \leq S_0 .
\end{align*}
\]

It is evident that an analogous decomposition exists, where

\[
S = \bigcup_{i=0}^{5} T_i
\]
\[ T_0 = \{ a \in S | S a \subseteq S \text{ and } \exists x \in S , x \neq 0 \text{ and } x a = 0 \} , \]
\[ T_1 = \{ a \in S | S a = S \text{ and } \exists y \in S , y \neq 0 , \text{ and } y a = 0 \} , \]

etc.

Our theorems concern for the decomposition /1/, but analogous results can be formulated for the decomposition /1'/.

**THEOREM 1.1.** \( S_5 \) is a right group.

**PROOF.** It is easy to see, that \( S_5 \) is right simple and left cancellative, whence the assertion follows.

Let \( S_0 \cup S_2 = \overline{S}_2 \) and \( S_1 \cup S_3 = \overline{S}_3 \).

**THEOREM 1.2.** \( \overline{S}_2 \) is a subsemigroup of \( S \).

**PROOF.** If \( s_o \in S_0 \) and \( s_2 \in S_2 \), then \( s_o s_2 \in \overline{S}_2 \). There are elements \( x, y \in S, x \neq y \) such that \( s_2 x = s_2 y \). We have

\[ s_2 s_2 \not\in S_3 \text{ and } s_2 s_2 \not\in S_5 \text{ because } s_{o2} s_2 = s_o (s_{o2} \subseteq S). \]

If \( s_{o2} \neq 0 \), then \( (s_{o2} s_2)x = (s_{o2} s_2)y \) (\( x \neq y \)), whence

\[ s_{o2} s_2 \in S_2 \not\subseteq \overline{S}_2 . \text{ Similarly } s_2 s_{o2} \in \overline{S}_2 . \text{ If } s_o \neq 0 \text{ then } s_{o2} s_2 \neq 0 \text{ because } s_2 \in S_2 . \text{ Since } s_o \in S_o , \text{ there is an element } z \neq 0 \text{ such that } s_o z = 0 , \text{ hence } (s_{o2} s_o)z = 0 . \text{ Therefore } s_{o2} s_2 \in S_o . \text{ Q.e.d. } \]

**THEOREM 1.3** \( \overline{S}_3 \) contains all the magnifying elements of \( S \) and only them.
PROOF. Let \( a \in S_1 \cup S_3 \). If \( a \in S \) and \( aS = S \), further there is an \( y \neq 0 \) so that \( ax = 0 \), then \( S' = S \setminus \{ 0 \} \subset S \) and \( aS' = S \), whence \( a \) is a magnifying element. If \( a \in S_3 \) and \( aS = S \), further there exist \( x, y \in S \) (\( x \neq y \)) such that \( ax = ay \), then \( a(S - \{ x \}) = S \) and \( a \) is a magnifying element.

Conversely, if \( a \in S \) is a magnifying element, then
\[
a \notin S_0 \cup S_2 \cup S_4 , \text{ and } aM = S (M \subset S).
\]
Thus there are \( m \in M \) and \( s \in S \setminus M \) such that \( am = as \). Hence it follows that \( a \in S_1 \cup S_3 \), q.e.d.

Remark. Theorem 1.2. and Theorem 1.3. imply

\[ S_0 S_2 \leq S_0 \cup S_2 ; \quad S_2 S_0 \leq S_0 \cup S_2 ; \]
\[ S_1 S_3 \leq S_1 \cup S_3 ; \quad S_3 S_1 \leq S_1 \cup S_3 . \]

In what follows we assume \( S \) is a regular semigroup, i.e. to every \( a \in S \) there is an \( x \in S \) such that \( a = axa \) and \( x = xax \) is the inverse of \( a \). The elements \( a, x, xa \) are idempotent and \( aS \supseteq axS \supseteq axaS = aS \) implies \( axS = aS \) and similarly \( xaS = xS \).

The regular semigroup \( S \) can contain a zero element hence the components \( S_0 \) and \( S_1 \) can exist in the decomposition \( /1/ \).

THEOREM 1.4. The inverses of elements of \( S_3 \) are in \( S_4 \) and the inverses of elements of \( S_4 \) are in \( S_3 \).

PROOF. Let \( a \in S_3 \) and \( x \in S \) an inverse of \( a \), that is
a \cdot a = a \quad \text{and} \quad x \cdot a = x. \text{ First of all, we show that } x \subseteq S.

Suppose that \( x \cdot S = S \), then there is a subset \( S' \subseteq S \) such that \( a \cdot S' = S \) because \( a \) is a magnifying element. Hence it follows that \( x \cdot a \cdot S' = x \cdot S = S \). But we have \((x \cdot a) \cdot S = x \cdot S = S\) and \( x \cdot a \) is idempotent, that is, \( x \cdot a \) is a left identity of \( S \), therefore \((x \cdot a) \cdot S' = S' \neq S\), which is a contradiction. Thus \( x \cdot S \subseteq S \), whence \( x \) is contained by \( S_0 \) or \( S_2 \). If \( x \in S_2 \), then

\[
x \cdot s_1 = x \cdot s_2 \quad (s_1 \neq s_2) \quad \text{and} \quad (a \cdot x) \cdot s_1 = (a \cdot x) \cdot s_2. \quad \text{Since} \quad (a \cdot x) \cdot S = a \cdot S = S
\]

and \( x \cdot a \) is idempotent we obtain that \( x \cdot a \) is a left identity of \( S \), i.e. \((a \cdot x) \cdot s_1 = (a \cdot x) \cdot s_2 \) implies \( s_1 = s_2 \), which is a contradiction. It can similarly be proved that \( x \notin S_0 \). It remains the case \( x \in S_4 \).

Conversely, let \( b \in S_4 \), that is, \( b \cdot S = S' \subseteq S \). Let \( y \) be an inverse of \( b \) in \( S \). Hence \( b \cdot y \cdot S = b \cdot S = S' \). Suppose that \( y \cdot S \subseteq S \).

Let \( y \cdot S = S'' \quad (\neq S) \). Hence \( b \cdot S'' = b \cdot y \cdot S = b \cdot S \). Thus there are elements \( s \notin S'' \), and \( s'' \in S'' \) such that \( b \cdot s'' = b \cdot s \). But every element \( a \) of \( S \) for which \( a \cdot x_1 = a \cdot x_2 (x_1 \neq x_2) \) is contained by \( S_0 \cup S_1 \) or \( S_2 \cup S_3 \), which is a contradiction with \( b \in S_4 \).

Thus necessarily \( y \cdot S = S \), that is, \( y \notin S_0 \cup S_2 \cup S_4 \). If \( y \in S_5 \), then \((y \cdot b) \cdot S = y \cdot S = y \cdot (b \cdot S) = y \cdot S' = S \quad (S' \neq S) \), i.e., \( y \in S_1 \cup S_3 \), which is a contradiction. It remains the only case \( y \in S_1 \cup S_3 = S_3 \), q.e.d.

It is easy to see, that the inverses of \( S_3 \) exhaust \( S_4 \) and
the inverses of the elements of \( S_4 \) also exhaust \( \bar{S}_3 \).

**COROLLARY 1.5.** If a regular semigroup \( S \) does not contain magnifying elements \( (\bar{S}_3 = \emptyset) \), then \( S_4 = \emptyset \), and conversely, \( S_4 = \emptyset \) implies \( \bar{S}_3 = \emptyset \).

**COROLLARY 1.6.** If a regular semigroup \( S \) does not contain left identity, then \( \bar{S}_3 = \emptyset \) and hence \( S_4 = \emptyset \).

For if \( a \in \bar{S}_3 \) and \( x \in S_4 \) is an inverse of \( a \), then \( a x \) is a left identity of \( S \).

**THEOREM 1.7.** \( \bar{S}_2 \) is a regular semigroup and the inverses of an element of \( \bar{S}_2 \) are contained by \( \bar{S}_2 \).

**PROOF.** Let \( a \in \bar{S}_2 \) and \( x \) an inverse of \( a \) in \( S \). Since \( a \in S_0 \cup S_2 \), we have \( a \in S \subseteq S \). Assume that \( x S = S \). Then \((x a)S = x(a S) = x S = S\), whence \( x \) is a magnifying element, i.e., \( x \in \bar{S}_3 \).

But every inverse of \( \bar{S}_3 \) is /by Theorem 1.4./ in \( S_4 \), thus \( a \in S_4 \), which is a contradiction. Therefore \( x S \subseteq S \). But \( x \in S_4 \) because \( a \in \bar{S}_2 \). We conclude that \( x \in S_0 \cup S_2 = \bar{S}_2 \), q.e.d.

The above results yield the following result.

**THEOREM 1.8.** A semigroup \( S \) is regular if and only if it has a decomposition /1/

\[
S = \bigcup_{i=0}^{5} S_i,
\]

where

a/ \( \bar{S}_2 = S_o \bigcup S_2 \) is regular;

b/ the inverses of elements of \( \bar{S}_3 = S_1 \bigcup S_3 \) are contained by \( S_4 \) and conversely;

c/ \( S_5 \) is a right group.

PROOF. The necessity follows from Theorems 1.1., 1.4, 1.7.

The sufficiency it follows from the fact that a right group is regular.

§. 2.

In this § we shall deepen our knowledge concerning the decomposition \( /1/ \) of a regular semigroup \( S \) as well as on the components \( \bar{S}_2, \bar{S}_3 \) and \( \bar{S}_4 \).

THEOREM 2.1. Let \( S \) be a regular semigroup without /left/ magnifying elements, Using the notations \( \bar{S}_2 = \bar{S}_2^1, S_5 = S_5^1 \) we obtain the following decompositions:

\[
S = \bar{S}_2^1 \bigcup S_5^1 \quad \text{and if} \quad \bar{S}_2^1 \quad \text{has no magnifying element,}
\]

\[
\bar{S}_2 = \bar{S}_2^2 \bigcup S_5^2 \quad \text{and if} \quad \bar{S}_2^2 \quad \text{has no magnifying element,}
\]

\[
\bar{S}_2 = \bar{S}_2^{k+1} \bigcup S_5^{k+1} \quad \text{............}
\]
where $\overline{s}_2^k$ are regular semigroups, $\overline{s}_5^k$ are right groups and the following inclusions hold:

$$s_5^k s_5^j \subseteq s_5^k \quad (k \geq j)$$
$$s_5^j s_5^k = s_5^k \quad (k \geq j)$$

(4/)

$$s_5^k s_2^j = s_2^j \quad (k \leq j)$$

$$s_2^j s_5^k \subseteq s_2^j \quad (k \leq j)$$

PROOF. It is enough to give a proof for the following cases:

$$s_5^1 s_5^k, \quad s_5^k s_5^1, \quad s_5^1 s_2^j, \quad s_2^j s_5^1$$

because the proof is similar in the semigroups $\overline{s}_2^j$.

The proof is by induction on $k$ and $j$. It is trivial that

$$s_5^1 s_5^1 = s_5^1, \quad s_5^1 s_2^1 = s_2^1, \quad s_5^2 s_2 = s_2^2, \quad (s_5^k \in s_5^k).$$

Hence

$$s_5^1 s_5^2 s_2^{-1} = s_2^1,$$

i.e., $s_5^1 s_5^2 \in s_5^2$ for all $s_5^1 \in s_5^1$ and $s_5^2 \in s_5^2$.

Since $s_5^1 s_2^{-1} = s_2^1$, furthermore $s_5^1 s_5^2 \subseteq s_5^2$ and $s_5^1(s_2^{-1} s_2) \subseteq s_2^{-1}$,

that is, $s_5^1 s_2 \in s_2^{-2}$, we conclude $s_5^1 s_2 = s_2$ and $s_5^1 s_2 = s_2$,

whence $s_5^1 s_5^2 = s_5^2$, $s_5^1 s_2 = s_2$.

Thus we have $s_5^1 s_5^1 = s_5^1$, $s_5^1 s_2^{-1} = s_2^{-1}$, $s_5^1 s_5^2 = s_5^2$,

$$s_5^1 s_2 = s_2, \quad s_5^2 s_5^1 \subseteq s_5^2$$
because $s_5^{-1} s_5^{-2} s_5^{-1} s_5^{-2} = s_5^{-2} s_5^{-2} = s_5^{-2}$ and thus $s_5^{-1} s_5^{-2} s_5^{-1} s_5^{-2}$.

The first step of the proof is true.

Now suppose that the following conditions hold:

$s_5^{-1} s_5^{-2} s_5^{-1} s_5^{-2} s_5^{-1} s_5^{-2} s_5^{-1} s_5^{-2} = s_5^{-2} s_5^{-2} s_5^{-2}$.

By the definition we have $s_5^{-1} s_5^{-2} = s_5^{-2}$. Hence

$(s_5^{-1} s_5^{-2}) s_5^{-2} = s_5^{-1} s_5^{-2} = s_5^{-2}$.

whence $s_5^{-1} s_5^{-2} = s_5^{-2}$.

Thus we obtain

$s_5^{-1} = (s_5^{-1} s_5^{-2}) s_5^{-2} = s_5^{-1} s_5^{-2} = s_5^{-2}$.

whence $s_5^{-1} s_5^{-2} = s_5^{-2}$.

It holds $(s_5^{-1} s_5^{-2}) s_5^{-2} = s_5^{-2}$, furthermore $s_5^{-1} s_5^{-2} = s_5^{-2}$, thus

$s_5^{-1} s_5^{-2} = s_5^{-2}$ implies $s_5^{-1} s_5^{-2} s_5^{-1} s_5^{-2}$.

We have also $(s_5^{-1} s_5^{-2}) s_5^{-2} = s_5^{-1} s_5^{-2} = s_5^{-2}$, whence $s_5^{-1} s_5^{-2} = s_5^{-2}$.

and $s_5^{-1} s_5^{-2} = s_5^{-2}$ implies $s_5^{-1} s_5^{-2} = s_5^{-2}$.

Finally we have $s_5^{-1} s_5^{-2} = s_5^{-2}$ and $s_5^{-1} s_5^{-2} = s_5^{-1} s_5^{-2}$, whence it follows that $s_5^{-1} s_5^{-2} = s_5^{-2}$ and $s_5^{-1} s_5^{-2} = s_5^{-2}$.

Q.e.d.
COROLLARY 2.2. If $S$ and $S_2^k$ ($k \geq 1$) are regular semigroups without magnifying elements, then $S$ has one of the following four types of decompositions:

a/ $S = (((...)) \cup S_2^4 \cup S_2^3 \cup S_2^2) \cup S_2^1$,
with infinite number of components;

b/ $S = S_2^{-1} \cup (((...)) \cup S_2^4 \cup S_2^3 \cup S_2^2) \cup S_2^1$,
where $S_2^{-1}$ is a semigroup of type $S_2$ and there are infinitely many components;

c/ $S = (((S_2^n \cup ...) \cup S_2^3) \cup S_2^2) \cup S_2^1$,
where the number of components is $n$;

d/ $S = (((S_2^{-m} \cup S_2^m) \cup ...) \cup S_2^3) \cup S_2^2) \cup S_2^1$,
where the number of components is $m + 1$.

We shall treat some properties of the semigroups $S_3$ and $S_4$.

THEOREM 2.3. Let $a, b \in S_3$, an inverse of $a$ is $x$, an inverse of $b$ is $y$ ($x, y \in S_4$). Then $xy$ is an inverse of $ba$.

PROOF. Since $ax$ and $by$ are left identities of $S$, we have

$ba \ xy \ ba = b (a \ xy) \ ba = by \ ba = ba,$

$xy \ ba \ xy = x \ y \ b (a \ xy) = x \ y \ b \ y = x \ y,$ q.e.d.

THEOREM 2.4. If $a, b \in S_4$ and $x$ is an inverse of $a$, $y$ is
an inverse of \( b \), then \( yx \) and \( ab \) are inverses of each other.

**PROOF.** By theorem 2.3. \((yb)(xa)a\) is an inverse of \( a \), i.e., \( ab = a(b)(xa)a \) \( ab = \)

\[
\begin{align*}
&= a(b)(yb)(xa)a \\
&= a(b)yx(aba) \quad b = a(b)(yb)(xa)a \\
&= yx \quad a = yb \quad yx = y
\end{align*}
\]

using that \( x, y \) are left identities of \( S \). Q.e.d.

By Theorem 1.4. \( S_3 \cup S_4 \) is a regular subset of \( S \), but it fails to be a subsemigroup, because, e.g., \( S_4S_3 \subseteq S_2 \) /cf.(2)/.

Let \( X_1 = \{ x \in S_4 \mid x \text{ is an inverse of some } a \in S_1 \} \)

\[
X_3 = \{ y \in S_4 \mid y \text{ is an inverse of some } b \in S_3 \}
\]

Then \( S_4 = X_1 \cup X_3 \). 

**COROLLARY 2.5.** \( X_1 \) and \( X_3 \) are subsemigroups of \( S_4 \).

In general, if \( A \subseteq S_3 \) is a subsemigroup, then the inverses of elements of \( A \) is a subsemigroup of \( S_4 \).

**PROOF.** This is an easy consequence of Theorem 2.3.

**COROLLARY 2.6.** \( S_3 \) and \( S_4 \) have no idempotent elements.

**PROOF.** Every element of \( S_3 \) is a magnifying one, thus \( a \neq a^2 \) \( (a \in S_3) \). Assume that \( e \in S_4 \) is idempotent. Since \( e \) is an inverse of \( e \), \( e \in S_3 \)/by Theorem 1.4/, which is a contradiction. Q.e.d.

**THEOREM 2.7.** Every element of \( S_3 \) and \( S_4 \) generates an
infinite cyclic semigroup.

**PROOF.** In opposite case \( S_3 \) or \( S_4 \) contains an idempotent element which is a contradiction by 2.6.

**THEOREM 2.8.**

1. \( S_3 \) has no proper right magnifying element.

2. \( S_4 \) has no left magnifying element.

3. If \( 1 \in S \) /i.e. \( S \) is a monoid/, then \( S_0 \cup S_2 \cup S_5 \) has no left or right magnifying element.

4. \( S_5 \) has no left magnifying element.

**PROOF.**

1/ Is a consequence of \([4]\), Chap.III. 5.6. (β)

Since in the product \( s_4 S \ (s_4 \in S_4) \) the representation of each element is uniquely, thus the same holds for \( s_4 S_4 \), and 2/ is true.

3/ If follows from \([4]\), Chap.III. 5.6. /γ/, because the union \( S_0 \cup S_2 \cup S_5 \) does not contain left or right magnifying elements of \( S \).

Finally, \( S_5 \) is a right group, and hence it has no left magnifying elements /cf. \([4]\), Chap. III. 5.3. (γ)/.
§ 3.

In this § results of the work [2] will be applied to the decomposition /1/ of regular semigroups.

For a regular semigroup $S$ we shall investigate the following cases based on Theorem 4 in [2] :

1/ $S$ has neither left nor right identity element;
2/ $S$ has identity element;
3/ $S$ has either left or right identity element.

In the case 3/ we may assume that $S$ has only left identity element. In the opposite case we need study the decomposition /1'/ instead of /1/.

As it is well known an idempotent element $e$ is $\Theta$-primitive if it is minimal among the idempotents $D_e$, where $D_e$ is the $\Theta$-class of $e$ (where $\Theta$ is a Green's relation).

In the case 1/ $S$ has no left magnifying elements (cf. corollary 1.6), that is, $S_1 \cup S_3 = \emptyset$ and $S_4 = \emptyset$, furthermore $S_5 = \emptyset$, because in the contrary case $S$ would have a left identity element. Hence $S = S_0 \cup S_2 = S_2$.

In the case 2/ suppose that $1 \in S$ is the identity element. Then: a/ if 1 is $\Theta$-primitive we have $S_1 \cup S_3 = \emptyset$, $S_4 = \emptyset$, ...
while $S_5 \neq \emptyset$ /e.g. $1 \in S_5/$. In this subcase we arrive

$$S = S_0 \cup S_2 \cup S_5.$$  

b/ If 1 is not \(\sym\)-primitive, then there are magnifying elements, that is $S_1 \cup S_3 \neq \emptyset$, $S_4 \neq \emptyset$, $S_5$ is equal to the subsemigroup of all invertable elements and thus it is nonempty. Since $S_4 S_3 \subseteq S_2$ and $S_4 S_1 \subseteq S_0$, at least one of the subsemigroups $S_0, S_2$ is nonempty. Hence we obtain

$$S = \overline{S}_2 \cup \overline{S}_3 \cup S_4 \cup S_5$$

where all the components are nonvoid.

In the case 3/ suppose that e is a left identity element of S. Then

a/ if e is \(\sym\)-primitive, then $S_1 \cup S_3 = \emptyset$, $S_4 = \emptyset$, while $S_5 \neq \emptyset$ /for example, $e \in S_5/$. Therefore

$$S = S_0 \cup S_2 \cup S_5.$$  

b/ If e fails to be \(\sym\)-primitive, then there are magnifying elements, that is $S_1 \cup S_3 \neq \emptyset$, $S_4 \neq \emptyset$, $S_5 \neq \emptyset$ and similarly to the case 2/b, we have $S_0 \cup S_2 \neq \emptyset$.

Hence

$$S = \overline{S}_2 \cup \overline{S}_3 \cup S_4 \cup S_5.$$
where all the components are nonempty.

Summing up the above statements:

**THEOREM 3.1.** Let $\mathcal{S}$ be a regular semigroup. Then

1/ if $\mathcal{S}$ has no left identity element:

$$
S_1 \cup S_3 = \emptyset, \quad S_4 = \emptyset, \quad S_5 = \emptyset.
$$

2/ If $\mathcal{S}$ has identity element:

a/ if 1 is $\emptyset$-primitive,

$$
S_1 \cup S_3 = \emptyset, \quad S_4 = \emptyset, \quad S_5 \neq \emptyset.
$$

b/ If 1 is not $\emptyset$-primitive, we have

$$
S_1 \cup S_3 \neq \emptyset, \quad S_4 \neq \emptyset, \quad S_5 \neq \emptyset, \quad S_0 \cup S_2 \neq \emptyset.
$$

3./ If $e$ is a left identity of $\mathcal{S}$:

a/ if $e$ is $\emptyset$-primitive, we get

$$
S_1 \cup S_3 = \emptyset, \quad S_4 = \emptyset, \quad S_5 \neq \emptyset.
$$

b/ if $e$ is not $\emptyset$-primitive, we have

$$
S_1 \cup S_3 \neq \emptyset, \quad S_4 \neq \emptyset, \quad S_5 \neq \emptyset, \quad S_0 \cup S_2 \neq \emptyset.
$$

Finally we make some remarks concerning the decomposition /1/.

If $x \in S_4$, let $B_x = \{a \in S \mid a$ is an inverse of $x\}$.

If $a \in \overline{S}_3$, let $C_a = \{x \in S \mid x$ is an inverse of $a\}$.

If $x \in S_4$ and $a \in B_x (a \in \overline{S}_3)$, then $ax$ is a left identity of $\mathcal{S}$, that is, $a \cdot x = e$ and $a \cdot y = e'$ ($a \in \overline{S}_3$, $y \in C_a$) are left identities of $\mathcal{S}$. 
THEOREM 4.1. /i/ If \( x \in S_4 \) then \( B \) fails to be a subsemigroup. /ii/ If \( a \in \overline{S}_3 \), then \( C \) fails to be a subsemigroup.

**PROOF.** Suppose that \( B \) is a semigroup and \( a, b \in B \). Then
\[
a \cdot a = a, \quad b \cdot b = b \quad \text{and} \quad b \cdot a \in B.
\]
Hence \( b \cdot (a \cdot b) = b \cdot a \).

Since \( a \cdot x \) is a left identity element, hence \( b(b \cdot a) = b \cdot a \).

On the other hand, \( b \cdot a \in \overline{S}_3 \), thus \( b \cdot a \cdot S = S \), whence \( b \cdot s = s \) for all \( s \in S \), which is a contradiction /b is a left magnifying element!/

Let \( x, y \in C \). If \( C \) is a semigroup, then \( a(x \cdot y) = (a \cdot x) \cdot y = y \cdot a \). But \( y \cdot a \neq a \), because \( y \cdot a \) is idempotent, while the element \( a \in \overline{S}_3 \) is not. Thus \( x \cdot y \notin C \). Q.e.d.

Let \( M \subset S \) be a subset of \( S \) such that \( a \cdot M = S \). Then the set \( M \) is left increasable by \( a \). Such left increasable set of \( a \) is not determined uniquely.

THEOREM 4.2. If \( a \in \overline{S}_3 \) then \( a(S_0 \cup S_2 \cup S_4) = S \).

**PROOF.** Let \( a \in \overline{S}_3 \) and \( x \in S_4 \) an inverse of \( a \). Then we have \( a \cdot x \cdot S = a \cdot S = S \) and \( x \cdot S \subset S \). On the other hand \( x \cdot S \subseteq S_4 \), furthermore, by making use the relations /2/ we get
\[
S_4 \cdot S = S_4(S_0 \cup S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5) \leq S_0 \cup S_2 \cup S_4.
\]
Hence \( x \cdot S \subseteq S_0 \cup S_2 \cup S_4 \) and thus
\[
a(S_0 \cup S_2 \cup S_4) = S.
\] Q.e.d.
Theorem 4.2. implies the existence $y_a \in S_o \cup S_2 \cup S_4$ to every $a \in \overline{S}_3$ such that $a y_a = a$.

**THEOREM 4.3.**

a/ If $a \in S_3$, then $y_a \notin S_o$.

b/ The elements $a \in \overline{S}_3$ for which $y_a \in S_4$ ($a y_a = a$) have a two-sided identity element in $S$.

**PROOF.**

a/ If $y_a \in S_0$, then there is $x \neq 0$ such that $y_a x = 0$. Thus $a x = (a y_a) x = a(y_a x) = a 0 = 0$, whence $a \in S_o \cup S_1$ which is a contradiction.

b/ If $y_a \in S_4$, then there exists $b \in \overline{S}_3$, such that $b a b = b$ and $y_a b y_a = y_a$. Then $a y_b = a b$, $a y_a y_a = a b y_a$, that is $a y_a = a b y_a$ whence it follows that $a = a(b y_a)$. On the other hand $b y_a$ is a left identity element of $S$, whence $b y_a a = a = a b y_a$. Q.e.d.