REMARK. - If G is an undirected graph, it is not necessary to contruct also the o-pattern to obtain a properly quasi-constant functio In this case the condition is reduced to $h(\sigma)$ = a vertex of $g(st^m(\sigma))$.

4) The third normalization theorem for homotopies.

Let $e,f\colon S\to G$ be two functions pre-cellular w.r.t. two finite decompositions C and K of S and $F\colon S\times I\to G$ a complete o-homotopy between e and f. Then, for every suffucently fine finite cellular decomposition Γ of $S\times I$, by Theorem 6, the function F can be replaced by a Γ-pre-cellular function $\hat{h}\colon S\times I\to G$. In order that the function $\hat{h}$ may also be a homotopy between e and f, the restrictions of $\hat{h}$ to $S\times\{0\}$ and $S\times\{1\}$ must coincide with e and f. Hence it is necessary that $\hat{h}$ characterizes on $S\times\{0\}$ and $S\times\{1\}$ two decompositions $\tilde{C}$ and $\tilde{K}$ finer than C and K, since e and f are properly quasi-constant (see Remark to Definition 10). Nevertheless, as, for example, the value of the function $\hat{h}$ on $S\times\{0\}$ depends from the value assumed by the function F on the maximal cells of the star $st(\tilde{C})$, in general the restriction $\hat{h}/_{|\tilde{C}|}$ is different from e. Consequently, at first, we must replace the homotopy F by a homotopy M given by:

$$M(x,t) = \begin{cases} e(x) & \forall x\in S, \quad \forall t\in\left[0,\frac{1}{3}\right] \\ F(x,3t-1) & \forall x\in S, \quad \forall t\in\left[\frac{1}{3},\frac{2}{3}\right] \\ f(x) & \forall x\in S, \quad \forall t\in\left[\frac{2}{3},1\right] \end{cases}$$

Then we have to costruct suitable cellular decompositions of the three cylinders $S\times\left[0,\frac{1}{3}\right]$, $S\times\left[\frac{1}{3},\frac{2}{3}\right]$ and $S\times\left[\frac{2}{3},1\right]$.

PROPOSITION 7. - *Let S be a compact triangulable space, C a finite cellular decomposition of S, G a finite graph and $e\colon S\to G$ a properly C-constant function. If we consider the decomposition $L=\{\{0\},\,]0,1[,\,\{1\}\}$ of I and the product decomposition $\Gamma = C\times L$ of the cylinder $S\times I$, then the function $F\colon S\times I\to G$, given by $F(x,t) = e(x)$, $\forall x\in S$, $\forall t\in I$, is properly Γ-constant .*

Proof. - We have only to remark that a cell τ is maximal in Γ iff $\tau = \tau'\times\,]0,1[$, where τ' is a maximal cell in C. Then it results $F(\tau) = e(\tau')$. □

REMARK. - Since the restrictions $F/_{S\times\{0\}}$ and $F/_{S\times\{1\}}$ coincide with e, they are obviously C-constant.

So we obtain:

THEOREM 8. -(The third normalization theorem for homotopies). *Let S be a compact triangulable space, G a finite directed graph, C,D two finite decompositions of S and $e,f\colon S\to G$ two functions pre-cellular w.r.t. C and D respectively, which are completely o-homotopic. Then, from any finite cellular decomposition Γ_2 of $S\times\left[\frac{1}{3},\frac{2}{3}\right]$ of suitable mesh which induces on the bases $S\times\left\{\frac{1}{3}\right\}$ and $S\times\left\{\frac{2}{3}\right\}$ decompositions $\tilde{C}$ and $\tilde{D}$ finer than C and D, we obtain a finite cellular decomposition Γ of $S\times I$ and a homotopy between f and g which is a Γ-pre-cellular function.*

Proof. - Let $F\colon S\times I\to G$ be a complete o-homotopy between e and f. We define the complete o-homotopy $M\colon S\times I\to G$ between e and f as in the introduction of this paragraph. Then, if we consider the restriction of M to $S\times\left[\frac{1}{3},\frac{2}{3}\right]$, we can determine the real number r, upper bound of the mesh. Now if Γ_2 is a finite cellular decomposition, satisfying the conditions of the theorem and with mesh $< r$, we can consider the cellular decomposition $\Gamma = \Gamma_1\cup\Gamma_2\cup\Gamma_3$ of the cylinder $S\times I$, such that:

i) Γ_1 is the product decomposition $\tilde{C}\times L_1$ of $S\times\left[0,\frac{1}{3}\right]$, where $L_1 = \left\{\{0\}, \left]0,\frac{1}{3}\right[, \left\{\frac{1}{3}\right\}\right\}$.

ii) Γ_3 is the product decomposition $\tilde{D}\times L_3$ of $S\times\left[\frac{2}{3},1\right]$, where $L_3 = \left\{\left\{\frac{2}{3}\right\}, \left]\frac{2}{3},1\right[, \{1\}\right\}$.

Then we define the function $\hat{g}\colon S\times I\to G$, given by:

$$\hat{g}(\sigma) = \begin{cases} M(\sigma), & \forall\sigma\in\Gamma-\Gamma_2,\\ \text{a vertex of } H(\{M(\bar{\sigma})\}), & \forall\sigma\in\Gamma_2.\end{cases}$$

Afterwards, by Theorem 6, we construct the o-pattern $\hat{h}$ of $\hat{g}$, by choosing as element of $H(\hat{g}(st^m(\sigma)))$, the value $\hat{g}(\sigma) = M(\sigma)$ if $\sigma\in\Gamma-\Gamma_2$. By construction $\hat{h}$ is a Γ-pre-cellular function. Hence $\hat{h}$ is the sought homotopy since $\hat{h}/_{S\times\{0\}} = e$ and $\hat{h}/_{S\times\{1\}} = f$. □

REMARK. - The finite cellular decomposition Γ induces on the bases $S\times\{0\}$ and $S\times\{1\}$ the decompositions $\tilde{C}$ and $\tilde{D}$.

5) *The second normalization theorem between pairs*.

Given a set A, a non-empty subset A' of A, a finite graph G and a subgraph $G'$of G, we can generalize Definition 4, by considering function $f\colon A,A'\to G,G'$ which are quasi-constant w.r.t. a partition $P = \{X_j\}$, $j\in J$, of A. In this case it follows that the image of every X_j, such that $X_j\cap A'\neq\emptyset$, necessary is a vertex of G'. Moreover, if A is a topo