We can order the a_i in such a way that $a_1,\dots,a_p \in \langle f(x)\rangle$ and $a_{p+1},\dots,a_n \in \langle f(x)\rangle - \langle g(x)\rangle$. Therefore it is $x \in \bar{A}_1^f \cap \dots \cap \bar{A}_p^f$, and so $W(x) \cap A_i^f \neq \emptyset$, $\forall i = 1,\dots,p$. Hence there are in $W(x)$ p points $x_1,\dots,x_p$ such that $x_i \in A_i^f$, $\forall i = 1,\dots,p$. Then it is $x \in W(x_1) \cap \dots \cap W(x_p) \subseteq V(A_1^f) \cap \dots \cap V(A_p^f)$. Moreover it is $x \in \bar{A}_{p+1}^g \cap \dots \cap \bar{A}_n^g$, and by *ii)* it follows $x \in V(A_{p+1}^f) \cap \dots \cap V(A_n^f)$. Hence we obtain the contradiction $x \in V(A_1^f) \cap \dots \cap V(A_n^f)$.

Now if we consider any o-pattern h of g, we obtain the sought function. In fact we have:

i') h: $S \to G$ is completely o-regular (see [5], Proposition 7).

ii') h is weakly p-constant by the definition of o-pattern of a quasi-constant function.

iii') h is completely o-homotopic to f. Since the homotopy F is completely quasi-regular by iii), there exists an o-pattern E of F (which is completely o-regular by [5], Proposition 7). Moreover we can choose E such that $E(x,0) = f(x)$, $E(x,1) = h(x)$, $\forall x \in S$, since f and h are completely o-regular i.e. $f(x) \in H(\langle f(x)\rangle) = H(\langle F(x,0)\rangle)$ and $h(x) \in H(\langle g(x)\rangle) = H(\langle F(x,1)\rangle)$, $\forall x \in S$. Then h is completely o-homotopic to f by E. □

REMARK 1. If W is a closed set, we can give the function g, by choosing as constant image of $X_j \in P$ any vertex of $H(\{f(\bar{X}_j)\})$.

REMARK 2. - *If S is a compact metric space, we can determine a real positive number r and choose partitions P with mesh $< r$.* In fact, we have just to calculate $enl(A_1,\dots,A_n)$, $\forall n$-tuple $a_1,\dots,a_n$ non-headed; so the real number r is given by $\frac{1}{2}\, inf(enl(A_1,\dots,A_n))$.

REMARK 3. - If G is an undirected graph, the function g can be choosen quasi-constant. Moreover if S is a compact metric space, by Remark to Definition 2, we have just to consider the couples of non-adjacent vertices a_h, a_k and then to find the distances $d(A_h,A_k)$ rather than the enlargabilities $enl(A_h,A_k)$. Consequently, if we put $r' = inf(d(A_h,A_k))$ and $r = \frac{1}{2} inf(enl(A_h,A_k))$, since by Remark 3 to Definition 3 it follows $r' \leq 4r$, we can choose a covering $P = \{X_j\}$, $j \in J$, with mesh $< \frac{r'}{4}$. So we obtain again Property 7 of [8].

3) The third normalization theorem.

By comparing the second normalization theorem for directed and undirected graphs, we remark an asymmetry since for the former we are able to construct a q.constant function, while for the latter we obtain only a weakly q.constant function Nevertheless, by choosing a particular compact space S, also for directed graphs we obtain results similar to those for undirected graphs. For this purpose we consider the compact triangulable spaces and its finite decompositions C by (open) CW-complexes (see [13], Cap. VII) which satisfy the condition:

(1) $\forall \sigma \in C$, $\bar{\sigma}$ is a subcomplex of C, i.e. $\forall \tau \in C,\ \tau \cap \bar{\sigma} \neq \emptyset \implies \tau \in \bar{\sigma}$. (*)

DEFINITION 6. - *Let C be a finite cellular complex and D a subset of cells of C. We denote by $|C|$ a realization of C and by $|D|$ the subspace of C constituted by the points of the cells of D.*

REMARK. - Nevertheless, if there is no ambiguity, we denote by σ both a cell and the subspace $|\sigma|$. So, for example, we write $\bar{\sigma}$ rather than $|\bar{\sigma}|$.

DEFINITION 7. - *Let D be a non-empty subset of cells of a finite complex C. We call* star of a point $x \in |D|$ *w.r.t. D, and write $st_D(x)$, the set of the cells of D whose closure in $|C|$, and therefore in $|D|$, includes x. Moreover we call* star of a subset *$X \subset |D|$ w.r.t. D, and write $st_D(X)$, the set of the cells of D, whose closure has a non-empty intersection with X. Similarly we can define the* star $st_D(\sigma)$ of a cell *of D and the* star $st_D(D')$ of a subset D' *of D. Then, if $D = C$, simply we write $st(x)$, $st(X)$ ..., rather than $st_C(x)$, $st_C(X)$,..., .*

REMARK 1. - *The stars are open sets in $|D|$*. In fact their complements are closed in $|D|$, since if for a cell τ it is $\tau \subset |D|$, also it follows $\bar{\tau} \subset |D|$. Then, if $D = C$, the complements of the stars are subcomplexes of C.

REMARK 2. - If x is any point of a cell $\sigma \in D$, then $st_D(x) = st_D(\sigma)$. In fact in $|D|$ it results $x \in \bar{\tau} \iff \sigma \subset \bar{\tau}$.

DEFINITION 8. - *Let D be a subset of cells of a finite cellular complex C. A cell $\tau \in D$ is said to be* maximal *in D if it is $\tau = st_D(\tau)$.*

REMARK. - *A cell is maximal in D iff it is an open set in $|D|$*. Consequently the cells maximal in a star are the cells maximal in C which are included in the star.

DEFINITION 9. - *Let D be a subset of cells of a finite complex C, x a point of $|D|$ and X a subset of $|D|$. We denote by $st^m_D(x)$ (resp.*

(*) We add (1), since we consider cellular subdivisions (triangulations and subdivisions into cubes) of this kind. Nevertheless we can obtain the same results also

$st_D^m(X)$) *the set of the maximal cells of* D, *whose closure includes* x (*resp. has non-empty intersection with* X). *If* $D = C$ *simply we write* $st^m(x)$ *and* $st^m(X)$, *rather than* $st_C^m(x)$ *and* $st_C^m(X)$.

REMARK. - *Let* x *be any point of a cell* $\sigma \in D$, *then obviously it results* $st_D^m(x) = st_D^m(\sigma)$.

DEFINITION 10. - *Let* C *be a finite cellular complex and* G *a finite graph. A function* $f\colon |C| \to G$ *is called* quasi-constant w.r.t. C *or* C-constant *if* f *is quasi-constant w.r.t. the partition determined by the cellular decomposition of* $|C|$. *Then, if* D *is a non-empty subset of cells of* C, *the function* $f\colon |C| \to G$ *is called* properly quasi-constant in D w.r.t. C *or* properly C-constant *in* D, *if, for all the cells* σ *non-maximal in* D, *there exists a cell* $\tau \in D$ (*different from* σ), *such that:*
i) the restrictions of f *to* σ *and to* τ *are identical.*
ii) $\sigma \subset \bar{\tau}$.
At least if $D = C$ *the function* $f\colon |C| \to G$ *is called* properly quasi-constant w.r.t. C *or* properly C-constant.

REMARK. - A function $f\colon |C| \to G$ (properly) C-constant is also (properly) quasi-constant w.r.t. a cellular decomposition C' finer than C.

PROPOSITION 4. - *Let* C *be a finite cellular complex,* D *a subset of cells of* C, G *a finite graph and* $f\colon |C| \to G$ *a* C-*constant function. Then it results* $\langle f(x) \rangle = f(st(x))$, $\forall x \in |C|$.
Moreover, the function f *is properly* C-*constant in* D *iff it is* $f(st_D(\sigma)) = f(st_D^m(\sigma))$, $\forall \sigma \in D$.
At least, if $D = C$, *the previous relation is equivalent to* $\langle f(x) \rangle = f(st^m(x))$, $\forall x \in |C|$.

Proof. - i) Let v be any vertex of G and σ any cell of C, then it follows:
$v \in \langle f(x) \rangle \Longleftrightarrow x \in \overline{V}^f \Longleftrightarrow \exists \sigma \,/\, x \in \bar{\sigma}$ and $f(\sigma) = v \Longleftrightarrow \exists \sigma / \sigma \in st(x)$ and $f(\sigma) = v \Longleftrightarrow v \in f(st(x))$.
ii) If it is $f(st_D(\sigma)) = f(st_D^m(\sigma))$, $\forall \sigma \in D$, the function f is properly C-constant in D, since, $\forall \sigma \in D$, from $f(\sigma) \in f(st_D^m(\sigma))$ we obtain there exists in D a maximal cell τ such that $\sigma \in \bar{\tau}$ and $f(\sigma) = f(\tau)$. The converse follows from the definition of properly quasi-constant function.
iii) By Remark 2 to Definition 7, by Remark to Definition 9 and by i), the condition $\langle f(x) \rangle = f(st^m(x))$, $\forall x \in |C|$, is equivalent to $f(st(\sigma)) = f(st^m(\sigma))$, $\forall \sigma \in C$. □

In order to employ briefer notations, we give the following:

DEFINITION 11. - *Let C be a finite cellular complex and G a finite directed graph. A completely o-regular function $f:|C|\to G$, which is properly C-constant is called a function* pre-cellular w.r.t. C *or a* C-pre-cellular *function.*

PROPOSITION 5. - *Let C be a finite cellular complex and G a finite directed graph. Then every C-pre-cellular function $f:|C|\to G$ is characterized, up to complete o-homotopy, by the restriction of f to the set of the maximal cells of C.*

Proof. -Let $g:|C|\to G$ be a C-pre-cellular function which takes the same values as f on all the maximal cells of C. By Proposition 4 it results $\langle f(x)\rangle = f(st^m(x)) = g(st^m(x)) = \langle g(x)\rangle$, $\forall x\in|C|$. Since g is c. o-regular, it is $g(x)\in H(\langle g(x)\rangle) = H(\langle f(x)\rangle)$, i.e. g is an o-pattern of f and then g is c.o-homotopic to f. (See [5], Proposition 7). □

THEOREM 6. - (The third normalization theorem). *Let S be a compact triangulable space, G a finite directed graph and $f: S\to G$ a completely o-regular function. Then, for every finite cellular decomposition C of S with suitable mesh, there exists a C-pre-cellular function $h: S\to G$ which is completely o-homotopic to f.*

Proof. -Let C be a cellular decomposition of S with mesh $< r$, where $r = \frac{1}{2}inf(enl(A_1,\ldots,A_n))$, $\forall a_1,\ldots,a_n$ non-headed n-tuple of G (see Remark 2 to Theorem 3). Then we construct the function g by choosing, $\forall \sigma_i\in C$, a vertex in $H(\{f(\bar{\sigma}_i)\})$ rather than in $H(\{f(\sigma_i)\})$ (see Remark 1 to Theorem 3). Hence, $\forall x\in|C|$, it is $H(g(st^m(x)))\subseteq H(\langle g(x)\rangle)$. Given, indeed, a vertex $a\in H(g(st^m(x)))$ and a cell $\tau\in st^m(x)$ such that $g(\tau) = a$, i.e. $a\in H(\{f(\bar{\tau})\})$, we prove that a is a predecessor of all the vertices of $\langle g(x)\rangle$. In fact if $b\in\langle g(x)\rangle$ and a is not a predecessor of b, b is the image of a non-maximal cell σ, while, by definition of g, we have $b\in H(\{f(\bar{\sigma})\})$. Since $\sigma\subset\bar{\tau}$, and also $\bar{\sigma}\subset\bar{\tau}$, it is $b\in f(\bar{\tau})$. Hence a is not a head of $f(\bar{\tau})$. Contradiction.
By remarking that, $\forall x\in\sigma$, it is $g(st^m(x)) = g(st^m(\sigma))$, we can define the o-pattern h in the following way:

$$h(\sigma) = \text{ a vertex of } H(g(st^m(\sigma))),\quad \forall\sigma\in C.$$

The function h is properly C-constant since, if τ is a maximal cell, from $g(st^m(\tau)) = \{g(\tau)\}$ it results $h(\tau) = g(\tau)$. Hence, by definition, we have $h(\sigma)\in g(st^m(\sigma)) = h(st^m(\sigma))$, $\forall\sigma\in C$. □

REMARK. - If G is an undirected graph, it is not necessary to contruct also the o-pattern to obtain a properly quasi-constant functio In this case the condition is reduced to $h(\sigma)$ = a vertex of $g(st^m(\sigma))$.

4) *The third normalization theorem for homotopies.*

Let $e,f\colon S\to G$ be two functions pre-cellular w.r.t. two finite decompositions C and K of S and $F\colon S\times I\to G$ a complete o-homotopy between e and f. Then, for every sufficently fine finite cellular decomposition Γ of $S\times I$, by Theorem 6, the function F can be replaced by a Γ-pre-cellular function $\hat{h}\colon S\times I\to G$. In order that the function $\hat{h}$ may also be a homotopy between e and f, the restrictions of $\hat{h}$ to $S\times\{0\}$ and $S\times\{1\}$ must coincide with e and f. Hence it is necessary that $\hat{h}$ characterizes on $S\times\{0\}$ and $S\times\{1\}$ two decompositions $\tilde{C}$ and $\tilde{K}$ finer than C and K, since e and f are properly quasi-constant (see Remark to Definition 10). Nevertheless, as, for example, the value of the function $\hat{h}$ on $S\times\{0\}$ depends from the value assumed by the function F on the maximal cells of the star $st(\tilde{C})$, in general the restriction $\hat{h}/_{|\tilde{C}|}$ is different from e. Consequently, at first, we must replace the homotopy F by a homotopy M given by:

$$M(x,t) = \begin{cases} e(x) & \forall x\in S, \quad \forall t\in\left[0,\frac{1}{3}\right] \\ F(x,3t-1) & \forall x\in S, \quad \forall t\in\left[\frac{1}{3},\frac{2}{3}\right] \\ f(x) & \forall x\in S, \quad \forall t\in\left[\frac{2}{3},1\right] \end{cases}$$

Then we have to costruct suitable cellular decompositions of the three cylinders $S\times\left[0,\frac{1}{3}\right]$, $S\times\left[\frac{1}{3},\frac{2}{3}\right]$ and $S\times\left[\frac{2}{3},1\right]$.

PROPOSITION 7. - *Let S be a compact triangulable space, C a finite cellular decomposition of S, G a finite graph and $e\colon S\to G$ a properly C-constant function. If we consider the decomposition $L=\{\{0\},]0,1[,\{1\}\}$ of I and the product decomposition $\Gamma = C\times L$ of the cylinder $S\times I$, then the function $F\colon S\times I\to G$, given by $F(x,t)=e(x)$, $\forall x\in S$, $\forall t\in I$, is properly Γ-constant .*

Proof. - We have only to remark that a cell τ is maximal in Γ iff $\tau=\tau'\times]0,1[$, where τ' is a maximal cell in C. Then it results $F(\tau) = e(\tau')$. □

REMARK. - Since the restrictions $F/_{S\times\{0\}}$ and $F/_{S\times\{1\}}$ coincide with e, they are obviously C-constant.