Notes: (1) $\frac{4}{45}(11p+10) < p - \frac{1}{4}\sqrt{p} + \frac{25}{16}$ for $p \geq 47$.

(2) $\frac{4}{45}(11p+10) < p - \sqrt{p}+1$ for $p \geq 2017$.

20. k-CAPS IN PG(n,q), $n \geq 3$.

A k-cap in PG(n,q) is a set of k points no 3 collinear. Let $m_2(n,q)$ be the maximum value that k can attain. From §19, $m(2,q) = m_2(2,q)$. For $n \geq 3$, the only values known are as follows:

$$m_2(3,q) = q^2+1, \qquad q > 2;$$

$$m_2(d,2) = 2^d;$$

$$m_2(4,3) = 20;$$

$$m_2(5,3) = 56.$$

See [8] for a survey on these and similar numbers. The sets corresponding to these values for $m_2(d,q)$ have been classified apart from (q^2+1)-caps for q even with $q \geq 16$.

As for the plane, let $m_2'(n,q)$ be the size of the second largest complete k-cap. Then, from [9], chapter 18,

$$m_2'(3,2) = 5 \quad , \quad m_2'(3,3) = 8.$$

We now summarize the best known upper bounds for $m_2'(n,q)$ and $m_2(n,q)$.

THEOREM 20.1: ([7]) For q odd with $q \geq 67$,

$$m_2'(3,q) \leq q^2 - \frac{1}{4}q\sqrt{q} + 2q.$$

THEOREM 20.2: ([10]) For q even with $q > 2$,

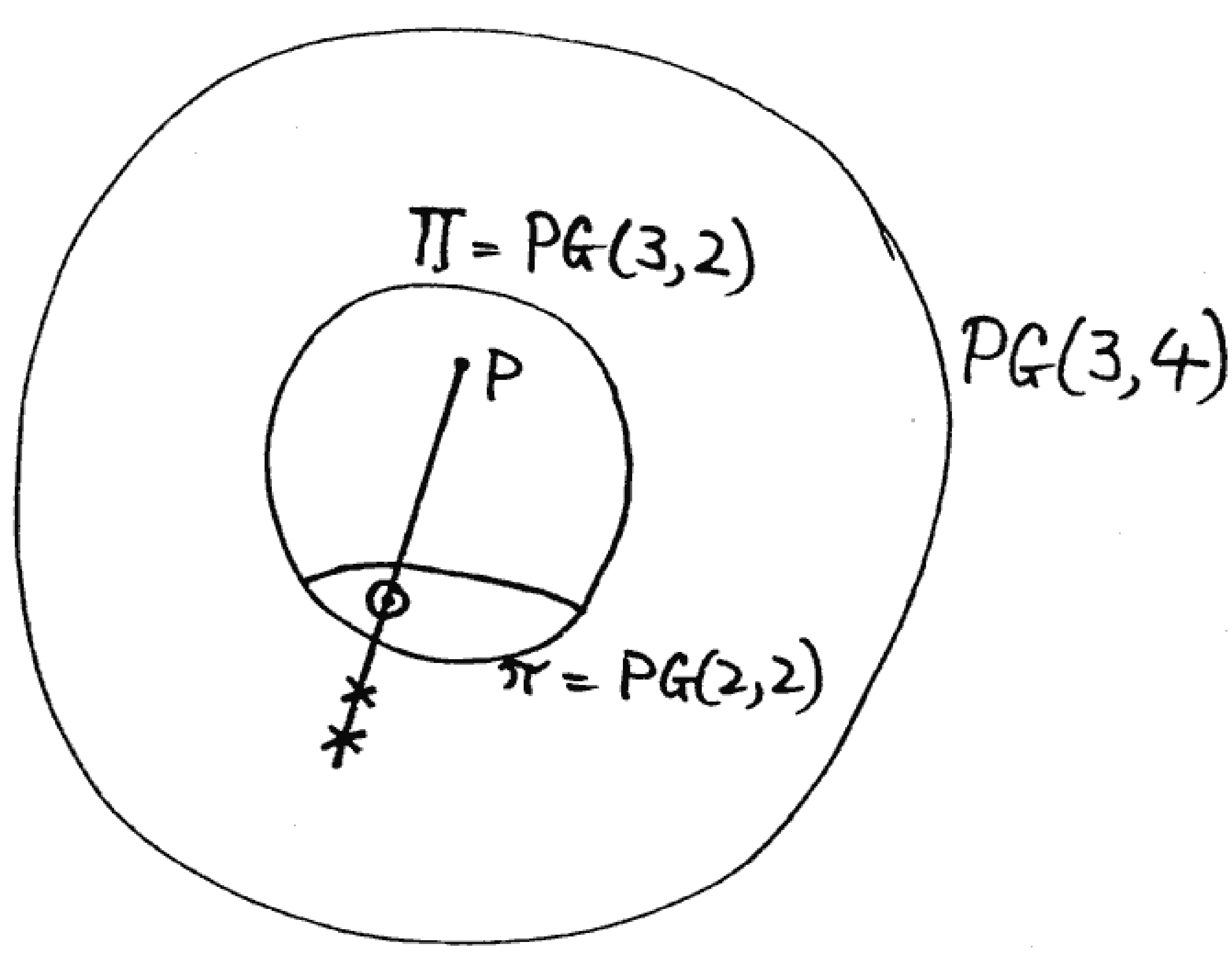
$$m_2'(3,q) \leq q^2 - \frac{1}{2}q - \frac{1}{2}\sqrt{q} + 2.$$

This gives that $m_2'(3,4) \leq 15$.

THEOREM 20.3: ([10]) $m_2'(3,4) = 14$.

In fact, a complete 14-cap in PG(3,4) is projectively unique and is obtained as follows.

Let π be a PG(2,2) in PG(3,4), let P be a point not in π, and let Π be a PG(3,2) containing P and π. Each of the seven lines joining P to a point of π contains three points in π and two points not in Π. The 14 points on the lines through P not in Π form the desired cap.



THEOREM 20.4: ([7]) For q odd, $q \geq 121$, $n \geq 4$,

$$m_2(n,q) < q^{n-1} - \frac{1}{4}q^{n-3/2} + 3q^{n-2}.$$

THEOREM 20.5: ([10]) For even, $q \geq 4$, $n \geq 4$,

$$m_2(n,q) \leq q^{n-1} - \frac{1}{2}q^{n-2} + \frac{5}{2}q^{n-3}.$$