19. AN IMPROVEMENT ON THE BOUND FOR m'(2,q) WHEN q IS PRIME

THEOREM 19.1: (Voloch [20]). For a prime $p \geq 7$,

$$m'(2,p) \leq \frac{44}{45}p + \frac{8}{9}.$$

Proof. A theorem of Segre (see [6], theorem 10.4.4) says that, for q odd with $q \geq 7$, we have $m'(2,q) \leq q - \frac{1}{4}\sqrt{q} + \frac{7}{4}$ and we follow the structure of this proof.

Let $\mathscr{K}$ be a complete k-arc with $k > \frac{44}{45}p+\frac{8}{9}$. Through each point P of $\mathscr{K}$ there are $t = p+2-k$ unisecants. The kt unisecants of $\mathscr{K}$ belong to an algebraic envelope Δ_{2t} of class 2t, which has a simple component Γ_n with $n \leq 2t$. For t=1, the envelope Δ_2 is the dual of a conic, $\mathscr{K}$ is a (q+1)-arc and so a conic. When $t \geq 2$, four cases are d stinguished.

(i) Γ_n <u>is a regular (rational) linear component</u>.

Here Γ_n is a pencil with vertex Q not in $\mathscr{K}$. Then $\mathscr{K}\cup\{Q\}$ is a (k+1)-arc and $\mathscr{K}$ is not complete.

(ii) Γ_n <u>is regular of class two.</u>

Here Γ_n is the dual of a conic $\mathscr{C}$, and $\mathscr{K}$ is contained in $\mathscr{C}$, [6] theorem 10.4.3.

(iii) Γ_n is irregular.

Suppose that Γ_n has M simple lines and d double lines, and let N=M+d. Then, by [6] lemma 10.1.1, it follows that $N \leq n^2$. Also by the definition of Δ_{2t} and Γ_n, there are at least $\frac{1}{2}n$ distinct lines of Γ_n through P; so $N \geq \frac{1}{k}$ kn. Therefore $k \leq 2N/n \leq 2n \leq 4t=$

$= 4(p+2-k)$. Thus $k \leq \frac{4}{5}(p+2) < \frac{44}{45}p + \frac{8}{9}$, a contradiction for $p \geq 5$.

(iv) Γ_n <u>is regular with</u> $n \geq 3$.

Either $n=2t \leq \frac{1}{2}p$ or $t > \frac{1}{4}p$. When $t > \frac{1}{4}p$, then $k=p+2-t < \frac{3}{4}p+2 < \frac{44}{45}p+\frac{8}{9}$ for $p \geq 5$.

When $n \leq \frac{1}{2}p$, then

$$N \leq \frac{2n}{5}\{5(n-2)+p\}$$

for $n \geq 5$ by theorem 14.1, note (3); for $n \geq 3$ it follows from theorem 11.5 when we note that $n \leq \frac{1}{2}p$ implies $v_i = i$ by theorem 11.4, corollary 1 (ii).

As in (iii), $N \geq \frac{1}{2}kn$. So

$$\frac{1}{2}kn \leq N \leq \frac{2n}{5}\{5(n-2) + p\},$$

$$k \leq \frac{4}{5}\{5(n-2)+p\},$$

$$k \leq \frac{4}{5}\{5(2t-2)+p\}.$$

Substituting $t = p+2-k$ gives

$$k \leq \frac{4}{5}\{10(p+1-k)+p\},$$

$$k \leq \frac{4}{45}(11p + 10),$$

the required contradiction.

COROLLARY: For any prime $p \geq 311$,

$$\frac{1}{2}(p+[2\sqrt{p}]) \leq m'(2,p) \leq \frac{4}{45}(11p+10).$$

Notes: (1) $\frac{4}{45}(11p+10) < p - \frac{1}{4}\sqrt{p} + \frac{25}{16}$ for $p \geq 47$.

(2) $\frac{4}{45}(11p+10) < p - \sqrt{p}+1$ for $p \geq 2017$.

20. k-CAPS IN PG(n,q), $n \geq 3$.

A k-cap in PG(n,q) is a set of k points no 3 collinear. Let $m_2(n,q)$ be the maximum value that k can attain. From §19, $m(2,q) = m_2(2,q)$. For $n \geq 3$, the only values known are as follows:

$$m_2(3,q) = q^2+1, \qquad q > 2;$$

$$m_2(d,2) = 2^d;$$

$$m_2(4,3) = 20;$$

$$m_2(5,3) = 56.$$

See [8] for a survey on these and similar numbers. The sets corresponding to these values for $m_2(d,q)$ have been classified apart from (q^2+1)-caps for q even with $q \geq 16$.

As for the plane, let $m_2'(n,q)$ be the size of the second largest complete k-cap. Then, from [9], chapter 18,

$$m_2'(3,2) = 5 \quad , \quad m_2'(3,3) = 8.$$

We now summarize the best known upper bounds for $m_2'(n,q)$ and $m_2(n,q)$.

THEOREM 20.1: ([7]) For q odd with $q \geq 67$,

$$m_2'(3,q) \leq q^2 - \frac{1}{4}q\sqrt{q} + 2q.$$

THEOREM 20.2: ([10]) For q even with $q > 2$,