19. AN IMPROVEMENT ON THE BOUND FOR $m'(2,q)$ WHEN $q$ IS PRIME

**THEOREM 19.1:** (Voloch [20]). For a prime $p \geq 7$,

$$m'(2,p) \leq \frac{44}{45}p + \frac{8}{9}.$$

**Proof.** A theorem of Segre (see [6], theorem 10.4.4) says that, for $q$ odd with $q \geq 7$, we have $m'(2,q) \leq q - \frac{1}{4}\sqrt{q} + \frac{7}{4}$ and we follow the structure of this proof.

Let $\mathcal{X}$ be a complete $k$-arc with $k > \frac{44}{45}p + \frac{8}{9}$. Through each point $P$ of $\mathcal{X}$ there are $t = p+2-k$ unisecants. The $kt$ unisecants of $\mathcal{X}$ belong to an algebraic envelope $\Delta_{2t}$ of class $2t$, which has a simple component $\Gamma_n$ with $n \leq 2t$. For $t=1$, the envelope $\Delta_2$ is the dual of a conic, $\mathcal{X}$ is a $(q+1)$-arc and so a conic. When $t \geq 2$, four cases are distinguished.

(i) $\Gamma_n$ is a regular (rational) linear component.

Here $\Gamma_n$ is a pencil with vertex $Q$ not in $\mathcal{X}$. Then $\mathcal{X} \cup \{Q\}$ is a $(k+1)$-arc and $\mathcal{X}$ is not complete.

(ii) $\Gamma_n$ is regular of class two.

Here $\Gamma_n$ is the dual of a conic $\mathcal{C}$, and $\mathcal{X}$ is contained in $\mathcal{C}$, [6] theorem 10.4.3.

(iii) $\Gamma_n$ is irregular.

Suppose that $\Gamma_n$ has $M$ simple lines and $d$ double lines, and let $N=M+d$. Then, by [6] lemma 10.1.1, it follows that $N \leq n^2$. Also by the definition of $\Delta_{2t}$ and $\Gamma_n$, there are at least $\frac{1}{2}n$ distinct lines of $\Gamma_n$ through $P$; so $N \geq \frac{1}{k}kn$. Therefore $k \leq \frac{N}{n} \leq 2n \leq 4t$.
= 4(p+2-k). Thus \( k \leq \frac{4}{5}(p+2) < \frac{44}{45}p + \frac{8}{9} \), a contradiction for \( p \geq 5 \).

(iv) \( \Gamma_n \) is regular with \( n \geq 3 \).

Either \( n=2t < \frac{1}{2}p \) or \( t > \frac{1}{4}p \). When \( t > \frac{1}{4}p \), then \( k=p+2-t < \frac{3}{4}p+2 < \frac{44}{45}p+\frac{8}{9} \) for \( p \geq 5 \).

When \( n \leq \frac{1}{2}p \), then

\[
N \leq \frac{2n}{5}(5(n-2)+p)
\]

for \( n \geq 5 \) by theorem 14.1, note (3); for \( n \geq 3 \) it follows from theorem 11.5 when we note that \( n \leq \frac{1}{2}p \) implies \( v_1 = i \) by theorem 11.4, corollary 1 (ii).

As in (iii), \( N \geq \frac{1}{2}kn \). So

\[
\frac{1}{2}kn \leq N \leq \frac{2n}{5}(5(n-2)+p),
\]

\[
k \leq \frac{4}{5}(5(n-2)+p),
\]

\[
k \leq \frac{4}{5}(5(2t-2)+p).
\]

Substituting \( t = p+2-k \) gives

\[
k \leq \frac{4}{5}(10(p+1-k)+p),
\]

\[
k \leq \frac{4}{45}(11p + 10),
\]

the required contradiction.

**COROLLARY:** For any prime \( p \geq 311 \),

\[
\frac{1}{2}(p+[2\sqrt{p}]) \leq m'(2,p) \leq \frac{4}{45}(11p+10).
\]
Notes: (1) \( \frac{4}{45} (11p+10) < p - \frac{1}{4}\sqrt{p} + \frac{25}{16} \) for \( p \geq 47 \).

(2) \( \frac{4}{45} (11p+10) < p - \sqrt{p} + 1 \) for \( p \geq 2017 \).

20. k-CAPS IN PG(n,q), \( n \geq 3 \).

A k-cap in PG(n,q) is a set of k points no 3 collinear. Let \( m_2(n,q) \) be the maximum value that k can attain. From §19, \( m(2,q) = m_2(2,q) \). For \( n \geq 3 \), the only values known are as follows:

\[
m_2(3,q) = q^2+1, \quad q \geq 2;
m_2(d,2) = 2^d;
m_2(4,3) = 20;
m_2(5,3) = 56.
\]

See [8] for a survey on these and similar numbers. The sets corresponding to these values for \( m_2(d,q) \) have been classified apart from \((q^2+1)\)-caps for \( q \) even with \( q \geq 16 \).

As for the plane, let \( m_2(n,q) \) be the size of the second largest complete k-cap. Then, from [9], chapter 18,

\[
m_2'(3,2) = 5, \quad m_2'(3,3) = 8.
\]

We now summarize the best known upper bounds for \( m_2'(n,q) \) and \( m_2(n,q) \).

**Theorem 20.1:** ([7]) For \( q \) odd with \( q \geq 67 \),

\[
m_2'(3,q) \leq q^2 - \frac{1}{4}q\sqrt{q} + 2q.
\]

**Theorem 20.2:** ([10]) For \( q \) even with \( q > 2 \),