9. THE GENERALIZED WRONSKIAN

Consider the <u>generalized Wronskian</u>

$$W = \det \begin{bmatrix} D^{(\varepsilon_o)} f_o & \dots & D^{(\varepsilon_o)} f_n \\ \vdots & & \vdots \\ D^{(\varepsilon_n)} f_o & \dots & D^{(\varepsilon_n)} f_n \end{bmatrix}$$

Here the derivations are taken with respect to a separating variable t (dt is the image of t under the map $d : \bar{K}(\mathscr{C}) \to \Omega_{\bar{K}}$; see Fulton [3] p. 203).

The ε_i are required to satisfy the conditions:

(i) $0 = \varepsilon_o < \varepsilon_1 < \dots < \varepsilon_n$;

(ii) $W \neq 0$;

(iii) given $\varepsilon_o, \dots, \varepsilon_{i-1}$, then ε_i is chosen as small as possible such that $D^{(\varepsilon_o)} f, \dots, D^{(\varepsilon_{i-1})} f$ are linearly independent.

Then

(iv) the ε_i are the $(\mathscr{D}, P)$-orders at a general point P;

(v) $\varepsilon_i \leq r_i$ for any $r_o < \dots < r_n$ with $\det (D^{(r_i)} f_j) \neq 0$;

(vi) $\varepsilon_i \leq j_i$ for any P in $\mathscr{C}$;

(vii) the ε_i are called the $\mathscr{D}$-<u>orders</u> of $\mathscr{C}$.

The divisor

$$R = \mathrm{div}(W) + (\sum_0^n \varepsilon_i)\ \mathrm{div}(dt) + (n+1) \sum_P e_P P,$$

where dt is the differential of t and $e_P = -\min_i \mathrm{ord}_P f_i$, is the <u>ramification divisor</u> of $\mathcal{D}$ and depends only on $\mathcal{D}$. Putting R = $= \Sigma r_P P$, we have

$$\deg R = \Sigma r_P = (2g-2)\Sigma\varepsilon_i + (n+1)d.$$

THEOREM 9.1: $r_P \geq \sum_{i=0}^{n}(j_i - \varepsilon_i)$ with equality if and only if det $C \neq 0 \pmod p$, where $C = (c_{is})$ and $c_{is} = \binom{j_i}{\varepsilon_s}$.

COROLLARY: (i) R is effective.

(ii) $r_P = 0$ if and only if $j_i = \varepsilon_i$ for $0 \leq i \leq n$.

The points P where $r_P=0$ are called $\mathcal{D}$-<u>ordinary</u>; the others are called $\mathcal{D}$-<u>Weierstrass.</u> The number r_P is the <u>weight</u> of P. When $\mathcal{D}$ is the canonical series, the $\mathcal{D}$-Weierstrass points are simply the <u>Weierstrass points</u>. This coincides with the classical definition.

When $\varepsilon_i = i$, $0 \leq i \leq n$, then $\mathcal{D}$ is <u>classical</u>. Next, the estimate $\varepsilon_i \leq j_i$ is improved.

THEOREM 9.2: (i) Let P on $\mathcal{C}$ have $(\mathcal{D},P)$-orders $j_0, \ldots, j_n$ and suppose that det $C' \not\equiv 0 \pmod p$, where $C' = (c'_{is})$ and $c'_{is} = \binom{j_i}{r_s}$, then $D^{(r_0)}f, \ldots, D^{(r_n)}f$ are linearly independent and $\varepsilon_i \leq r_i$.

(ii) If $\prod_{i>s} (j_i - j_s)/(i-s) \not\equiv 0 \pmod p$, then $\mathcal{D}$ is classical and $r_P = \sum_{i=0}^{n} (j_i - i)$

(iii) If $p > d$ or $p=0$, then $r_p = \sum_0^n (j_i - i)$ for all P in $\mathscr{C}$.

(iv) If ε is a $\mathscr{D}$-order and μ is an integer with $\binom{\varepsilon}{\mu} \not\equiv 0 \pmod p$, then μ is also a $\mathscr{D}$-order.

(v) If ε is a $\mathscr{D}$-order and $\varepsilon < p$, then $0, 1, \ldots, \varepsilon - 1$ are also $\mathscr{D}$-orders.

Entering into this theorem is the classical result of Lucas.

LEMMA 9.3: Let $A = a_o + a_1 p + \ldots + a_m p^m$ and $B = b_o + b_1 p + \ldots + b_n p^m$ be p-adic expansions of A and B with respect to the prime p; that is, $0 \leq a_i, b_i \leq p-1$. Then

(i) $\binom{A}{B} \equiv \binom{a_o}{b_o}\binom{a_1}{b_1}\ldots\binom{a_m}{b_m} \pmod p$;

(ii) $\binom{A}{B} \not\equiv 0 \pmod p$ if and only if $a_i \geq b_i$, all i;

Proof: $(1+x)^A = (1+x)^{\Sigma a_i p^i}$

$$= (1+x)^{a_o}(1+x^p)^{a_1} \ldots (1+x^{p^m})^{a_m} .$$

Now, the result follows by comparing the coefficient of x^B on both sides.