# CONGRUENCES ON REGULAR SEMIGROUPS

H. Mitsch

University of Vienna/Austria

## 1. Generalities.

Let $(S,.)$ be a semigroup; an element $a \in S$ will be called <u>regular</u> if there is some $x \in S$ such that $axa = a$. $S$ is called regular if each element of $S$ is regular. Notice that if $a = axa$, then for $y = xax$ we have that

$$a = aya \text{ and } y = yay.$$

For every $a \in S$, denote $V(a) = \{ x \in S \mid a = axa,\ x = xax \}$; hence $S$ is regular iff $V(a) \neq \emptyset$ for all $a \in S$. Furthermore, if $a = axa$, then clearly $ax$ and $xa$ are idempotents; denote by $E_X$ the set of all idempotents in $X$ for any subset $X$ of $S$.

<u>Examples</u> for regular semigroups are:

Idempotent semigroups (bands); groups; unions of groups (completely regular semigroups); inverse semigroups (i.e. $|V(a)| = 1$ for all $a \in S$); $(T_X, \circ)$ the semigroup of all mappings of the set $X$ into itself with respect to composition of functions; $(P_X, \circ)$ the semigroup of all partial mappings of $X$ into itself; $(L_V, \circ)$ the semigroup of all linear mappings of the vector space V into itself; $(M_n(F),\cdot)$ the semigroup of all $(n\times n)$-matrices over the field F; direct products and homomorphic images of regular semigroups, but not subsemigroups (the subsemigroup of all natural numbers of the additive group of all integers is not regular), hence the class of all regular semigroups does not form a variety.

Not only since appearance of the book of M.Petrich [22] on "Inverse Semigroups", the theory of regular semigroups has attracted wide attention. This is particularly true for the study of congruences. They play a central role in many of the structure theorems and various considerations of semigroups in general. The efficient handling of the congruences is a basic prerequisite for their useful application. For this reason, the most important facts concerning congruences on regular semigroups are collected here, with particular emphasis on

- the construction of general congruences, and
- the explicite form of special types of congruences.

An equivalence relation $\rho$ on a semigroup $(S,.)$ is called a congruence if

$$a\,\rho\, b\,(a,b\in S) \text{ implies that } ac\rho bc \text{ and } ca\rho cb \text{ for all } c\in S.$$

The set $S/\rho$ of all congruence classes $a\rho$ $(a\in S)$ of $\rho$ forms a semigroup with respect to the multiplication

$$(a\,\rho) * (b\,\rho) = (ab)\rho$$

and is a homomorphic image of $(S,.)$. Conversely, every homomorphic image of $(S,.)$ is obtained by a congruence on $(S,.)$.

With respect to the partial ordering

$$\rho \leq \tau \text{ iff } a\rho\, b \text{ implies that } a\tau\, b \;(a,b \in S),$$

the set C(S) of all congruences on (S,.) forms a complete lattice with least element $\varepsilon$, the identity relation, and greatest element $\omega$, the universal relation. It is easily seen that if $\rho,\tau \in \mathcal{C}(S)$ such that $\rho \leq \tau$ then $(S/\tau, *)$ is a homomorphic image of $(S/\rho, *)$.

In general, particular homomorphic images of a given semigroup S are of special interest; thus particular congruences on S have to be found. If $\mathcal{C}$ denotes any class of semigroups, then a congruence $\rho$ is called a $\underline{\mathcal{C}\text{- congruence}}$ if the semigroup $(S/_\rho, *)$ belongs to the class $\mathcal{C}$. For example, let $\mathcal{C}$ the class of all groups, semilattices, bands,resp.; then a $\mathcal{C}$ - congruence is called a group congruence, semilattice congruence, band congruence, respectively. In particular, we will be interested in the least or the greatest congruence for some given class $\mathcal{C}$ (with respect to the partial order $\leq$ above):

$\beta$ ....... the least band congruence
$\sigma$ ....... the least group congruence
$\gamma$ ....... the least right-group congruence
$\eta$ ....... the least semilattice congruence
$Y$ ....... the least inverse congruence
$\nu$ ...... the least semilattice of groups congruence
$\pi$ ....... the greatest idempotent-pure congruence (i.e. $a\pi\, e$, $a \in S$, $e \in E_S \rightarrow a \in E_S$)
$\mu$ ....... the greatest idempotent separating congruence (i.e. $e\mu f, e,f \in E_S \rightarrow e=f$)

For details see section 3. below. Note that for general semigroups not all of these congruences exist; but they do exist for any regular semigroup (see Howie-Lallement [8]).

If, for example, we consider the least group congruence on a semigroup $(S,.)$ - if exists, then we have for every $\tau \in C(S)$ satisfying $\tau \geq \sigma$ that $(S/_{\tau}, *)$ is again a group, a homomorphic image of the group $(S/_{\sigma}, *)$. Thus, one can say that the least group congruence on S gives the greatest group homomorphic image of S.

A very useful result on congruences on regular semigroups is the following

<u>Lemma 1.1.</u> (Lallement [9]) Let $(S,.)$ be a regular semigroup and $\rho$ any congruence on S. If $a\rho \in S/_{\rho}$ is idempotent then there is some $e \in E_S$ such that $a\rho = e\rho$.

## 2. General congruences.

Our first aim will be the description of an arbitrary congruence on a regular semigroup. For this, let us consider first the special case of a group.

If G is a group then it is known that for every normal subgroup N of G, the relation $\rho_N$ on G defined by

$$a \ \rho_N \ b \leftrightarrow ab^{-1} \in N$$

is a congruence on G, and conversely that for a congruence $\rho$ on G the

$\rho$-class containing the identity e of G is a normal subgroup N of G such that $\rho_N = \rho$. Furthermore, all of $\rho$ can be reconstructed from any one of its classes, in particular the class $e\rho = N$. For semigroups, such a reconstruction of a congruence $\rho$ from a single $\rho$-class is not possible, in general. If S is regular, we have at least the following result:

Lemma 2.1. (Clifford-Preston [1]). For a regular semigroup S, any congruence $\rho$ on S is uniquely determined by the $\rho$-classes containing idempotents.

Note that this result does not tell us how to reconstruct all of the congruence $\rho$ from the set of all idempotent $\rho$-classes. Various attempts have been made to find an analog of the connection between congruences on groups and normal subgroups. For *inverse* semigroups, G.B.Preston abstractly characterised the set of all idempotent classes of a congruence on S and gave a construction of the congruence associated with such a *kernel normal system* (see Clifford-Preston [1]). Meakin [14] generalized this result to regular semigroups:

Definition. A set $A = \{A_i \mid i \in I\}$ of disjoint subsets of a regular semigroup S is called a *kernel system on S* if

(1) $A_i \cap A_j = \emptyset$ for all $i \neq j$ in I

(2) each $A_i$ contains an idempotent of S and each idempotent of S belongs to some $A_j (j \in I)$

(3) $x A_i y \cap A_j \neq \emptyset$ implies that $xA_iy \subseteq A_j$ for $x, y \in S^1, i, j \in I$.

The construction of the unique congruence $\rho$ having every $A_i \in A$ as idempotent $\rho$-class, is now the following:

Theorem 2.2. (Meakin [14]) Let $(S,.)$ be a regular semigroup and $A = \{A_i | i \in I\}$ a kernel system on S. Then the unique congruence on S with all $A_i$ $(i \in I)$ as idempotent classes is given by

$$a \, \rho_A \, b \leftrightarrow a' \in V(a),\ b' \in V(b):\ aa', ba' \in A_i,\ b'a,\ b'b \in A_j \text{ for some } i,j \in I.$$

This result suffers - as in the inverse case - from the disadvantage that the conditions imposed on a kernel system are very difficult to utilize. For inverse semigroupsanother approach proved very useful: it is possible to reconstruct any congruence $\rho$ from the set-theoretical union of all the idempotent $\rho$-classes (the kernel of $\rho$) taking into account the partition on the set of all idempotents induced by $\rho$ (the trace of $\rho$); see M.Petrich [23]. Following this idea, Pastijn-Petrich [19] introduced the concept of congruence pair for a regular semigroup generalizing the corresponding notion for inverse semigroups. The exact definitions are the following:

Definition. Let S be a regular semigroup; then for any congruence $\rho$ on S consider the following two characteristic concepts:

1) $\mathrm{t\,r}\,\rho = \rho|E_S$ is called the trace of $\rho$,

2) $\ker\rho = \{a \in S |\ a\rho e \text{ for some } e \in E_S\}$ is called the kernel of $\rho$.

Note that $\mathrm{tr}\rho$ is the restriction of $\rho$ to the subset $E_S$ of S and thus yields a certain partition of $E_S$. Furthermore, $\ker\rho$ is the set-theoretical union of all idempotent $\rho$-classes.

It was shown by R.Feigenbaum [4] that every congrüence $\rho$ can be reconstructed from its trace and kernel. We give this result in the formulation of Pastijn-Petrich [18] which uses Green s relation $\mathcal{L}$ and $\mathcal{R}$; recall that for a regular semigroup S, $a\mathcal{L}b$ iff $Sa = Sb$ and $a\mathcal{R}b$ iff $aS = bS$.

Theorem 2.3. (Pastijn-Petrich [19]. Any congruence $\rho$ on a regular semigroup S with $\ker\rho = K$ and $\operatorname{tr}\rho = \tau$ can be described in the following way:

$a\rho b \leftrightarrow a(\mathcal{L}\tau\mathcal{L}\tau\mathcal{L} \cap \mathcal{R}\tau\mathcal{R}\tau\mathcal{R})b$, $ab' \in K$ for some (all) $b' \in V(b)$.

Note. It was proved by G. Gomes [5] that $\rho$ can be obtained also in the following way:

$a\rho b \leftrightarrow aa'\rho\, bb'aa'$, $b'b\rho\, b'ba'a$, $ab' \in K$ for some (all) $a' \in V(a)$, $b' \in V(b)$.

As it was observed above, to every congruence $\rho$ on S there can be associated the pair $(\ker\rho, \operatorname{tr}\rho)$. But the problem is to find conversely all congruences on S. In general, for a pair $(K, \tau)$ with $K \subseteq S$ and $\tau$ an equivalence on $E_S$, there is not always a congruence $\rho$ on S such that $K = \ker\rho$ and $\tau = \operatorname{tr}\rho$. Thus, the pairs $(\ker\rho, \operatorname{tr}\rho)$, $\rho \in C(S)$, have to be characterized abstractly in order to give all pairs $(K, \tau)$ by means of which a congruence on S can be defined.

For inverse semigroups S this attempt was successful in the following way (Petrich [23]): if $K \subseteq S$, $\tau \in C(E_S)$, then the pair $(K, \tau)$ is called a congruence-pair if

1) K satisfies: (i) $E_S \subseteq K$; (ii) $a \in K \rightarrow a^{-1} \in K$, (iii) $a^{-1}Ka \subseteq K$ $\forall a \in S$

   (where $a^{-1}$ denotes the unique element $a' \in V(a)$)

2) $\tau$ satisfies: $e\tau f$, $a \in S$ imply that $a^{-1}ea\,\tau\, a^{-1}fa$

3) $ae \in K$, $a^{-1}a\tau e$ $(a \in S, e \in E_S) \rightarrow a \in K$

4) $aa^{-1}\tau a^{-1}a$ for all $a \in S$.

Then for any <u>inverse</u> semigroup S and every congruence $\rho$ on S the pair $(\ker \rho, \operatorname{tr}\rho)$ is a congruence pair, and conversely, for every congruence pair $(K,\tau)$ the relation

$$a\ \rho_{(K,\tau)}\ b \leftrightarrow a^{-1}a\ \tau\ b^{-1}b,\ ab^{-1} \in K$$

is a congruence on S such that $\ker \rho_{(K,\tau)} = K$, $\operatorname{tr}\rho_{(K,\tau)} = \tau$.

For <u>regular</u> semigroups S, Pastijn-Petrich [19] found an abstract characterization of those pairs $(K, \tau)$ for which a congruence $\rho$ on S can be defined in an analogous way. The first trivial observation is that K has to be the kernel of some congruence, which is equivalent to say that $K = \ker \pi_K$, where $\pi_K$ is defined on S by

$$a\ \pi_K\ b \leftrightarrow xay \in K \text{ is equivalent } xby \in K (x, y \in S^1).$$

Also, $\tau$ has to be the trace of some congruence, which is equivalent to the requirement that $\tau = \operatorname{tr} \tau^*$ (where $\tau^*$ denotes the congruence on S generated by the equivalence $\tau$ on $E_S$): see Pastijn-Petrich [19].

The key to the theory-similar to the inverse case-is the following concept.

<u>Definition</u>. (Pastijn-Petrich [19]). Let S be a regular semigroup, $K \subseteq S$, $\tau$ an equivalence on $E_S$; then a pair $(K, \tau)$ is called a <u>congruence-pair</u> if

(i) K is a normal subset of S(i.e. K is the kernel of some congruence on S)

(ii) $\tau$ is a normal equivalence on $E_S$ (i.e. $\tau$ is the trace of some congruence on S)

(iii) $K \subseteq \ker (\mathcal{L}\tau\mathcal{L}\tau\mathcal{L} \cap \mathcal{R}\tau\mathcal{R}\tau\mathcal{R})^\circ$ (where for any equivalence $\xi$ on S, $\xi^\circ$ denotes the greatest congruence on S contained in $\xi$ )

(i) $\tau \leq \operatorname{tr} \pi_K$.

Note that in case that S is an inverse semigroup, this definition of congruence-pair reduces to that given above.

With this concept we are ready for the construction of all congruences on a regular semigroup, which is completely analogue to the situation in the inverse case.

<u>Theorem 2.4</u>. (Pastijn-Petrich [ 19 ]). If $(K,\tau)$ is a congruence-pair of the regular semigroup S, then the relation $\rho_{(K,\tau)}$ defined as in Theorem 2.3 is the unique congruence $\rho$ on S for which $\ker\rho = K$, $\operatorname{tr}\rho = \tau$.
Conversely, if $\rho$ is a congruence on S then $(\ker\rho, \operatorname{tr}\rho)$ is a congruence pair of S and $\rho_{(\operatorname{Ker}\rho, \operatorname{tr}\rho)} = \rho$.

An obvious, but very useful consequence is the following

<u>Corollary 2.5</u>. (Pastijn-Petrich [ 19 ]). Let $(C(S),\leq)$ be the lattice of all congruences and $Cp(S)$ the set of all congruence-pairs of a regular semigroup S, partially ordered by: $(K,\tau) \leq (K',\tau')$ iff $K \subseteq K'$, $\tau \leq \tau'$.
Then the mappings $\rho \to (\ker\rho, \operatorname{tr}\rho)$, $(K,\tau) \to \rho_{(K,\tau)}$ are mutually inverse isomorphisms of the lattices $C(S)$ and $Cp(S)$.

The special case of <u>orthodox</u> semigroups is worthy of note. A regular semigroup $(S,.)$ is called orthodox if $E_S$ forms a subsemigroup of S.
Note that an inverse semigroup S can be characterized as a regular semigroup, in which all idempotents commute; thus every inverse semigroup is orthodox. The concept of congruence-pair reduces in this case to a set of axioms which is strongly reminiscent to the inverse case.

Gomes [5] called a pair $(K,\tau)$ a congruence-pair of the orthodox semigroup $(S,.)$ if

1) $K$ satisfies: (i) $E_S \subseteq K$; (ii) $a \in K \rightarrow a' \in K$ for some [all] $a' \in V(a)$; (iii) $a'Ka \subseteq K$ for all $a \in S$, $a' \in V(a)$

2) $\tau$ satisfies: $e \tau f$, $a \in S$ imply that $a'ea \tau a'fa$ (note that $a'ea \in E_S$ for all $a \in S$, $a' \in V(a)$, $e \in E_S$)

3) $ae \in K$, $a'a \tau e$ $(a \in S,\ e \in E_S) \rightarrow a \in K$

4) $a'ea \ \tau \ a'a'eaa$ for all $a \in S$, $e \in E_S$.

Then Gomes [5] showed that for any orthodox semigroup $S$, if $(K,\tau)$ is a congruence-pair of $S$ then $\rho(K,\tau)$ defined by:

$$a \, \rho_{(K,\tau)} b \leftrightarrow \exists a' \in V(a),\ b' \in V(b):\ aa' \tau bb'\ \ aa', b'b \tau b'b\ a'a\ ,\ ab' \in K$$

is a congruence on $S$ with kernel $K$ and trace $\tau$. Conversely, if $\rho$ is a congruence on $S$ then $(\ker\rho, \operatorname{tr}\rho)$ is a congruence-pair of $S$ and $\rho = \rho(\ker\rho, \operatorname{tr}\rho)$. Also, the mappings $\rho \rightarrow (\ker\rho, \operatorname{tr}\rho)$, $(K,\tau) \rightarrow \rho_{(K,\tau)}$ are mutually inverse lattice isomorphisms between $(C(S), \leq)$ and $(Cp(S), \leq)$.

In order to illustrate the construction of all congruences on a regular semigroup $S$, some special cases will be considered. Compare also with the explicite form of certain congruences given in section 3. below.

a) $K = E_S$, $\tau = \varepsilon$

It is easily seen that $(E_S, \varepsilon)$ is a congruence-pair of $S$ defining the identity relation on $S$: $\rho_{(E_S,\varepsilon)} = \varepsilon$ .

b) $K = S, \tau = \omega$

It is immediate that $(S, \omega)$ is a congruence-pair of S defining the universal relation on S: $\rho_{(S,\omega)} = \omega$ .

c) $K = E_S, \tau = \omega$

$(E_S, \omega)$ is a congruence-pair of S iff tr $\pi_{E_S} = \omega$, i.e. iff for all $e,f \in E_S$

$$xey \in E_S \text{ is equivalent to } xfy \in E_S \quad (x,y \in S^1).$$

In this case, S is orthodox (put $x = e$, $y = 1$). Note that conversely, if S is orthodox then $(E_S, \omega)$ is not necessarily a congruence-pair. In fact, consider $S=T_2$ , the semigroup of all transformations on the set $X = \{1,2\}$. Then S is orthodox, but for $x = \alpha$, $e = \alpha_1$, $f = y = id$ (where $\alpha(1) = 2$, $\alpha(2) = 1$; $\alpha_1(x) = 1$; $\alpha_2(x) = 2$ for all $x \in X$), $\alpha \circ \alpha_1 \circ id = \alpha_2 \in E_S$ and $\alpha \circ id \circ id = \alpha \notin E_S$. Furthermore, if $(E_S, \omega)$ is a congruence-pair then by 2.5. $\rho_{(E_S,\omega)} = \sigma$ – the least group congruence on S (since for every group congruence $\rho$ on S, tr $\rho = \omega$ ); it is given explicitely by

$$a \sigma b \leftrightarrow ab' \in E_S \text{ for some (all) } b' \in V(b).$$

Also, in this case $\rho(E_S, \omega) = \pi$ , the greatest idempotent-pure congruence on S (since for every such congruence $\rho$, ker $\rho = E_S$).

d) $K$ normal, $\dot{\tau} = \omega$

$(K, \omega)$ is a congruence pair of S iff tr $\pi_K = \omega$, i.e. iff for all $e,f \in E_K$

$$xey \in K \text{ is equivalent to } xfy \in K (x,y \in S^1).$$

In this case, $\rho_{(K,\omega)}$ is a group congruence given by

$$a\, \rho_{(K,\omega)} b \leftrightarrow ab' \in K \quad \text{for some (all) } b' \in V(b).$$

Thus, by 2.5, the least group congruence $\sigma$ on S is defined by the least normal subset K of S satisfying the condition at the beginning of this paragraph.

e) $K = E_S, \tau$ normal

$(E_S, \tau)$ is a congruence-pair of S iff $\tau \leq \mathrm{tr}\,\pi_{E_S}$ , i.e. iff for all $e,f \in E_S$,

$$e\, \tau f \rightarrow xey \in E_S \text{ is equivalent to } xfy \in E_S \ (x,y \in S^1).$$

In this case, $\rho_{(E_S,\tau)}$ is an idempotent-pure congruence on S (since for every such congruence $\rho$ on S, $\ker \rho = E_S$). Thus by 2.5, the greatest idempotent-pure congruence $\pi$ on S is defined by the greatest normal equivalence $\tau$ on $E_S$ satisfying the condition at the beginning of this paragraph.

f) $K = S$, $\tau = \varepsilon$

$(S,\varepsilon)$ is congruence-pair of S iff $\ker \mathcal{H}^\circ = S$ (where $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$).
We shall see that this is the case iff S is a band of groups (i.e. S is a union of groups and $\mathcal{H}$ is a congruence on S; see Petrich [ 21 ] , IV.1.7).

In fact, if S is a band of groups, then $\mathcal{H}^\circ = \mathcal{H}$ and for every $a \in S$, $a \in H_e$ for some $e \in E_S$; thus $a \in \ker \mathcal{H} = \ker \mathcal{H}^\circ$, i.e. $\ker \mathcal{H}^\circ = S$. Conversely, suppose that $\ker \mathcal{H}^\circ = S$. Then for every $a \in S$ there is some $e \in E_S$ such that $a\, \mathcal{H}^\circ e$, hence $a \mathcal{H} e$ and S is the union of the groups $H_e$ (see Clifford-Preston [1 ], 2.16). Let $a \mathcal{H} b$ $(a,b \in S)$; by Lallement [9], $\mathcal{H}^\circ = \mu$ hence $\ker \mu = S$ ($\mu$ the greatest idempotent-separating congruence on S). Thus, $a\, \mu\, e$ for some $e \in E_S$, and $b \mu f$ for some $f \in E_S$. Consequently, $a \mathcal{H} e$ and $b \mathcal{H} f$, thus $e \mathcal{H} f$ and by Clifford-Preston [ 1] , 2.16, $e = f$. Hence, $a \mu e$ and $b \mu e$, thus $a \mu b$. Since $\mu$ is a congruence, it follows that $ac \mu bc$, $ca \mu cb$ for all $c \in S$. Now by Lallement [9] $\mu \leq \mathcal{H}$ , hence $ac \mathcal{H} bc$ and $ca \mathcal{H} cb$ for all $c \in S$. Consequently, $\mathcal{H}$ is a congruence and S is a band of groups.

In this case, $\rho_{(S,\varepsilon)} = \beta = \mu = \mathcal{H}$, where $\beta$ denotes the least band congruence on S. In fact, $\rho_{(S,\varepsilon)} = \beta$ since $\ker \mathcal{H}^{\circ} = S$ implies that for every $a \in S$ there is $e \in E_S$ such that $a \rho_{(S,\varepsilon)} e$, thus each $\rho_{(S,\varepsilon)}$ - class is idempotent. This means, that $\rho_{(S,\varepsilon)}$ is a band congruence. It is the least such (by 2.5), since for every band congruence $\rho$, $\ker \rho = S$ (by 1.1.). Furthermore, by Lallement [ 9 ], $\mathcal{H}^{\circ} = \mu$. Hence by hypothesis, $\ker \mu = S$ and $\mu = \beta$ by 2.5 (since $\ker \beta = S = \ker \mu$ and $\operatorname{tr} \beta = \operatorname{tr} \rho_{(S,\varepsilon)} = \varepsilon = \operatorname{tr} \mu$). But by Howie-Lallement [ 8 ], 1.3, $\mu = \mathcal{H}^{\circ} \leq \mathcal{H} \leq \beta$ , so that $\mathcal{H} = \beta = \mu$ (which again implies that $\mathcal{H}$ is a congruence).

g) K normal, $\tau = \varepsilon$

$(K, \varepsilon)$ is a congruence-pair of S iff $K \subseteq \ker \mathcal{H}^{\circ} = \ker \mu$. Recall (Latorre [ 10 ],12) that for every regular semigroup S,

$$\ker \mu = \{ a \in S \mid \exists\, a' \in V(a):\ a' ea = e \text{ for each idempotent } e \leq aa' \} .$$

Hence, $(K, \varepsilon)$ is a congruence-pair iff for every $a \in K$ there is some $a' \in V(a)$ such that $a'ea = e$ for each idempotent $e \leq aa'$. In this case, $\rho_{(K,\varepsilon)}$ is an idempotent separating congruence on S (since for every such congruence $\rho$ on S, $\operatorname{tr} \rho = \varepsilon$), explicitely given by

$$a \,\rho_{(K,\varepsilon)}\, b \leftrightarrow a \mathcal{H} b \text{ and } ab' \in K \text{ for some (all) } b' \in V(b).$$

Thus by 2.5, the greatest idempotent separating congruence $\mu$ on S is defined by the greatest normal subset K of S satisfying the above condition.

## 3. Particular congruences

The knowledge of a simple, explicite form of a particular congruence on a semigroup is of special importance when applying it in certain considerations. For regular semigroups, useful descriptions of some important congruences are known. A survey of these will be given including several different methods of characterization, which have been found up to now. Note that in section 2. some particular congruences have appeared already, given explicitely by their kernel and their trace .

### a) Group congruences

Croisot [ 2 ] found a description of all group congruences on an arbitrary semigroup S by means of particular subsemigroups of S. For every subset H of S and any $a \in S$ denote

$$a:H = \{ (x,y) \in S \times S \mid xay \in H \} .$$

Theorem 3.1. (Croisot [ 2 ]). Let S be a semigroup and H be a subsemigroup of S such that (i) $a:H \neq \emptyset$ for all $a \in S$, and (ii) $a:H \cap b:H \neq \emptyset$ $(a,b \in S) \rightarrow$ $\rightarrow a:H = b:H$. Then the relation $a \, \rho_H \, b \leftrightarrow a:H = b:H$ is a group congruence on S. Conversely, if $\rho$ is any group congruence on S then the identity class E of $S/\rho$ is a subsemigroup of S satisfying (i),(ii) and $\rho_E = \rho$ .

For regular semigroups, numerous other characterizations of group congruences are known. Note first that every cancellative congruence $\rho$ on a regular semigroup S is a group congruence, and conversely (since $S/\rho$ is a regular, cancellative semigroup, thus a group). Since the universal congruence on S is cancellative and since the intersection of all cancellative congruences on S is again a cancellative congruence on S, the least group-congruence can be described in the following way.

Theorem 3.2. (Masat [ 12 ] ). Let S be a regular semigroup; then the least group congruence $\sigma$ on S is given by $\sigma = \rho^t$ where

$$a \rho b \leftrightarrow e a e = e b e \text{ for some } e \in E_S$$

and $\rho^t$ means the transitive closure of $\rho$.

Note. If S is a conventional semigroup (i.e. S is regular and $a E_S a' \subseteq E_S$ for all $e \in S$, $a' \in V(a)$) then the unpleasant transitive closure of $\rho$ can be omitted and $\sigma = \rho$ (Masat [12]). In particular, this is true for every orthodox semigroup which was proved already by Meakin [14].

A more convenient description of $\sigma$ on a general regularsemigroup was given by Masat[ 12] by means of the reflexive subsemigroup of S generated by its idempotents $E_S$. A subset T of S is called reflexive if $ab \in T (a,b \in S)$ implies $ba \in T$.

Theorem 3.3. (Masat [ 12]). Let S be a regular semigroup; denote by T the reflexive subsemigroup of S generated by $E_S$ and by $T\omega = \{a \in S \mid ta \in T$ for some $t \in T\}$. Then the least group congruence $\sigma$ on S is given by

$$a \sigma b \leftrightarrow xa,\ xb \in T\omega \text{ for some } x \in S.$$

Note. If S is conventional, then $T\omega = \{ a \in S \mid ea \in E_S \text{ for some } e \in E_S \}$ (Masat [12] ). In particular, if S is E-unitary and regular (i.e. $ea, e \in E_S \rightarrow a \in E_S$) then S is orthodox (see Howie-Lallement [ 8 ], 2.1) and $T\omega = E_S$. Hence, in this case

$$a \sigma b \leftrightarrow xa, xb \in E_S \text{ for some } x \in S.$$

An other approach to the characterization of all group congruences on a regular semigroup S was found by Feigenbaum [4] using full and self-conjugate subsemigroups of S: a subset T of S is called full if $E_S \subseteq T$, and self-conjugate if $a'Ta \subseteq T$ for all $a \in S$, $a' \in V(a)$. Let C denote the set of all full and self-conjugate subsemigroups of S and let U be the intersection of all semigroups in C.

Theorem 3.4. (Feigenbaum [4]). For each $H \in C$, the relation

$$a \rho_H b \leftrightarrow xa = by \text{ for some } x, y \in H$$

is a group congruence on the regular semigroup S. The least group congruence $\sigma$ on S is given by $\sigma = \rho_U$.

Defining the closure of a subset H of a regular semigroup S as the set

$$H\omega = \{ a \in S \mid h a \in H \text{ for some } h \in H \},$$

Feigenbaum [4] showed that also for each $H \in C$

$$a \rho_H b \leftrightarrow ab' \in H\omega \text{ for some (all) } b' \in V(b).$$

Further details for the description of $\rho_H$ can be found in Latorre [10] .

In particular, he showed that

$$a\,\sigma\, b \leftrightarrow aub' \in U \text{ for some } u \in U \text{ and some (all) } b' \in V(b).$$

Now let $\bar{C}$ be the set of all closed subsemigroups in C, i.e. consider those full and self-conjugate subsemigroups H of S such that $H\omega = H$. Note that a closed subsemigroup H of a regular semigroup is necessarily regular. Feigenbaum [4] proved that the mapping

$$\bar{H} \to \rho_{\bar{H}}, \text{ where } a\rho_{\bar{H}}\, b \leftrightarrow ab' \in \bar{H} \text{ for some } b' \in V(b),$$

is a bijektive and inclusion preserving function of $\bar{C}$ onto the set of all group congruences on S. (For every group congruence $\rho$ on S, $\rho = \rho_{\bar{H}}$ with $\bar{H} = \ker\rho$). It is easily seen that the intersection $\bar{U}$ of all semigroups $\bar{H}$ in $\bar{C}$ is again closed. Consequently, we obtain that $\sigma = \rho_{\bar{U}}$ and

$$a\,\sigma\, b \leftrightarrow ab' \in \bar{U} \text{ for some (all) } b' \in V(b).$$

Note. For the much larger class of E-inversive semigroups an explicite description of all group congruences was given by Mitsch[17]. A semigroup S is called E-inversive if for every $a \in S$ there is some $x \in S$ such that $ax \in E_S$. This is equivalent to the condition that $I(a) = \{x \in S \mid ax, xa \in E_S\} \neq \emptyset$ for all $a \in S$. The characterization is strongly reminiscent to that given by Feigenbaum for the regular case (see Theorem 3.4).

b) Right group congruences

A group is right- and left-simple and also right- and left-cancellative. Weakening these properties one may ask for those homomorphic images of a semigroup S which are right groups, i.e. which are right-simple and left-cancellative (for several equivalent definitions see Clifford-Preston [1]).

A description of all right-group congruences on an arbitrary semigroup S was given by Massant [13] by means of group-congruences and left-zero congruences on S. Numerous characterizations of group congruences were given in a). Concerning right-zero congruences $\rho$ (i.e. such that $S/\rho$ is a right-zero semigroup: $xy = y \ \forall x,y \in S/\rho$) a description for arbitrary semigroups S can be found in Petrich [22], III. 1:

Let $L_S$ be the set of all left ideals $L \neq S$ of S such that $ab \in L$ implies $b \in L (a,b \in S)$; denoting by $L_x$ the least left ideal of S in $L_S$ containing $x \in S$, the following characterization of right-zero congruences on S holds: Let S be a semigroup and $\emptyset \neq A \subseteq L_S$; then the relation

$$a \rho_A b \leftrightarrow \text{for every } L \in A : \text{ either } a,b \in L \text{ or } a,b \notin L$$

is a right-zero congruence on S. The least right-zero congruence $\xi$ on S is given by $\xi = \rho_{L_S}$, or equivalently by $a \xi b \leftrightarrow L_a = L_b$.

Theorem 3.5. (Massat [13]). Let S be a semigroup; then a congruence $\rho$ on S, which is not a group- nor a right-zero congruence, is a right-group congruence iff is the intersection of a non-trivial group congruence on S and a right-zero congruence on S.

Since on regular semigroup S the least group congruence exists (see a) above) we obtain the following

Corollary 3.6. Let S be a regular semigroup; then the least right-group congruence on S is given by

$$a\ \gamma\ b \leftrightarrow L_a = L_b \text{ and } xa = by \text{ for some } x,y \in U,$$

where U is the intersection of all full and self-conjugate subsemigroups of S.

For the special case that S is regular with $E_S$ a rectangular band (i.e. $e\,f\,e = e$ for all $e,f \in E_S$), Massat [12] gave the following description of the least right-group congruence on S:

$$a\,\rho\,b \leftrightarrow ea = eb \text{ for all } e \in E_S.$$

Conversely, he showed that if the congruence $\rho$ so defined on a regular semigroup S is a right-group congruence on S then $E_S$ is a rectangular band.

c) The least inverse congruence

Reducing the condition that the homomorphic image of S has to be a group one can ask for those congruences $\rho$ on S, for which $S/\rho$ is an inverse semigroup. In the general case, there is no description of such congruences similar to the group case. Even the characterization of the least inverse congruence Y is not very satisfactory. It is based on the fact that a regular semigroup S is inverse iff the idempotents of S commute (see Petrich [22]).

Theorem 3.7. (Hall [6]). Let S be a regular semigroup; then the least inverse congruence Y on S is given by $Y = \rho^*$, where

$$a\,\rho\,b \longleftrightarrow a = ef,\quad b = fe \quad \text{for } e,f \in E_S$$

and $\rho^*$ denotes the congruence on S generated by $\rho$.

In the particular case that S is orthodox, Hall [6] gave the following explicite description of Y: $a\ Y\ b \leftrightarrow V(a)=V(b)$.

Also, he showed conversely that if for a regular semigroup S, Y is an inverse congruence on S then S is orthodox.

Using the concept of congruence pair G.Gomes (R-unipotent congruences on regular semigroups, Semigroup Forum 31 (1985), 265-280) found a description of all inverse congruences on an arbitrary regular semigroup S. She called a pair $(K,\tau)$ an inverse congruence pair of S if

a) K satisfies: (i) K is a regular subsemigroup of S; (ii) $E_S \subseteq K$; (iii) $a'Ka \subseteq K$ for all $a \in S$, $a' \in V(a)$;

b) $\tau$ is a congruence on $\langle E_S \rangle$, the subsemigroup of S generated by $E_S$, such that (i) $\langle E_S \rangle / \tau$ is a semilattice, (ii) $x\,\tau\,y$, $x,y \in \langle E_S \rangle \rightarrow$ $\rightarrow a'xa\ \tau\ a'ya$, whenever $a'xa, a'ya \in \langle E_S \rangle$ for $a \in S$, $a' \in V(a)$;

c) (i) $ax \in K$, $a'a\ \tau\ x$ ($a \in S$, $a' \in V(a)$, $x \in \langle E_S \rangle$) $\rightarrow a \in K$

(ii) $ab \in K (a,b \in S) \rightarrow axb \in K$ for all $x \in \langle E_S \rangle$

(iii) $axa'\,\tau\,aa'x$, whenever $axa' \in \langle E_S \rangle$ for $a \in S$, $a' \in V(a)$, $x \in \langle E_S \rangle$

Given such an inverse congruence pair the unique inverse congruence on S, whose kernel is K and whose restriction to $\langle E_S \rangle$ is $\tau$, is given by

$$a\ \rho_{(K,\tau)}\, b \Longleftrightarrow \exists\ a' \in V(a),\ b' \in V(b): aa'\tau\ bb',\ a'b \in K.$$

Conversely, if $\rho$ is an inverse congruence on S then $(\ker\rho, \tau)$ with $\tau = \rho|_{\langle E_S \rangle}$ is an inverse congruence pair of S and $\rho_{(\ker\rho\ ,\ \tau)} = \rho$.

Remark. As a consequence, the particular case of group congruences on a general regular semigroup S now can be described in the following way (G. Gomes, loc.cit.):

If $K \subseteq S$ satisfies (a) above and (d) $ax \in K (a \in S,\ x \in \langle E_S \rangle) \rightarrow a \in K$, then the relation

$$a \, \rho_K b \Leftrightarrow \exists b' \in V(b) \text{ such that } ab' \in K$$

is a group congruence on S with kernel K. Conversely, if $\rho$ is a group congruence on S then ker $\rho$ satisfies (a) and (d) above and $\rho_{\ker \rho} = \rho$ .

d) The least semilattice of groups congruence

A semilattice of groups (or: Clifford semigroup) can be defined as a regular semigroup with central idempotents (i.e. ea = ae for every $a \in S$ and every $e \in E_S$). Thus, such a semigroup is a special inverse semigroup and also a particular union of groups (see Clifford-Preston [ 1] ). An explicite form of the least congruence $\rho$ on a regular semigroup S such that $S/\rho$ is a semilattice of groups was found by Latorre [ 11 ] . It is a characterization by means of the least group congruence $\sigma$ on S (see Theorem 3.4 above) and the least semilattice congruence $\eta$ on S (see paragraph f) below).

Theorem 3.8. (Latorre [11] ). Let S be a regular semigroup; then the least semilattice of groups congruence on S is given by

$$a \, \nu \, b \rightarrow a \, \eta \, b \text{ and } xa = by \text{ for some } x,y \in U \cap (a\eta),$$

where U is the intersection of all full and self-conjugate subsemigroups of S.

In the particular case that S is orthodox, J. Mills [ 16] showed that

$$a \, \nu \, b \longleftrightarrow a \, \eta \, b \text{ and } eae = ebe \text{ for some } e \in E_S \cap (a\eta).$$

Latorre [11] described $\nu$ on an orthodox semigroup S in a slightly different way:

$$a \nu b \leftrightarrow a \eta b \text{ and } ea = bf \text{ for some } e,f \in E_S \cap (a\eta).$$

e) Orthodox congruences

An inverse congruence $\rho$ on a regular semigroup S yields a (regular) homomorphic image $S/\rho$, in which the idempotents commute. Generalizing, one may ask for those congruences $\rho$ on S, for which the idempotents of $S/\rho$ form a subsemigroup, only. Gomes [5] gave a description of all these orthodox congruences by means of so called orthodox congruence-pairs, specializing the general concept of congruence-pair on a regular semigroups defined by Pastijn-Petrich [18] (see section 2, above).

Definition (Gomes [5]). Let S be a regular semigroup.

1) A subset K of S is said to be a normal subsemigroup of S if K is a regular subsemigroup of S such that $E_S \subseteq K$ and $aKa' \subseteq K$ for every $a \in S$, $a' \in V(a)$.
2) A congruence $\xi$ on $\langle E_S \rangle$, the subsemigroup of S generated by $E_S$, is called normal if $x \xi y \rightarrow a'xa \; \xi \; a'ya$ for all $a \in S$, $a' \in V(a)$, whenever $a'xa, a'ya \in \langle E_S \rangle$.
3) The restriction of a congruence $\rho$ on S to $\langle E_S \rangle$ is called the hypertrace (core) of $\rho$, denoted by $\operatorname{htr} \rho$.

Those congruence-pairs, which yield all the orthodox congruences on a regular semigroup, are characterized abstractly in the following

Definition (Gomes [5]). Let S be a regular semigroups, K a normal subsemigroup of S and $\xi$ a normal congruence on $\langle E_S \rangle$ such that $\langle E_S \rangle / \xi$ is a band. Then the pair $(K, \xi)$ is called an orthodox congruence-pair of S if for all $a,b \in S$, $a' \in V(a)$, $x \in \langle E_S \rangle$ and $f \in E_S$,

(i) $xa \in K, \quad x \,\xi\, aa' \rightarrow a \in K$

(ii) $ab \in K, \; a'a \,\xi\, bb'\, a'a \rightarrow axb \in K$

(iii) $a \in K, \; aa' \,\xi\, f \rightarrow fxf \,\xi\, fa'xaf$, whenever $fa'xaf \in \langle E_S \rangle$.

Theorem 3.9 (Gomes [5]). Let S be a regular semigroup. If $(K, \xi)$ is an orthodox congruence-pair of S then the relation

$$a \; \rho_{(K,\xi)} b \leftrightarrow aa' \,\xi\, bb'aa', \; b'b \,\xi\, b'ba'a, \; ab' \in K \text{ for some (all) } a' \in V(a), \; b' \in V(b)$$

is an orthodox congruence on S such that $\ker \rho_{(K,\xi)} = K$, $\operatorname{htr} \rho_{(K,\xi)} = \xi$. Conversely, if $\rho$ is an orthodox congruence on S, then $(\ker \rho, \operatorname{htr} \rho)$ is an orthodox congruence-pair of S and $\rho_{(\ker\rho, \operatorname{htr} \rho)} = \rho$.
Furthermore, the mappings $\rho \rightarrow (\ker \rho, \operatorname{htr} \rho)$, $(K, \xi) \rightarrow \rho_{(K,\xi)}$ are mutually inverse order-preserving between the lattice of all orthodox congruences on S and the set of all orthodox congruence-pairs of S partially ordered by

$$(K, \xi) \leq (K', \xi') \text{ iff } K \subseteq K', \; \xi \leq \xi'.$$

Remark 1. For the special case that S is orthodox itself, this result yields a description of all congruences on S (see section 2. above).

2. The least orthodox congruence $\lambda$ on a regular semigroup S can be descripted also in the following evident way: $\lambda = \rho^*$, where

$$a \,\rho\, b \;\leftrightarrow\; a = ef, \; b = efef \text{ for } e,f \in E_S .$$

f) Semilattice congruences

A semilattice is defined as a commutative and idempotentsemigroup, i.e. as a special band. Band congruences on a semigroup S are of particular interest, because all the congruence classes form subsemigroups of S. For general semigroups a construction of all band congruences is known, as are descriptions of all rectangular band congruences and of all (right,left) normal band congruences (see Petrich[ 22], III, IV). The least band congruences on a regular semigroup satisfies

$$\mathcal{H} \leq \beta \leq \mathcal{R}^* \cap \mathcal{L}^*,$$

$\mathcal{H}, \mathcal{R}, \mathcal{L}$ are Greens's relations (see Howie-Lallement [8]).

For the important special case of semilattice congruences, the construction found by Petrich [20] for arbitrary semigroups will be given now. Recall that a filter F of a semigroup S is a subsemigroup of S such that $ab \in F$ implies that $a,b \in F$. Note that $\emptyset \neq F \subseteq S$ is a filter of S iff $I = S \setminus F$ is empty or a completely prime ideal of S (i.e. an ideal I of S such that $ab \in I$ implies that $a \in I$ or $b \in I$).Denote by $\mathfrak{F}$ the set of all filters of S and $F_x$ the least filter of S containing $x \in S$.

Theorem 3.10 (Petrich [20]) Let S be a semigroup and $A \subseteq \mathfrak{F}$ be a set of filters of S. Then the relation

$$a \rho_A b \quad \leftrightarrow \quad \text{for every } F \in A \text{ either } a,b \in F \text{ or } a,b \notin F$$

is a semilattice congruence on S. Conversely, for every such congruence $\rho$

on S there is some $A \subseteq \mathfrak{F}$ such that $\rho = \rho_A$. The least semilattice congruence $\eta$ on S is given by $\eta = \rho_{\mathfrak{F}}$, or equivalently by

$$a \eta b \leftrightarrow F_a = F_b \leftrightarrow \text{for every filter } F: a \in F \text{ iff } b \in F.$$

For regular semigroups S, $\eta$ can be described by means of Green's relation $\mathcal{D}$ or $\mathcal{J}$ on S (where $\mathcal{D} = \mathcal{L} \circ \mathcal{R}$ and $\mathcal{J}$ is defined by : $a \mathcal{J} b$ iff $SaS = SbS$):

Theorem 3.11 (Howie-Lallement [ 8 ] Let S be a regular semigroup. Then the least semilattice congruence $\eta$ on S is given by $\eta = \mathcal{D}^* = \mathcal{J}^*$ (where $\mathcal{D}^*$ denotes the congruence on S generated by $\mathcal{D}$ ).

g) The greatest idempotent-pure congruence

A congruence $\rho$ on a semigroup S is called idempotent-pure (also: idempotent-determined) if

$$a \rho e, \quad a \in S, \quad e \in E_S \rightarrow a \in E_S,$$

i.e. each $\rho$-class containing an idempotent consists entirely of idempotents. Evidently, the identity relation on S is an idempotent-pure congruence. For general semigroups, the greatest such congruence can be described in the following way.

Theorem 3.12. (Theissier [24 ]). If S is a semigroup, then the relation

$$a \pi b \longleftrightarrow xay \in E_S \text{ if and only if } xby \in E_S (x,y \in S^1)$$

is the greatest idempotent-pure congruence on S.

## h) Idempotent-separating congruences

In a certain sense opposite to the idempotent-pure congruences are those congruences $\rho$ for which each congruences class contains at most one idempotent, i.e.

$$e \rho f, \quad e, f \in E_S \rightarrow e = f.$$

Clearly, the identity relation on S is always idempotent-separating. It was noted by Lallemenet [9] that for a regular semigroup every such congruence is contained in Green's relation $\mathcal{H}$.

Theorem 3.13. (Lallement [9] ) Let S be a regular semigroup. Then a congruence $\rho$ on S is idempotent-separating iff $\rho \leq \mathcal{H}$. Thus, the greatest idempotent-separating congruence on S is given by $\mu = \mathcal{H}^{\circ}$ (the greatest congruence contained in $\mathcal{H}$ ), i.e.

$$a \mu b \leftrightarrow xay \mathcal{H} xby \ \forall x, y \in S^1 .$$

Note that the hypothesis of the regularity of S cannot be removed: if $S = \{0, a\}$ is the two-element zero semigroup ($a^2 = a0 = 0a = 00 = 0$) then $\mathcal{H} = \varepsilon$, the identity relation, and $\mu = \omega$, the universal relation, hence $\mu \nsubseteq \mathcal{H}$.

Another characterization of $\mu$ on a regular semigroup was given by Hall [7] and Meakin [15] , independently:

Theorem 3.14. (Hall [7] ) Let S be a regular semigroup; then

$a\mu b \leftrightarrow \exists\, a' \in V(a), b' \in V(b): aa'=bb',\ a'a=b'b,\ a'ea=b'eb$ for each idempotent $e \leq aa'$.

For the special case that S is orthodox, Meakin [15] found the following description of $\mu$:

$a\mu b \leftrightarrow \exists\, a' \in V(a),\ b' \in V(b):\ a'ea=b'eb,\ aea'=beb'$ for all $e \in E_S$.

Note. For the much larger class of eventually regular semigroups an explicite description of $\mu$ was found by Edwards [3] . A semigroup S is called eventually regular if for every $a \in S$ there is some positive integer n such that $a^n \in S$ is regular. The greatest idempotent-separating congruence on such a semigroup is given by

$a\ \mu\ b \leftrightarrow$ if $x \in S$ is regular then each of $x\mathcal{R}xa$, $x\mathcal{R}xb$ implis $xa\mathcal{H}xb$,
and each of $x\mathcal{L}ax$, $x\mathcal{L}bx$ implies $ax\mathcal{H}bx$.

It is noted also, that the hypothesis on S to be eventually regular cannot be removed. An example of a semigroup is given for which the greatest idempotent-separating congruence is different from $\mu$ described above (see Edwards [ 3 ], Ex. 3).

Remark. There is still another approach of characterizing particular congruences on a regular semigroup S. Since every congruence on S is uniquely determined by its kernel and trace, one can define the following equivalence relations on the lattice C(S) of all congruences on S:

$$\rho K \tau \leftrightarrow \ker\rho = \ker\tau\ ;\ \rho T \tau \leftrightarrow \operatorname{tr}\rho = \operatorname{tr}\tau\ .$$

Then each K-class and each T-class is an interval in $(C(S), \leq)$. Using these two relations, P. Alimpić-D.Krgović(Some congruences on regular semigroups, Proceedings Oberwolfach 1986, Lect. Notes Math. 1320(1988), 1-10) gave an alternative description of some special congruences; for example:

(i) the least band of groups congruence on S is the least element of the T-class containing $\beta$;

(ii) the least semilattice of groups congruence on S is the least element of the T-class of $\eta$ ;

(iii) the least E-unitary congruence on S is the least element of the K-class containing $\sigma$.

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