NEAR-RINGS OF GROUP MAPPINGS

by

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I. Introduction

It is a basic result of ring theory that the set of endomorphisms of an abelian group is a ring under function addition and composition and furthermore every ring is isomorphic to a subring of a ring of this type. If the group is not abelian then the set of endomorphisms is no longer closed under addition. This leads one to the study of near-rings. It is the purpose of this paper to present a survey of some of the more recent results in the area of near-rings of group mappings. We start with some basic definitions and concepts to be used throughout the paper. For further details about these concepts and other results in near-ring theory we refer the reader to the books of Meldrum, [14] and Pilz, [17].

We recall that a near-ring $N := (N,+,\cdot)$ is a set N with binary operations of addition + and multiplication $\cdot$ such that

(i) $(N,+)$ is a group (not necessarily abelian) with neutral element 0;

(ii) $(N,\cdot)$ is a semigroup;

(iii) $(a+b)c = ac + bc, \forall a,bc \in N$.

More precisely we have defined a right near-ring. Using

(iii)' $a(b+c) = ab + ac, \forall a,b,c \in N$

one gets a left near-ring. Henceforth we consider only right near-rings and refer to them as "near-rings". Examples of near-rings are abundant. They arise in a natural manner when one deals with "non-linear" mappings.

Examples: Let $(G,+)$ be a group with neutral element 0, let T be a topological group, V a vector space and R a commutative ring. With respect to function addition, +, and function composition, $\cdot$, the following are near-rings:

(a) $M(G) := \{f:G \to G\}$;

(b) $M_0(G) := \{f \in M(G) \mid f(0) = 0\}$;

(c) $M_{cont}(T) := \{f \in M(T) \mid f \text{ is continuous on } T\}$;

(d) $M_{aff}(V) := \{f \in M(V) \mid f \text{ is an affine map on } V\}$;

(e) $R[x] := \{f \mid f \text{ is a polynomial over R in a single indeterminant, } x\}$.

Further every ring is a near-ring and if we define $*$ on any group $(G,+)$ by $a*b = a$, $a,b \in G$ then we get a near-ring $(G,+,*)$, i.e., every group can be made into a near-ring.

A near-ring N is said to be zero-symmetric if $a \cdot o = o \cdot a = o \quad \forall a \in N$. A near-ring N is a near-ring with identity if $\exists i \in N$ such that $i \cdot a = a \cdot i = a$, $\forall a \in N$. In the sequel all near-rings will be zero-symmetric with identity.

Let G be a group, End G the monoid of endomorphisms of G and let $S \subseteq$ End G be any semigroup of endomorphisms of G such that the zero map and identity map are in S. We discuss two ways of associating near-rings with the pair (G,S).

Distributively generated near-rings. Let dg S denote the subgroup of M(G) generated by S. Thus dg $S = \{f = \sum_{i=1}^{n} \pm \sigma_i \mid \sigma_i \in S\}$. It is straightforward to verify that dg S is a near-ring, zero-symmetric and with identity. We call dg S the near-ring distributively generated by S.

Centralizer near-rings. Let $M_S(G) = \{f \in M(G) \mid f\sigma = \sigma f, \forall \sigma \in S\}$. Since S contains the zero map we see that $M_S(G)$ is a zero-symmetric near-ring with identity. We call $M_S(G)$ the centralizer near-ring determined by (G,S).

Our main focus in the remainder of this paper will be on various centralizer near-rings although distributively generated near-rings will reappear.

A near-field is a near-ring N with the property that ($N^* := N - \{0\}, \cdot$) is a group. Historically near-fields were the first class of near-rings investigated. In 1905, L.E. Dickson gave the first example of a near-field which is not a field. In 1936, H. Zassenhaus determined all finite fields. He found that, except for seven isomorphism types, all finite near-fields can be constructed by a method going back to Dickson.

Subnear-rings and homomorphisms are defined in the usual manner. The ideals of a near-ring N are defined as kernels of near-ring homomorphisms. This gives rise to the internal characterization that a subset I of a near-ring N is an ideal of N if

(i) (I,+) is a normal subgroup of (N,+);

(ii) $\forall a \in I, \forall n,m \in N, n(a+m) \cdot nm \in I$;

(iii) $\forall a \in I, \forall n \in N, an \in I$.

A subset A of N satisfying (i) and (ii) of the above definition is called a left ideal of N and a subgroup (B,+) of (N,+) is an N-subgroup if $nb \in B, \forall n \in N, \forall b \in B$.

We define the J_2 radical of a near-ring N as the intersection of all ideals of N which are maximal as N-subgroups and we denote this radical by $J_2(N)$. When N is a ring the J_2 radical corresponds to the Jacobson radical of the ring.

A near-ring N is simple when the only ideals of N are {0} and N. A near-ring is 2-semisimple when $J_2(N) = \{0\}$. When N is finite, $J_2(N)$ is the intersection of all maximal ideals of N and N is 2-semisimple if and only if N is the direct sum of simple near-rings.

The interest in centralizer near-rings stems from the following result which shows that such near-rings are general in the sense that every near-ring (as usual, zero-symmetric with identity) arises as a centralizer near-ring.

Theorem I.1. Let N be a near-ring. Then there exists a group G and a semigroup S of endomorphisms of G such that $N \cong M_S(G)$.
Proof. For each $a \in N$ the map $\beta_a : N \to N$ defined by $\beta_a(x) = xa\ \forall x \in N$ is an endomorphism of (N,+). Then, for $S = \{\beta_a \mid a \in N\}$, one finds $N \cong M_S(N)$.

Therefore, since $M_S(G)$ is as general as possible, in order to obtain specific structural results, one must put some restrictions on the pair (G,S). In the next section we indicate structural results for certain choices of (G,S).

II. Structure of the centralizer near-ring $M_S(G)$

When S is a group of automorphisms of G one can make use of the theory of groups acting on sets. This situation has received a great deal of attention. Hence we first consider (G,A) where A is group of automorphisms of G with zero adjoined.

Recall that in this situation, for each $a \in G$, we have a subgroup $st(a) := \{\alpha \in A \mid \alpha(a)=a\}$, the A-stabilizer of a. Also for $a \in G$, the orbit Aa of a is defined by $Aa := \{\alpha(a) \mid \alpha \in A\}$. The next result, due to G. Betsch and known as Betsch's Lemma is fundamental to the study of $M_A(G)$.

Lemma II.1. Let A be a group of automorphisms of the group G and let $x,y \in G$. There exists a function $f \in M_A(G)$ such that $f(x) = y$ if and only if $st(x) \subseteq st(y)$.

When G is finite several definitive structural results can be given.

Theorem II.2. [10] Let G be a finite group and A an automorphism group of G.
1. The following are equivalent:
    a) $M_A(G)$ is a near-field;
    b) A acts transitively on $G^* = G - \{0\}$;
    c) G^* is a single orbit under the action of A on G.
2. $M_A(G)$ is a simple near-ring if and only if all A-stabilizers of non-zero elements of G are A-conjugate, i.e., for $a,b \in G^*$ there exists $\gamma \in A$ such that $st(a) = \gamma\, st(b)\gamma^{-1}$.
3. $M_A(G)$ is 2-semisimple if and only if all A-stabilizers of elements in G^* are maximal, i.e., for $a,b \in G^*$, $st(a) \subseteq st(b)$ implies $st(a) = st(b)$.

In particular, if A is a group of fixed point free automorphisms (only the identity of A has more than one fixed point) then $st(a) = \{id\}$ for each $a \in G^*$ so in this case $M_A(G)$ is simple.

Much more is known. When G is finite and $M_A(G)$ is not 2-semisimple the J_2 radical has been characterized and the structure of $M_A(G)/J_2(M_A(G))$ determined ([10]).

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We now consider the case in which G is an infinite group. Recall that a near-ring N is regular if for every $a \in N$, $a = aba$ for some $b \in N$. In [15], Meldrum and Oswald obtain a very nice characterization for regular centralizer near-rings.

Theorem II.3. [15] Let A be a group of automorphisms of a group G. The near-ring $M_A(G)$ is regular if and only if for $a,b \in G^*$, $st(a) \subseteq st(b)$ implies $st(a) = st(b)$.

We remark that in the finite case regularity coincides with 2-semisimple. Further, if A is fixed point free then $M_A(G)$ is regular. If the pair (G,A) satisfies the condition of Theorem II.3 then we say (G,A) is regular.

When G is infinite, it seems to be a rather difficult problem to determine in general whether or not $M_A(G)$ is a simple near-ring. If $A = \{0, id\}$ (recall the groups of automorphisms have zero adjoined) then it is a classical result of Berman and Silverman (see [14] or [17]) that $M_0(G)$ is a simple near-ring. The investigation of the general situation was initiated by Meldrum and Oswald [15] and continued in [16] and [2]. When dealing with regular pairs, Meldrum and Zeller [16] showed that it suffices to restrict A to be fixed point free. They prove the following result

Theorem II. 4. [16] If (G,A) is regular and the stabilizers in A form a single conjugacy class then there exists a subgroup H of G and a fixed point free group of automorphisms, B, of H such that $M_B(H) \cong M_A(G)$.

Thus one focuses on fixed point free automorphism groups A. Let $\{w_\lambda \mid \lambda \in \Lambda\}$ be a complete set of A-orbit representatives in G and define for $v \in G$,
$\Lambda_v = \{\lambda \in \Lambda \mid Aw_\lambda + v \not\subseteq Aw_\lambda\}$.

Lemma II.5. [16] Let A be fixed point free on G. If there exists $v \in G^*$ such that $|\Lambda_v| = |\Lambda|$, then $M_A(G)$ is a simple near-ring.

Using this result Meldrum and Zeller then prove

Theorem II.6. [16] If A is fixed point free on G and $|A| < |G|$ then $M_A(G)$ is a simple near-ring.

Given a function $f \in M_A(G)$, define the rank of f, rk(f), to be the cardinality of the set of A-orbits in the range of f. For a nonempty subset B of G, define the rank of B, rk(B), to be the cardinality of the set of A-orbits in G which intersect B nontrivially. For each cardinal $\aleph_\alpha$, define $R_\alpha = \{f \in M_A(G) \mid rk(f) < \aleph_\alpha\}$. It was proven by Meldrum and Zeller [16] that these sets R_α are the only candidates for ideals in $M_A(G)$.

Theorem II.7. [16] Let A be fixed point free on G. If I is an ideal of $M_A(G)$ then $I = R_\alpha$ for some ordinal α.

This result was recently improved.

Theorem II.8. [2] Let A be fixed point free on G. Then MA(G) has at most one nontrivial ideal I. Specifically, $I = \{f \in M_A(G) \mid rk(f) < |A|\}$ is the only possible nontrivial ideal of $M_A(G)$.

In [2] several conditions on the pair (G,A) are given which force $M_A(G)$ to be a simple near-ring. Moreover it is shown that if a nonsimple near-ring $M_A(G)$ exists then A and G have rather unusual properties. But that is where the matter now stands. It remains an open question if $M_A(G)$ is simple.

Question: If A is fixed point free on G, is $M_A(G)$ a simple near-ring?

We leave the case of automorphisms and return to the situation in which S is a monoid of endomorphisms with zero. We discuss a particular situation.

Definition II.8. [12] A semigroup S of endomorphisms of a group G is fixed point free if

(a) $\bigcap_{\alpha \in S} \text{Ker } \alpha = \{0\}$;

(b) $\forall \beta \in S$, Ker β = Ker $\beta^2 = \ldots$;

(c) $\forall \alpha,\beta \in S, \forall a \in G$, if $\alpha a = \beta a \neq 0$ then $\alpha = \beta$.

It is clear that if S is a group of automorphisms then this concept agrees with the previous use of fixed point free.

Theorem II.10. [12] Let N be a finite near-ring. Then N is 2-semisimple near-ring with its simple summands being non-rings or fields if and only if $N \cong M_S(G)$ for some finite group G and S a semigroup of fixed point free endomorphisms of G.

If S is a fixed point free semigroup of endomorphisms of a finite group G then S is a completely regular inverse semigroup, [12]. Thus the previous theorem suggests a study of near-rings of the form $M_S(G)$ where S is a completely regular inverse semigroup. In [12] it was determined for finite groups when such a near-ring is 2-semisimple. There are also other isolated results on the structure of $M_S(G)$ when S has certain properties (see e.g., [7]). However much more work needs to be done in this area.

We mentioned above that $M_S(G)$ is indeed general. However, one has been able to characterize those pairs (G,S) such that $M_S(G)$ is a near-field. Not surprisingly, the discussion breaks into the cases in which S is a group and when it is not.

Theorem II.11. [6] Let A be a group of automorphisms of a group G. The following are equivalent:

(i) $M_A(G)$ is a near-field;

(ii) $G = \{0\} \cup Ax$ and (G,A) is regular;

(iii) $G = \{0\} \cup Ax$ and (G,A) satisfies the property (F.C.): If $st(x) \subseteq st(\alpha x)$, $x \in G$, $\alpha \in A$, then $st(x) = st(\alpha x)$.

When G is finite, (F.C.) is always satisfied so we obtain Theorem II.2, (1). Moreover, if the action of A on G is fixed point free then regularity is equivalent to (F.C.) and in this case both conditions hold trivially.

Corollary II.12. [6] If A is a group of fixed point free automorphisms of G then $M_A(G)$ is a near-field if and only if $G = \{0\} \cup Ax$.

We mention here that we know of no example of a group G and a group A of automorphisms of G such that $G = \{0\} \cup Ax$ but (G,S) does not satisfy (F.C.).

Now let S be a semigroup of endomorphisms of G as usual with zero and identity. For any $x \in G$, $x \in Sx$ so we have $G = \bigcup_{i \in I} Sy_i$. We call $Y = \{y_i \mid i \in I\}$ a generating set. Henceforth we take $Y = \{y_i \mid i \in I\}$ as an arbitrary but fixed generating set and we consider I well ordered by the relation "$\leq$". Hence we consider Y as an I-sequence $\{y_i\}$.

For $u,v \in G$ define the relation $F(u,v) := \{(\alpha,\beta) \in S{\times}S \mid \alpha u = \beta v\}$. Further let $H = \{$I-sequences $\{x_i\} \mid x_i \in G, F(y_i,y_j) \subseteq F(x_i,x_j), i \leq j\}$. If π_i is the i-th projection map then clearly $\pi_i(H) \subseteq G$. We define another relation R on G^* by $(x,y) \in R$ if there exists $\alpha \in S$ such that $\alpha(x) = y$. Let $\tilde{R}$ denote the equivalence relation generated by R. We call the equivalence classes of $\tilde{R}$ the connected components of G and we say G is S-connected provided G^* is a connected component.

Equivantly $u,v \in G^*$ are S-connected if and only if there exist $x_1,x_2,\ldots,x_{n-1} \in G^*$, $\sigma_1,\ldots,\sigma_n,\rho_1,\ldots,\rho_n \in S$ such that

$$\begin{aligned} \sigma_1 u &= \rho_1 x_1 \neq 0 \\ \sigma_2 x_1 &= \rho_2 x_2 \neq 0 \\ &\vdots \\ \sigma_n x_{n-1} &= \rho_n v \neq 0. \end{aligned}$$

We now introduce a concept needed in the next theorem but also used very much in the following section.

Definition II.13. Let G be a group and $\mathcal{F} = \{G_\alpha\}$ a collection of subgroups of G such that

(i) $\{0\} \subsetneq G_\alpha \subsetneq G$;

(ii) $\cup G_\alpha = G$;

(iii) $G_\alpha \cap G_\beta = \{0\}$ if $\alpha \neq \beta$.

Then $\mathcal{F}$ is called a fibration of G and $(G,\mathcal{F})$ is called a fibered group.

If $\mathcal{F} = \{G_\alpha\}$ is a fibration of G, we say $\sigma \in S$ is a $\mathcal{F}$-isomorphism if for each $G_\alpha \in \mathcal{F}$, $\sigma(G_\alpha) = \{0\}$ or $\mathrm{Ker}\ \sigma \cap G_\alpha = \{0\}$ and $\sigma(G_\alpha) = G_\beta$ for some $G_\beta \in \mathcal{F}$. Thus $\sigma \in S$ is a $\mathcal{F}$-isomorphism if and only if for each $G_\alpha \in \mathcal{F}$, σ is the zero map on G_α or σ is an isomorphism on G_α with image in $\mathcal{F}$. The characterization result is as follows.

Theorem II.14. [6] Let S be a semigroup of endomorphisms of a group G. Then $M_S(G)$ is a near-field if and only if

(i) G is S-connected,

(ii) G has a fibration, say $\mathcal{F} = \{H_j \mid j \in J\}$ and each $\sigma \in S$ is an $\mathcal{F}$-isomorphism,

(iii) if $y_i \in H_j^*$ then $\pi_i(H) = H_j$.

III. Geometry and Near-rings

From the time of Descartes, early in the 17th century , mathematicians have been interested in associating algebraic structures with geometric structures and investigated the transfer of information. In this section we introduce a geometric structure, associate two near-rings to the geometry and indicate how the geometry influences the algebra. We start with a definition due to André, [1].

Definition III.1. [1] Let $\Sigma = (\mathcal{P},\mathcal{L},\|)$ where $\mathcal{P}$ is a set of points, $\mathcal{L}$ a collection of subsets of $\mathcal{P}$ called lines, with the incidence relation "belongs to", and a parallelism relation $\|$ defined on $\mathcal{L}$ such that

(A1) Every two points in $\mathcal{P}$ determine a unique line;

(A2) $|\mathcal{L}| \geq 2$ and for each $A \in \mathcal{L}$, $|A| \geq 2$;

(A3) Parallelism is an equivalence relation;

(A4) $\forall x \in \mathcal{P}$, $\forall A \in \mathcal{L}$, there exists a unique $B \in \mathcal{L}$ such that $x \in B$ and $B \| A$.

Further there exists a one-one map $\Phi: \mathcal{P} \to \mathrm{Coll}\ \Sigma$ such that $\Phi(\mathcal{P})$ is a point transitive group of fixed point free collineations. We say (Σ,Φ) is a translation structure. (See [1] and [4].)

Let $(G,\mathcal{F} = \{G_i\})$ be a fibered group (see Definition II.13). By taking $\mathcal{P}(G) = G$, $\mathcal{L}(G) = \{x + G_i \mid G_i \in \mathcal{F}, x \in G\}$ and setting $a + G_i \| b + G_j$ if and only if i=j one gets an incidence structure $\Sigma(G) = (\mathcal{P}(G), \mathcal{L}(G), \|)$ satisfying (A1) - (A4). Further define $\Phi(G) : \mathcal{P}(G) \to \mathrm{Coll}\ \Sigma\ (G)$ by $\Phi(G) : a \to \lambda_a$ where λ_a denotes the left translation of G determined by $a \in G$. We then find we have a translation structure $(\Sigma(G), \Phi(G))$.

Conversely, every translation structure arises in this manner. That is, if (Σ,Φ), $\Sigma = (\mathcal{P},\mathcal{L},\|)$ is a translation structure, then there is a fibered group $(G,\mathcal{F})$ such that $\mathcal{P} = \mathcal{P}(G)$,

$\mathcal{L} = \mathcal{L}(G)$, $| \ |$ is as defined above and $\Phi = \Phi(G)$. Hence a translation structure may be considered as a fibered group and we henceforth do so.

If the translation structure $(G, \mathcal{F} = \{G_i\})$ has the property that $G_i + G_j = G$ for each $G_i, G_j \in \mathcal{F}$, $i \neq j$ then $\mathcal{F}$ is called a congruence fibration and in this case one obtains the classical translation planes.

Thus congruence fibrations tighten the structure of the geometry. We next tighten the structure in an alternative fashion. Let $(G, \mathcal{F} = \{G_i\})$ be a translation structure and let S be a semigroup of endomorphisms of G such that

(01) The identity map and zero map are in S;

(02) For each $\sigma \in S$, for each $G_i \in \mathcal{F}$, $\exists G_j \in \mathcal{F}$ such that $\sigma(G_i) \subseteq G_j$.

Then S is called a semigroup of operators for $(G, \mathcal{F})$ and $(G, \mathcal{F}, S)$ is a translation structure with operators, TSO. We mention that operators can also be defined in a geometric manner, ([4]).

We now show how to associate near-rings with TSO's, $(G, \mathcal{F}, S)$. First we consider the set $\mathrm{Dil}(G, \mathcal{F}) = \{\sigma \in \mathrm{End}\ G \mid \sigma(G_i) \subseteq G_i, \forall G_i \in \mathcal{F}\}$. (Note that the operators play no role here.) Under function composition $\mathrm{Dil}(G, \mathcal{F})$ is a semigroup with zero and identity, called the semigroup of dilitations of $(G, \mathcal{F}, S)$. Our first near-ring is d.g. $\mathrm{Dil}(G, \mathcal{F})$ called the kernel of $(G, \mathcal{F}, S)$. For our second associated near-ring we take $M_S(G, \mathcal{F}) = \{f \in M_0(G) \mid f(G_i) \subseteq G_i, \forall G_i \in \mathcal{F}, f\sigma = \sigma f, \forall \sigma \in S\}$, a near-ring under funtion addition and composition called the centralizer of $(G, \mathcal{F}, S)$.

We restrict now to the case in which G is a finite group and look at various properties of these associated near-rings.

III.A: Kernel of $(G, \mathcal{F}, S)$.

The structure of $\mathrm{Dil}(G, \mathcal{F})$ is well-known, ([3], [9]).

Theorem III.2. For a finite fibered group $(G, \mathcal{F})$, $\mathrm{Dil}(G, \mathcal{F}) \setminus \{0\}$ is a cyclic group of fixed point free automorphisms of G.

Proof. To illustrate some of the ideas we show that each $0 \neq \sigma \in \mathrm{Dil}(G, \mathcal{F})$ is a monomorphism. Hence, since G is finite σ is an automorphism. Suppose $\sigma \in \mathrm{Dil}(G, \mathcal{F})$ and $\sigma(x) = 0$ for some $x \in G$, say $x \in G_i$. Let $y \in G_j$, $j \neq i$. Then $x+y \in G_k$, $i \neq k \neq j$. Now $\sigma(y) \in G_j$ and $\sigma(y) = \sigma(x+y) \in G_k$. Hence $\sigma(y) = 0$. For any $w \in G_i$, use w and y to get $\sigma(w) = 0$. Thus σ is the zero map.

A classical result states that when $\mathcal{F}$ is a congruence fibration, G is an abelian group, therefore $\mathrm{Dil}(G, \mathcal{F})$ is a finite field. Thus when G is an abelian group dg $\mathrm{Dil}(G, \mathcal{F}) = \mathrm{Dil}(G, \mathcal{F})$ is a field. We now turn to the non-abelian case.

Theorem III.3. [3] If $(G,\mathcal{F})$ is a finite fibered group with $\mathrm{Dil}(G,\mathcal{F}) \neq \{0,id\}$ then G is a p-group for some prime p, is of exponent p and of nilpotency class at most 2.

Using this result and the known structure of $\mathrm{Dil}(G,\mathcal{F})$ the following rather surprising result has been obtained.

Theorem III.4. [9] If $(G,\mathcal{F})$ is a finite fibered group then dg $\mathrm{Dil}(G,\mathcal{F})$ is a commutative ring. If further, $\mathrm{Dil}(G,\mathcal{F}) \neq \{0,id\}$ then dg $\mathrm{Dil}(G,\mathcal{F})$ is a field.

Clearly if $\mathrm{Dil}(G,\mathcal{F}) = \{0,id\}$ then dg $\mathrm{Dil}(G,\mathcal{F}) = Z_n$ where n is the exponent of G. The above theorem shows that whether or not G is abelian, whenever $\mathrm{Dil}(G,\mathcal{F}) \neq \{0,id\}$ there is a field associated with the geometry $(G,\mathcal{F})$ is a natural manner. We also mention that in the abelian case the field has geometric significance. The significance of the field dg $\mathrm{Dil}(G,\mathcal{F})$ in the non abelian case is still unknown.

III.B. Centralizer of $(G,\mathcal{F},S)$.

As above, to obtain definitive structural results one places some restrictions on the semigroup of operators. One first considers the case where S is a group of automorphisms (with 0). As one might expect from the previous discussion on centralizer near-rings, the orbits of the action and the stabilizers play an important role. For results in this situation see [8].

Next one considers the situation in which S is a cyclic semigroup, say $S = \langle\alpha\rangle \cup \{0,id\}$. We write $M_\alpha(G,\mathcal{F})$ for $M_S(G,\mathcal{F})$. We are mainly interested as to when $M_\alpha(G,\mathcal{F})$ is a simple near-ring. If α is an automorphism, using the results in [8], one notes when $M_\alpha(G,\mathcal{F})$ is simple. In other cases we have the following.

Theorem III.5. [8] If $S = \langle\alpha\rangle \cup \{0,id\}$, α not invertible and α not nilpotent, then $M_\alpha(G,\mathcal{F})$ is not a simple near-ring.

Proof. Since α is not invertible, Ker $\alpha \neq \{0\}$. Thus there is some fiber, G_i, of the fibration such that $G_i \cap \mathrm{Ker}\,\alpha \neq \{0\}$. For $f \in M_\alpha(G,\mathcal{F})$, $f(\mathrm{Ker}\,\alpha \cap G_i) \subseteq \mathrm{Ker}\,\alpha \cap G_i$ so $A = (\{0\}: \mathrm{Ker}\,\alpha \cap G_i)$ is an ideal in $M_\alpha(G,\mathcal{F})$. Using the fact that α is not nilpotent one gets $A \neq \{0\}$, hence $M_\alpha(G,\mathcal{F})$ is not simple.

We henceforth restrict our attention to nilpotent endomorphisms. We recall the concepts of generating set and connected components as discussed after Corollary II.12.

Lemma III.6. There are k-1 connected components of G^* where $|\mathrm{Ker}\,\alpha| = k$.

Proof. Let C_i be a connected component and let $y \in C_i$. Since α is nilpotent there exists some s such that $\alpha^s(y) \in \mathrm{Ker}\ \alpha$ and since $\alpha^s(y) \in C_i$, $\alpha^s(y) \in \mathrm{Ker}\ \alpha \cap C_i$. Thus there exists a kernel element in each connected component. Suppose $x_1, x_2 \in \mathrm{Ker}\ \alpha \cap C_i$. We then find $x_1 = \alpha^k(x_2)$ for some integer $k \geq 0$. If $k \neq 0$, $x_1 = 0$, a contradiction. Thus each connected component has a unique kernel element.

If we let $\{0\}$ be a connected component then we say the number of connected components is the cardinality of Ker α. In particular $(G, \mathcal{F})$ is S-connected if and only if Ker $\alpha = \{0, x\}$.

Suppose $M_\alpha(G, \mathcal{F})$ is a simple near-ring. We know there is some fiber G_i such that $\mathrm{Ker}\ \alpha \cap G_i \neq \{0\}$. If $\mathrm{Ker}\ \alpha \cap G_j \neq \{0\}$, $i \neq j$ then one finds there exists a component with more than one kernel element which contradicts the above lemma. This gives the following result.

<u>Lemma III.7.</u> If $M_\alpha(G, \mathcal{F})$ is simple and α is nilpotent then Ker α is contained in a single fiber of $\mathcal{F}$, say G_0.

<u>Lemma III.8.</u> If $M_\alpha(G, \mathcal{F})$ is a simple near-ring and α is nilpotent there is a unique generating set $Y = G \setminus \mathrm{Ker}\ \alpha^{n-1}$ where $\alpha^n = 0$ but $\alpha^{n-1} \neq 0$.

When G is S-connected much can be said.

<u>Theorem III.9.</u> [8] Let α be a nilpotent operator on $(G, \mathcal{F})$ and let G be S-connected, $S = \langle\alpha\rangle \cup \{0, id\}$ with $\mathrm{Ker}\ \alpha \subseteq G_0$. Let Y be any generating set for G. The following are equivalent.

(i) $Y \cap G_0 = \emptyset$;
(ii) $M_\alpha(G, \mathcal{F})$ is a near-field;
(iii) $M_\alpha(G, \mathcal{F})$ is a simple near-ring;
(iv) $M_\alpha(G, \mathcal{F})$ is a 2-semisimple near-ring;
(v) $M_\alpha(G, \mathcal{F}) \cong Z_2$.

When G is not S-connected necessary and sufficient conditions, in terms of the geometry, are known for $M_\alpha(G, \mathcal{F})$ to be simple, [8]. Instead of stating these we give an external characterization.

<u>Theorem III.10.</u> [8] Let α be a nilpotent operator on $(G, \mathcal{F})$. Then $M_\alpha(G, \mathcal{F})$ is a simple near-ring if and only if $M_\alpha(G, \mathcal{F}) \cong M_0(\mathrm{Ker}\ \alpha)$.

In [8] an example is given where $G := (F)^6$, F a finite field $\mathcal{F}$ a fibration of G and α a nilpotent operator such that $M_\alpha(G,\mathcal{F}) \cong M_D(F \oplus F)$. Thus simple near-rings, not rings, actually arise.

IV. Rings and Near-rings

Let R be a ring with identity and let G be a (right) unitary R-module. Then R determines a semigroup of endomorphisms of G so we have a centralizer near-ring $M_R(G)$ $= \{f \in M_D(G) \mid f(xr) = (fx)r, \forall x \in G, \forall r \in R\}$. In this section we discuss some of the interplay between the properties of the ring R, the R-module, G_R, and the near-ring $M_R(G)$.

We recall that a <u>cover</u> for an R-module G is a collection $\mathcal{C} = \{G_\alpha\}$ of submodules of G sucht that

(i) $\{0\} \subsetneqq G_\alpha \subsetneqq G$;

(ii) $G_\alpha \nsubseteq G_\beta$ for $\alpha \neq \beta$;

(iii) $\cup G_\alpha = G$.

Let $R := Z$ and $G := Z^2$ and let $\mathcal{C}$ be a cover by maximal cyclic submodules. Further let $f \in M_Z(Z^2)$ be determined on $G_\alpha = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} Z$ by $f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$. Since G_α is a maximal submodule we have $\gcd(x_1,x_2) = 1$ so $\exists h,k \in Z$, $hx_1 + kx_2 = 1$. But then f can be represented on G_α by the matrix $\begin{bmatrix} c_1h & c_1k \\ c_2h & c_2k \end{bmatrix}$, i.e., f/G_α can be extended to an endomorphism of G. Equivalently, every $f \in M_Z(Z^2)$ is piecewise an endomorphism of Z^2 in the sense that for each $G_\alpha \in \mathcal{C}$, $\exists \varphi \in End_Z(Z^2)$ with $f/G_\alpha = \varphi$.

In general, let $\mathcal{C} = \{G_\alpha\}$ be a cover of G by maximal cyclic submodules of G and let $N := \{f \in M_R(G) \mid f/G_\alpha$ can be extended to an endomorphism of G$\}$, a subnear-ring of $M_R(G)$ which we call the near-ring of piecewise endomorphisms determined by $(R,G,\mathcal{C})$. We ask, "When is $N = M_R(G)$?". The next example shows that in general, $N \neq M_R(G)$.

<u>Example IV.1.</u> [5] Let $R := Z[x]$, $G := R^2$ and let $\mathcal{C}$ be a cover by maximal cyclic submodules. One verifies that $\begin{bmatrix} x \\ x+2 \end{bmatrix} R \in \mathcal{C}$. Further, $\exists f \in M_R(G)$ with

$$f\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{cases} \begin{bmatrix} 1 \\ 1 \end{bmatrix} r, \text{ if } \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} x \\ x+2 \end{bmatrix} r, r \in R, \\ \\ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \text{otherwise,} \end{cases}$$

However, there is no $\varphi \in End_R(G)$ with $\varphi \begin{bmatrix} x \\ x+2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Hence $N \neq M_R(G)$.

Note that in the above example R is not a PID. For PID's the situation is quite different. In fact we have the next rather interesting result.

Theorem IV.2. [5] Let G be a finitely generated module over a PID, D, let $\mathcal{C} = \{G_\alpha\}$ be a cover by maximal cyclic submodules and let $N = \{f \in M_R(G) \mid f/G_\alpha$ can be extended to an endomorphism of $G\}$. Then $N = M_D(G)$.

We mention that it is an open question whether or not the requirement that G be finitely generated can be omitted.

In the next theorem we present some further relationships between the ring module G_R and the near-ring $M_R(G)$.

Theorem IV.3. [13] (a) If D is an integral domain, not necessarily commutative then $M_D(D^2)$ is a near-ring, not a ring.

(b) Let R be a commutative ring. $M_R(R^2)$ is a simple near-ring if and only if R is an integral domain.

(c) Let R be a left Artinian ring. Then $M_R(R^2)$ is 2-semisimple if and only if R is semisimple.

It should be pointed out that rings do arise as $M_R(G)$. In fact if D is a commutative integral domain and Q(D) its field of fractions, then $M_D(Q(D))$ is a ring. Further, if R is a complete nxn matrix ring over a ring S then for each R-module, G, $M_R(G)$ is a ring, in fact $M_R(G) = End_R(G)$.

On the other hand if R is the field of real numbers, for $G := R$, $M_R(R)$ is a ring while for $G := R^2$, $M_R(R^2)$ is not a ring.

This raises the questions:

(Q1): Which rings R have the property that $M_R(G)$ is a ring for each R-module G ?

(Q2): Which rings R have the property that $M_R(G) = End_R(G)$?

For finite rings R the above questions have been shown to be equivalent and those rings R such that $M_R(G)$ is a ring for each R-module have been characterized, (see [11]). However the general problem remains open.

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