Chapter 13

Foulser's Theorem: Baer-Elation Incompatibility.

In this chapter, we demonstrate the high degree of incompatibility between Baer p-elements and affine elations, acting on a translation plane π of order p^{2r}. Among the most startling of such results is Foulser's theorem, asserting that non-trivial Baer p-elements and non-trivial affine elations cannot simultaneously act on π if p is odd. The first section of this chapter establishes striking constraints of this type, all due to Foulser, that apply to translation planes of odd order. The second section is concerned with the even order versions of Foulser's theory: here affine elations and Baer 2-elements *are* compatible, but they constrain each otehr quite severely.

13.1 Baer-Elation Theory: Odd Order Case.

We begin with a theorem that allows us to use Ostrom's theorem for generalsied elations due to Foulser.

Theorem 13.1.1 *Let π be a translation plane of order p^{2k} for $p > 3$.*

If σ and τ are Baer p-collineations in the translation complement whose axes are distinct then $Fix\sigma \cap Fix\tau = 0$.

Proof: Sketch. Suppose not! Then there exist σ and τ as Baer p-collineations such that $Fix\sigma \cap Fix\tau = X$ has maximum dimension r over $GF(p)$. We note that if X is a proper subplane of $Fix\sigma$ then $r \leq k/2$ and if X is a part of a line of $Fix\sigma$ this restriction is still valid.

Exercise 13.1.5 *Change basis by* $\begin{bmatrix} A_1 & 0 & 0 & 0 \\ A_2 & A_4 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}$ *and realize that the general form of* τ *does not change to obtain that, without loss of generality,* $A_1 = A_4 = I$ *and* $A_2 = 0$. *Then ,again using the fact that* σ *and* τ *commute, show that that* $B_1 = B_4$.

Exercise 13.1.6 *Show that* $\sigma\tau$ *is a generalized elation by computing* $\sigma\tau$ *and its fixed point space.*

Exercise 13.1.7 *Compute* $(\sigma\tau - 1)^2$ *and show that the following matrix is obtained:*

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2B_1 & 0 & 0 & 0 \end{bmatrix}.$$

Now since $(\sigma\tau - 1)^2 = 0$, it follows that $B_1 = 0$. From the above exercise, it turns out that the fixed point space of $\sigma\tau$ is too large to be either a line or a Baer subplane.

This proves case (3). Actually, this same proof can be adapted to show hat Baer p-elements and elations cannot coexist when $p > 2$. We shall come back to this in a later section.

Case (2) both σ_1 and τ_1 are non-trivial on V_1 and $Fix\sigma_1 \cap Fix\tau_1 \neq 0$.

Suppose that $Fix\sigma_1 = Fix\sigma/X$ and $Fix\tau_1 = Fix\tau/X$. Then $Fix\sigma/X \cap Fix\tau/X = X$ or rather $Fix\sigma_1 \cap Fix\tau_1 = 0$.

Hence, assume without loss of generality, that $y + X$ is in $Fix\sigma_1 - Fix\sigma/X$ and write $y = v + u$ where v is in $Fix\sigma$ and u is in $Fix\tau$. Since σ_1 fixes $y + X$, it follows that σ also fixes $u + X$. Since τ fixes $X = Fix\sigma \cap Fix\tau$ pointwise, it follows that $\langle u, X\rangle \subseteq Fix\tau$. Note that u is nonzero by assumption. But, $\langle u, X\rangle \subseteq \sigma(Fix\tau) = Fix\tau^{\sigma^{-1}}$. But, $Fix\tau^{\sigma^{-1}} \neq Fix\tau$ since if it were this would imply that $\sigma_1 = 1$ by an exercise above. Hence, τ and $\tau^{\sigma^{-1}}$ are generalized elations of V both of whose fixed point space properly contain X which is contrary to the maximality condition.

Hence, it remains to consider

Case (1) $Fix\sigma_1 \cap Fix\tau_1 = 0$ on V_1.

We give the proof in a series of lemmas.

Lemma 13.1.8 *$(\sigma - 1)Fix\tau \oplus X = Fix\sigma$ and $(\tau - 1)Fix\sigma \oplus X = Fix\tau$*

Proof: Consider $(\sigma-1)v$ for v in $Fix\tau$. If $(\sigma-1)v \neq 0$ then v is not in X. If $(\sigma-1)v$ is in $Fix\sigma \cap Fix\tau$ then σ fixes $v+X$ and clearly τ fixes $v+X$ so that σ_1 and τ_1 fix a common nonidentity element and hence $Fix\sigma_1 \cap Fix\tau_1 \neq 0$.

Notice that the kernel of $\sigma - 1$ in $Fix\tau$ is $Fix\tau \cap Fix\sigma$ and $Fix\tau/X \simeq (\sigma - 1)Fix\tau$. By the rank-nullity theorem, the result now follows.

Lemma 13.1.9 *$\langle\sigma, \tau\rangle = G$ leaves $(\sigma-1)Fix\tau \oplus (\tau-1)Fix\sigma = V_2$ invariant.*

Proof: Note that $(\sigma - 1)^2 = (\tau - 1)^2 = 0$ and apply $(\sigma - 1)$ to $(\sigma - 1)v + (\tau - 1)u$ for v in $Fix\tau$ and u in $Fix\sigma$ realizing that $(\tau - 1)w$ is in $Fix\tau$ for any w in V. Hence, $\sigma-1$ and $\tau-1$ and thus σ and τ leave the given subspace invariant.

Exercise 13.1.10 *Check that the sum is a direct sum.*

Lemma 13.1.11 *Let $\rho_2 = \rho \mid V_2$. Let $G_2 = \langle\sigma_2, \tau_2\rangle$. Then $G_2 \simeq SL(2,p^z)$ for some positive integer z.*

Proof: The idea of the proof is to show that the set $\{Fix\sigma_2^g\ ,\ Fix\tau_2^h$ for g,h in $G_2\}$ is a partial spread and then apply Ostrom's theorem. Note that σ_2 and τ_2 are generalized elations of V_2.

Note that $Fix\sigma_2 = Fix\sigma \cap V_2 = (\sigma - 1)Fix\tau$ and $Fix\tau_2 = Fix\tau \cap V_2 = (\tau-1)Fix\sigma$. These subspaces are both of dimension $k-r$ and since we have a direct sum above, these particular fixed point spaces are disjoint so that V_2 has dimension $2(k - r)$ and the generalized elations are of type $k - r$.

Now assume there exist ρ and γ in G which are conjugate to σ and/or τ such that $Fix\rho_2 \neq Fix\gamma_2$ but $Fix\rho_2 \cap Fix\gamma_2 \neq 0$. Then, it follows that $Fix\rho \cap Fix\gamma \subseteq X \oplus Fix\sigma_2 \cap Fix\rho_2$ contrary to the maximality condition. Hence, $G_2 \simeq SL(2,p^z)$. In particular, -1 is in G_2 acting on V_2. This proves the lemma.

Lemma 13.1.12 *Let θ be in G such that $\theta_2 = -1$. Then $\theta^2 = 1$.*

Exercise 13.1.13 *Note that any nonidentity collineation can pointwise fix a subspace of dimension $\leq k$ (one half the dimension of the translation plane). Prove the above lemma by considering $X \oplus V_2$ and realizing that G fixes X pointwise and show that the dimension of $X \oplus V_2$ is $2k - r > k$.*

Lemma 13.1.14 $G \simeq G_2$.

Proof: Since G fixes V_2, the group induced on V_2 is isomorphic to $G/G[V_2]$ where $G[V_2]$ is the subgroup which fixes V_2 pointwise. The above exercise shows that $G[V_2] = \langle 1 \rangle$.■

Remark 13.1.15 *A result of Baer's states that in any finite affine plane, an involution either fixes pointwise a line or a Baer subplane. Thus, the dimension of a pointwise fixed subspace by an involution of a translation plane is half the dimension of the translation plane*

Note that $(-\theta)^2 = 1$ so that $-\theta$ is an involution.

Lemma 13.1.16 *The subspace fixed pointwise by $-\theta$ contains V_2. Then $r = k/2$.*
Furthermore, θ is in $Z(G)$.

Proof: From the preceding, we have $2(k-r) \leq k$ so that $k/2 \leq r$ but $r \leq k/2$ since X is either contained with a line of $Fix\sigma$ or is a subplane of it (note that the intersections of subplanes is either contained within a line or is a subplane of each containing subplane). So, $r = k/2$. Note that $(w\theta w^{-1})_2 = \theta_2^w = \theta_2 = -1$. It follows that $Fix\theta^{-1}w\theta w^{-1}$ contains $X \oplus V_2$ since G fixes X pointwise. Hence, $\theta^{-1}w\theta w^{-1} = 1$ which proves the lemma.

Thus, it follows that $Fix\theta$ is left invariant by G. Represent $Fix\theta = X \oplus W$ where both X and W are $k/2$-dimensional subspaces.

Lemma 13.1.17 $W \oplus (Fix\sigma + Fix\tau) = V$.

Proof: By the previous notes on dimension, it suffices to show that the indicated direct sum is, in fact, direct.

If $\theta(v+u) = v+u$ for v in $Fix\sigma$ and u in $Fix\tau$ then recalling that θ is in $Z(G)$, we have $\sigma\theta(v+u) = \theta(v+\sigma(u)) = v+\sigma(u)$. It then follows that $\sigma(u) - u = (\sigma - 1)u$ is fixed by θ. But, θ acts as -1 on V_2 so that $u = 0$. Similarly, $v = 0$.

Now let $\phi_3 = \phi \mid Fix\theta$. Then σ_3 and τ_3 are generalized elations of $Fix\theta$ with identical fixed point spaces X since σ does not fix a nonidentity element of W.

Hence, we obtain

Lemma 13.1.18 *$\langle \sigma_3, \tau_3 \rangle$ is an elementary Abelian p -group (of order p^2).*

Exercise 13.1.19 *Show that the commutator subgroup G' of G fixes $Fix\theta$ pointwise.*

However, $G' = G$ as $G \simeq SL(2, p^z)$. On the other hand, G leaves invariant V_2 and θ acts on V_2 as -1, $V_2 \cap Fix\theta = 0$. Hence, there exists an element g of order p which fixes a nonzero point of V_2 which implies that $Fixg$ has dimension strictly larger than k –a contradiction. Hence, this completes the proof of case (3) and consequently the proof of the theorem.

It might be pointed out that both Ostrom's and Foulser's theorems can be stated for $p = 3$ also and in this case, it is possible that $SL(2, 5)$ is generated. Furthermore, the full group generated by elations or Baer p -collineations is completely determined by the work of Ostrom, Hering and Foulser.

We mentioned above that an adaption of the proof of case (3) will show that it is not possible to have both Baer p-collineations and elations acting on a translation plane of odd order. We state this formally. We note that this case only requires that p is odd.

Theorem 13.1.20 *Let π be a finite translation plane of odd order p^r.*

Then the collineation group of π does not contain both Baer p -collineations and elations.

Furthermore, Foulser shows that all Baer axes of p-collineations share their parallel classes.

Theorem 13.1.21 *Let π be a finite translation plane of odd order p^{2k} for $p > 3$.*

If B denotes the set of axes of Baer p-collineations in the translation complement then each subplane of B lies in the same net of degree $p^k + 1$.

Proof: In this case, the group generated by any pair of Baer p -collineations is $SL(2, p^z)$ for some positive integer z. Since any two distinct axes π_0 and π_1 share exactly the zero vector, we may decompose the space as $\pi_0 \oplus \pi_1$ so that the collineation group has the form

$$\left\langle \begin{bmatrix} a & b \\ c & d \end{bmatrix}; ad - bc = 1 \text{ for all } a, b, c, d \text{ in } K \simeq SL(2, p^z) \right\rangle.$$

In particular, we have the subgroup $\left\langle \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}; a \text{ in } K - \{0\} \right\rangle$. Choose a in the prime subfield $F \simeq GF(p)$ of K and since $p > 3$, we may assume that $a \neq a^{-1}$. We note that a field of $2k \times 2k$ matrices over a field $GF(p)$ contains the scalars αI_n. Hence, $a = \lambda I_{2k}$ for λ in $GF(p) \subseteq$the kernel of the translation plane.

In other words, $\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$ is a kernel homology if a is in the prime subfield of K. Hence, it follows that $\begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} = g_a$ acts as a scalar group on each subplane π_0 and π_1 so fixes each line of π_o and each line of π_0 incident with the zero vector. But, $\begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} = \begin{bmatrix} a^2 & 0 \\ 0 & 1 \end{bmatrix} = h \neq I$ fixes each line of π_0 and fixes π_1 pointwise. Since the fixed lines of h are exactly the lines of π_1, it follows that each line of π_0 extending to a line of π is a line of π_1. Hence, each line of π_0 incident with the zero vector is a line of π_1 and conversely. Hence, the lines of π_0 incident with the zero vector are exactly the lines of π_1 which are incident with the zero vector.

Furthermore, more can be said about the structure of the net containing the Baer axes and we shall come back to this in the next section in more generality both for even order and for infinite order.

13.2 Incompatibility Theory: Even Order Translation Planes.

We have seen in the previous section that, when p is odd, it is not possible that elations and Baer p-collineations can coexist in translation planes of order p^r. This is definitely not the case in planes which are not translation planes. For example, there exist semi-translation planes of order q^2 derived from dual translation planes for which there is a Baer group of order q and an elation group of order q as well. Furthermore, it is possible that Baer involutions and elations exist even in Desarguesian affine planes of even order. If π is Desarguesian of order q^2 coordinatized by $GF(q^2)$ then the field automorphism of order 2 which fixes $GF(q)$ pointwise induces a Baer involution.

When π is a semifield plane of even order, Ganley [14] has shown that if there is a Baer involution then the full group which fixes the Baer axis pointwise has order 2.

Exercise 13.2.1 *Let π^+ be any projective plane and π_o^+ a projective subplane. Let σ be a central collineation. Then show that σ leaves π_o^+ invariant if and only if the center and axis of σ are in π_o^+ and for some point P of π_o^+ then σP is also a point of π_o^+.*

Exercise 13.2.2 *Let π be a semifield plane with special point (∞) on the line at infinity. Let π_o be an affine subplane of order h of π one of whose parallel classes is (∞). Show there exists an elation group of order h which leaves π_o invariant.*

Note that, in a semifield plane of even order q^2, if there exists a Baer subplane sharing the special point on the line at infinity then there exists an elation group of order q which leaves the subplane invariant.

13.2.1 Maximal Elation Groups and Baer involutions.

Here we consider this more generally. The reader is referred to Jha and Johnson [21] for more details.

Theorem 13.2.3 *Let π be a translation plane of even order q^2 for $q = 2^r$. Let π_o be a Baer subplane of π which is fixed pointwise by a Baer 2 -group B. If π admits an elation group $\mathcal{E}$ of order q which normalizes $\mathcal{B}$ then $\mid \mathcal{B} \mid \leq 2$. If $\mid \mathcal{B} \mid = 2$ then the full collineation group which fixes π_o pointwise has order 2.*

Proof: The proof will be given as a series of lemmas. In particular, we shall require a more-or-less standard representation of the translation plane and Baer subplane.

Represent π is the form $\{(x_1, x_2, y_1, y_2);\ x_i, y_i$ are r -vectors over $GF(2)$ for $i = 1, 2\}$. Represent with equation $x_1 = y_1 = 0$ and consider a spread for π is the form l$x = 0, y = 0, y = xM$ where x is a $2r$ -vector and M is a nonsingular $2r \times 2r$ matrix. We also assume, with loss of generality, that $x = 0, y = 0, y = x$ are components of π_o also and that the axis of $\mathcal{E}$ is $x = 0$.

This first lemma depends on the previous representations and should be clear by now.

Lemma 13.2.4 *Let the kernel of π_o be denoted by K_o where K_o is consider as the set of $r \times r$ matrices centralizing the slopes of π_o.*

(i) $\mathcal{B}$ may be represented in the following form:

$$\left\langle \begin{bmatrix} I & B & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & B \\ 0 & 0 & 0 & I \end{bmatrix}; B \in \lambda \text{ and } 0, I \in \lambda \right\rangle.$$

(ii) The components of π_o may be represented in the form

$$x = 0, y = x \begin{bmatrix} C & f(C) \\ 0 & C \end{bmatrix}$$

for C in a set Ω of matrices where $f : \Omega \longmapsto Hom_{GF(2)}(V_{2r}, V_{2r})$ where V_{2r} is a $2r$-dimensional vector space over $GF(2)$ such that $f(I) = f(0) = 0$.

Exercise 13.2.5 *Prove that λ is contained in the kernel K_o of π_o.*

Note that since we are assuming that $\mathcal{E}$ normalizes $\mathcal{B}$, it follows that E acts transitively on the non-axis components of π_o . Hence, we have

Lemma 13.2.6 *$\mathcal{E}$ may be represented in the form*

$$\left\langle \begin{bmatrix} I & 0 & C & f(C) \\ 0 & I & 0 & C \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}; C \in \Omega \right\rangle.$$

Exercise 13.2.7 *Prove that if $\mathcal{B}$ has order > 2 then we may take λ to include $\{0, I, B, B+I\}$ for some fixed $B \neq 0$ or I.*

Now let $\sigma_D = \begin{bmatrix} I & D & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & D \\ 0 & 0 & 0 & I \end{bmatrix} \in \mathcal{B}$ and let $\tau_E = \begin{bmatrix} I & 0 & E & f(E) \\ 0 & I & 0 & E \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \in \mathcal{E}$.

Exercise 13.2.8 *Show that $\sigma_D\tau_E$ is a Baer involution and a component $y = x\begin{bmatrix} m_1 & m_2 \\ m_3 & m_4 \end{bmatrix}$ is fixed by $\sigma_D\tau_E$ if and only if $m_3 = D^{-1}E$ and $Dm_4 = f(E) + ED + m_1D$. (Hint: Write out what the conditions are for a component to be fixed by $\sigma_D\tau_E$ recalling that D is in the kernel of K_o and hence commutes with E).*

Lemma 13.2.9 *Let $S_D = \{\sigma_D\tau_C;\ C \in \Omega\}$. The components by elements of S_D cover π. Hence, this implies that, for each $C \in \Omega$, $B^{-1}C$ is also in Ω and furthermore, B^i and B^jC is in Ω for all integers i, j.*

Exercise 13.2.10 *Prove the previous lemma.*

Thus, we have:

Lemma 13.2.11 *$\langle\sigma_I\tau_{B^{-1}C}, \sigma_B\tau_C\rangle$ fixes the same Baer subplane pointwise (namely, $\{(0, y_2B^{-1}C, y_1, y_2)\}$).*

Hence, $Dm_4 = f(E) + ED + m_1D$ for $(D, E) \in \{(B, C), (I, B^{-1}C), (B + 1, (B^{-1} + 1)C\}$.

Choose $(D, E) = (I, B^{-1}C)$, we obtain $m_4 = f(B^{-1}C) + B^{-1}C + m_1$. Now reapplying (B, C), we obtain

$$Bm_4 = B(f(B^{-1}C) + B^{-1}C + m_1) = f(C) + CB + m_1B$$

which implies that

$$Bf(B^{-1}C) + (B + I)C + f(C) = Bm_1 + m_1B.$$

Let $g_0(C) = Bf(B^{-1}C) + f(C)$.

Exercise 13.2.12 *For $k \geq 1$ if*

$$g_k(C) = g_{k-1}(C)B^{2k-1} + B^{2k-1}g_{k-1}(C)$$

then

$$g_k(C) = B^{2^k}m_1 + m_1B^{2^k}.$$

(Hint: Recall that $BC = CB$.)

Since B is in the kernel of the subplane of order q , it follows that $B^q = B$.
Hence, when $q = 2^r$, it follows that $g_r(C) = Bm_1 + m_1 B = g_0(C) + (B+I)C$.

Lemma 13.2.13 *Let* $J(C) = \sum_{i=1}^{q-1} B^i f(C) B^{-i}$.
Then
(i) $J(C) + BJ(B^{-1}C) = (I+B)C$ *and*
(ii) $J(B^2C) = B^2 J(C)$.

Proof:

Exercise 13.2.14 *Show that* $g_k(C) = \sum_{i=o}^{2^k-1} B^i f(C) B^{(2^k-1)-i} + \sum_{i=1}^{2^k} B^i f(B^{-1}C) B^{2^k-i}$.
Then let $k = r$ *and using the fact that* $g_r(C) + g_{0(C)} = (I+B)C$ *conclude that (i) is valid.*

Exercise 13.2.15 *Since* BC *is in* Ω, *replace* C *by* BC *in (i) to conclude (ii).*

Since the above lemma is valid for all elements C of Ω, letting $C = I$, we obtain by induction that

$$J(B^{2^k}) = B^{2^k} J(I) = 0.$$

Letting $r = k$, we have that $J(B) = 0$. In (i) above, let $C = B$ to obtain $(I+B)B = 0$. Hence, $B = 0$ or I contrary to our assumptions. Hence, the Baer 2-group has order 2 or 1. If the order is 2 then since the group fixing the Baer axis normalizes the 2-group fixing it pointwise, it follows that any Baer group must commute with a given Baer involution which cannot occur unless the group has order 2 itself. This completes the proof of the theorem.

13.2.2 Large Baer groups and Elations.

Considering possible incompatibility relations, we consider the co-existence of a 'large' Baer group and an elation group of order > 2. Recall that it follows from the previous subsection that the existence of a Baer group of order $> \sqrt{q}$ shows that the Baer axis is a Desarguesian subplane. In this subsection, we consider the possible incompatibility with Baer groups of order $> \sqrt{q}$ and elation groups of order > 4.

Previously, we required that a given elation group normalizes a Baer group and hence centralizes it. A result of Dempwolff [9] shows that if a Baer group of order $> \sqrt{q}$ normalizes an elation group E then it must centralize it.

Exercise 13.2.16 *Let π be a translation plane of order 2^r that admits a Baer group $\mathcal{B}$ of order $> \sqrt{q}$. Let E be any affine elation group. Let S_2 be a Sylow 2-subgroup containing the full elation group E^* with axis E. Show that there exists a Baer group $\mathcal{B}^*$ of order $\mid \mathcal{B} \mid$ contained in S_2. Show that $\mathcal{B}^*$ normalizes the full group E^*.*

Hence, if we use the result of Dempwolff, we may assume the existence of an elation group E and a Baer group $\mathcal{B}$ of order $> \sqrt{q}$ which centralizes each other.

Theorem 13.2.17 *Let π be a translation plane of order $q^2 = 2^{2r}$ that admits a Baer group of order $\geq 2\sqrt{q}$. If E is any elation group of π then $\mid E \mid \leq 2$.*

Proof: We formulate the proof in a manner similar to the above. In particular, we take the representation exactly as in the previous subsection. However, now we know that the elements of λ belong to a field $K \simeq GF(q)$ that coordinatizes the Baer subplane so that we may assume that the elements of Ω belong to the field K. ∎

Lemma 13.2.18 *For each C of Ω, then $\mid C\lambda \cap \lambda \mid \geq 4$.*

Proof: Note that λ is a vector space over $GF(2)$ as it is additive. Similarly, $C\lambda$ is a vector space over $GF(2)$. Furthermore, dim$\lambda > r/2$ so $\geq r/2+1$. Hence, $C\lambda + \lambda$ is a subspace of K so that the dimension of the intersection $C\lambda \cap \lambda$ is at least 2. Hence, the order is at least 2^2.

The impact of the previous lemma is that there are at least two Baer groups of order 4 which come from the same element τ_C.

Lemma 13.2.19 *For each C in Ω, there exist distinct nonzero elements E and F such that*

$\langle \sigma_B \tau_I, \sigma_{BC} \tau_C \rangle$ fixes a Baer subplane $\{(0, y_2 B, y_1, y_2)\}$ pointwise for $B \in \{E, F\}$.

Proof: Let CE and CD be in $C\lambda \cap \lambda$. Recall that $\sigma_D \tau_E$ fixes $y = x \begin{bmatrix} m_1 & m_2 \\ m_3 & m_4 \end{bmatrix}$ if and only if $m_3 = D^{-1}E$ and $Dm_4 = f(E) + ED + m_1 D$. Thus, the indicated group must fix the same Baer subplane pointwise.

Noting that $f(I) = 0$, let $D = B$ and $E = I$ to obtain $Bm_4 = B + m_1 B$. But, also we may let $D = BC$ and $E = C$ to obtain $BCm_4 = f(C) + BC^2 + m_1 BC$.

Thus, $C(B + m_1 B) = f(C) + BC^2 + m_1 BC$.

Exercise 13.2.20 *Show that* $C^2 m_1 B + m_1 B C^2 = f(C)C + Cf(C)$.

Exercise 13.2.21 *Let* $f(C) = f_0(C), f_1(C) = f(C)C + Cf(C)$ *and, in general, let*

$$f_k(C) = C^{2^{k-1}} f_{k-1}(C) + f_{k-1}(C) C^{2^{k-1}}.$$

Show that $f_k(C) = C^{2^k} m_1 B + m_1 B C^{2^k}$.

Now let $k = r$ where $q = 2^r$. Then, $f_r(C) = C^{2^{r-1}} f_{r-1}(C) + f_{r-1}(C) C^{2^{r-1}} = C m_1 B + m_1 B C$. From $C(B + m_1 B) = f(C) + BC^2 + m_1 BC$, we obtain $C m_1 B + m_1 BC = CB + CB^2 + f(C)$.

Hence, $(C + C^2)B = C^{2^{r-1}} f_{r-1}(C) + f_{r-1}(C) C^{2^{r-1}} + f(C)$. Since B can take on either of the nonzero elements E or F, this can only occur when $C + C^2 = 0$ and hence that $C = 0$ or I. Hence, we have shown that the only possible elations τ_C are τ_0 and τ_I. That is, the elation group has order at most 2.