Chapter 12

Hering-Ostrom Theory: Elation-Generated Groups.

The celebrated Lenz-Barlotti theory describes maximal groups of central collineations of arbitrary projective and affine planes. Similarly, one might ask for a description of groups that are generated by groups of central collineations of a plane that share neither an axis nor a center. For example, in affine Desarguesian planes of order p^r, if E_1 and E_2 are groups of elations with distinct affine axes they generate the group $G \cong SL(2, p^s)$ whenever s divides r.

A fundamental theorem of Ostrom asserts that the same conclusion holds for arbitrary finite translation planes with characteristic $p > 3$. The case $p \leq 3$ has been completely resolved by Hering, where the conclusions are slightly more complicated: for example, in the spreads associated with the even order Lüneburg planes, elation groups might generate Suzuki groups. Taken together, the Hering-Ostrom theorem provides a complete description of groups G generated by affine elations of [partial] spreads and has proven be a powerful tool for the investigation of finite translation planes.

It is thus natural to seek to generalise this theorem. Major results on finite translation planes have been obtained by Foulser based on extending the Hering-Ostrom theorem to generalised elations. It turns out that a generalised elation, in the context of a spread, is either an affine elation or a Baer p-element, and this leads to a Baer analogue of the Ostrom's theorem. In the next chapter, we use this to establish striking incompatibilty results concerning Baer p-elements and affine elations, and also incompatibility among Baer p-elements that have different slopesets in odd characteristic. These results are due to Foulser in the odd characteristic case. In characteristic 2, Foulser's results do not apply as there are counterexamples. However, as demonstrated by the authors via group-theoretic results of Dempwolff, there is still still a high degree incompatibility between Baers and elations, even in spreads of even order. In all cases, the incompatibilities indicated have a profound influence on the collineation group of a translation plane. For example, it implies that semifield planes of odd order cannot admit Baer p-elements.

One of the main goals of the present chapter is to prove Ostrom's result, describing the groups G generated by elations acting on finite partial spreads of characteristic $p > 3$; we refer the reader to Lüneburg's immaculate treatment [31] for the full Hering-Ostrom theorem. In addition to Ostrom's theorem, and its generalization to finite-dimensional spreads, we shall establish Foulser's analogue of Ostrom's result that applies to generalised elations: this will be applied in the next chapter to establish the incompatibility theorems indicated above.

12.1 Field Extensions and Spreads.

Let $V = F^n \oplus F^n$, and let $K \supset F$ be an extension field of the finite field F. Rather than using tensor product notation, we shall write: $V_K = K^n \oplus K^n$, $X_K = K^n \oplus \mathbf{0}$, $Y_K = \mathbf{0} \oplus K^n$; in general if W is an F-subspace of V then W_K denotes the K-subspace of V_K generated by W; so W_K consists of all the K-linear combination of any F-basis of W. This follows by noting that F-independent subsets in V are also K-independent: look at the rank of the matrix M_W of any F-linear basis of W: the rank of M_W, whether viewed as an F-matrix or as a K-matrix is always the same.

Next consider $g \in Hom(V, F)$; g_K is the unique extension g_K of g to $Hom(V, K)$ and the two maps have the same matrix relative to any F-basis of V, in particular relative to the canonical basis. So $g \in GL(V, F)$ if and only if $g_K \in GL(V_K, K)$.

We shall be particularly concerned with the action that a group $G \leq GL(V, F)$ induces on a K-subspace $U \leq V_K$ that is G_K-invariant, sometimes when $V \cap U = \mathbf{O}$. In all cases, the action of G_K on U is just the action associated with the matrix group representing G, and we write G^U to mean G_K^U, the action of G_K on U.

A spread Γ on V corresponds in the obvious way to a *partial* spread Γ_K

of V_K, and Γ includes the standard components X and Y of $F^n \oplus F^n$ iff Γ_K does: that is X_K and Y_K lie in Γ_K, and a similar comment applies to the unit line I. We always assume that we are dealing with spreads and partial spreads containing the standard components X and Y, as well as the unit line I. Let $\mathcal{M}$ be the spreadset of matrices defining Γ; so $\mathcal{M}$, viewed as a set of K-matrices is a *partial* spreadset defining the partial spread Γ_K.

Next focus on a rank two K-subspace $U \leq V_K$ that meets non-trivially the subsaces X and Y of V, and let Γ_U be the set of all components $\gamma_K \in \Gamma_K$ that meet U non-trivially. Since U has rank 2 over K, Γ_U is a Desarguesian K-spread on U, and it meets non-trivially each of X, Y and I, in three distinct components.

Next suppose $G \leq GL(V, F)$ preserves Γ and such that G_K leaves U-invariant. So G_K is a K-linear automorphism group of the partial spread Γ_K and also leaves U invariant. Thus $G_K^U \leq GL(2, K)$. Moreover, the given elation groups continue to act as elation elation groups on the Desarguesian spread Γ_U, so $G_K^U \cong SL(2, K')$ for some $K' \subseteq K$. The close connection between G_K^U and G leads to a similar conclusion for G, as required.

This suggests a strategy: take any F-spread admitting G, then seek an extension field K over which G fixes a 2-space made up of distinct eigenvectors of some normal subgroup of G and then apply the above argument.

Returning to the main theme, assume G acts transitively on the non-zero points of U. Now the components of Γ that meets U non-trivially do so in at least one non-zero point of V, so the components of Γ_K induce the standard Desarguesian spread on U. Note that the point of this claim is that the components of the standard Desarguesian spread that U carries, simply because it is a 2-dimensional space, must *extend* to components of Γ_K.

Suppose now that the p-Sylow subgroups of G are non-trivial but not planar. So if P is such a group then V_P is a component of Γ. By the conjugacy of Sylow subgroups it follows that the associated components, which we call p-axes [of G] form a G-orbit Now P certainly fixes a component of the Desarguesian spread Δ_U. Also wlog X is the axis of P. So if more than one axis is involved then the transitivity of P on the axes implies that the axes all meet U non-trivially and each corresponds to the axis of a shears group of Δ_U. The non-planarity hypothesis means that P acts faithfully on Δ_U and hence is elementary abelian. All these groups generate $SL(2, L)$ on Δ_U, where $F \leq L \leq K$.

12.2 Algebra Generated By Matrix A.

Let A be an $n \times n$ matrix over a field F, and define the F-algebra generated by A to be the smallest ring $< A >$ of matrices containing A and $F\mathbf{1}$. Since we have finite dimension, A satisfies a unique monic minimum polynomial $f(x) = \sum_{i=0}^{k-1} f_i x^i$ over F; thus

$$A^k = \sum_{i=0}^{k-1} f_i A^i = 0,$$

and we have an algebra isomorphism:

$$< A > \cong F[x]/(f(x)).$$

Thus we have:

Remark 12.2.1 *$< A >$ is a field iff its minimum polynomial $f(x)$ is irreducible and now $< A >$ is isomorphic to an extension field of F by any of the roots of $f(x) = 0$.*

Now, even in the general case, if λ is an eigenvalue of A then $f(\lambda) = 0$, so if $f(x)$ is *irreducible* then the algebra $F(\lambda) \cong F[x]/(f(x))$ is the extension field of F by λ. But the eigenvalues of A are just the roots of $f(x) = 0$, since the minimum and the characteristic polynomials have the same roots. In particular, the eigenvalues of A are all congugate in the algebraic closure of F. Hence the previous remark may be restated as:

Remark 12.2.2 *The F-algebra $< A >$ is a field iff its minimum polynomial $f(x)$ is irreducible and now $< A >$ is an extension field of F such that $< A > \cong F(\lambda)$, where λ is any eigenvalue of A; the fields $F(\lambda)$ are isomorphic as λ ranges over the eigenvalues of A.*

We can now consider the the case of interest: when the F-algebra $< A >$ does *not* contain any non-zero singular matrices. In this case, if for some non-zero $T \in < A >$ the minimum polynomial $f_T(x) = g_T(x)h_T(x)$, where $\min[\partial g, \partial h] \geq 1$, and $T \in < A >$ then $g_T(T)$ and $h_T(T)$ are both non-zero and singular matrices since ther product $f_T(T)$ is zero. This contradicts our assumption that the non-zero elements in $< A >$ are non-singular, so we have:

Proposition 12.2.3 *If the F-algebra $< A >$, i.e. the polynomial ring $F[A]$, is a partial spreadset of matrices then it is a field of matrices isomorphic to the field $F(\lambda)$, where λ may be chosen to be any eigenvalue of A: these are all conjugate over F.*

12.3 Properties of $SL(n, K)$.

In this section, we mention a couple of properties of the unimodular group $SL(2, q)$. The first property is will be tacitly assumed in several places.

Theorem 12.3.1 *Let $GL(n, K)$ be the group of non-singular maps of an n-dim-ensional vector space over a finite field K and let $SL(n, K)$ be its full unimodular subgroup.*

If H is a subgroup of $GL(n, K)$ such that $H \cong SL(n, K)$ then $H = SL(n, K)$.

Proof: Let p denote the characteristic of K. Then every $SL(n, K)$ in $GL(n, K)$ is generated by the set of all Sylow p-subgroups of $GL(n, K)$, and these are all in the 'standard' unimodular group $SL(n, K)$ since this group is normal in $GL(n, K)$ and contains at least one of the Sylow p-groups of $GL(n, K)$. ∎

In the infinite case the Sylow 'p-subgroups' may be identified with the maximal groups that have characteristic polynomial $(x - 1)^n$, and these groups are generated by all the transvections, and all transvections are conjugate by a basis-change argument. Such considerations permit the extension of the above theorem to the case where K is any infinite field.

We record for convenience:

$$\rho := \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}; \quad \sigma = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}. \quad \Longrightarrow \quad \tau := \sigma^{-1}\rho^{-1}\sigma^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{12.1}$$

12.4 Ostrom's Theorem.

We adopt the notation:

$$\rho = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}; \quad \rho_M = \begin{pmatrix} 1 & M \\ 0 & 1 \end{pmatrix}.$$

From now on until Ostrom's theorem has been established we shall assume:

Hypothesis 12.4.1 *$\mathcal{N} \supset \{X, Y, 1\}$ is a partial spread on $V = F^n \oplus F^n$ admitting an automorphism group $G = <\rho, \rho_A>$, for some $A \neq \mathbf{0}$.*

The following elementary observations associated with the above hypothesis will be frequently used:

Remark 12.4.2 *The maps ρ and ρ_A are non-trivial elations of $\mathcal{N}$ with axis X and Y respectively. Moreover:*

1. *The map $\rho_A \in Aut\mathcal{N}$ maps X to $y = xA$; more generally an elation of $\mathcal{N}$ with axis Y mapping X onto a component $y = xM$ must be the map ρ_M, and conversely if $\rho_M \in Aut\mathcal{N}$ then it is an elation of the type just mentioned.*

2. *If $\rho_M \in Aut\mathcal{N}$ then M is non-singular; so A is non-singular.*

3. *The group G_Y of all Y-axes elations in G is isomorphic to an additive group of matrices $\mathcal{E}$ contained in the full set of slopes of $\mathcal{N}$. In fact*

$$\mathcal{E} = \{E \in F_n \mid \rho_E \in G\},$$

or equivalently

$$\mathcal{E} = \{E \mid y = xE \in Orb_{G_Y}(X)\} \cong G_Y.$$

4. *The elation ρ maps Y onto the unit line: $y = x := \mathbf{1}$.*

5. *The Y-orbit under G includes the unit line $\mathbf{1}$ among its components.*

Proof: The maps ρ and ρ_A are both elations of $\mathcal{N}$ since their fixed spaces are precisely components, viz., X and Y respectively. All the listed items are equally trivial to verify. ■

Now suppose U is any G-invariant rank 2 K-subspace of V_K, using our standard notation, see page 197. So U cannot be part of a component since G contains non-trivial elation groups with distinct axes. Moreover, both ρ and ρ_A are elations of U, viewed as a K-spread, and this spread is Desarguesian because it has order $|K|$ and K is in the kern. So G_K induces a unimodular group $\tilde{G}$ of U. Furthermore, distinct elation axes associated with non-trivial elations in G must meet U in *distinct* components:

Lemma 12.4.3 *Suppose α and β are components of $\mathcal{N}$ such that each is the axes of a non-trivial elation in G. Then $\alpha_K \cap U$ and $\beta_K \cap U$ are distinct components of U.*

Proof: Let A and B be the groups of elations of $\mathcal{N}$ whose axes are respectively the components $\alpha, \beta \in \mathcal{N}$. Since A and B are both non-trivial p-groups and are K-linear each fixes a 1-dimensional K-subspace of U elementwise.

These spaces are disjoint since α_K and β_K are distinct components of $\mathcal{N}_K$. ∎
The following proposition shows that the group $G =< \rho, \rho_A >$ leaves invariant a rank 2 K-subspace U of V_K and induces on U the group $SL(2, K)$, *when* K *is taken to be* $F(\lambda)$, where λ is an eigenvalue of A. Thus establishing Ostrom's theorem will mainly involve showing that the G induces $SL(2, K)$ *faithfully* on U.

Proposition 12.4.4 *Assume $F = GF(p)$ is a prime field, $p > 3$, and fix the extension field $K = F(\lambda)$, where λ is any eigenvalue of an F-matrix A, in the algebraic closure of F. Then the group $G =< \rho, \rho_A >$ leaves invariant a rank two K-space U such that $G^U = SL(2, K)$.*

Proof: There is a K-matrix B such that

$$B^{-1}AB = \begin{pmatrix} \lambda & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}, \tag{12.2}$$

now by a direct computation

$$\beta\rho\beta^{-1} = \rho, \tag{12.3}$$

where $\beta = Diag(\mathbf{B}, \mathbf{B})$.

Similarly the β-conjugate of ρ_A is given by:

$$\begin{pmatrix} \mathbf{B} & \mathbf{0} \\ \mathbf{0} & \mathbf{B} \end{pmatrix}\begin{pmatrix} \mathbf{1} & \mathbf{A} \\ \mathbf{0} & \mathbf{1} \end{pmatrix}\begin{pmatrix} \mathbf{B}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{B}^{-1} \end{pmatrix} = \begin{pmatrix} \mathbf{1} & \mathbf{BAB}^{-1} \\ \mathbf{0} & \mathbf{1} \end{pmatrix}, \tag{12.4}$$

and by eqn 12.2 the RHS above has top row of form:

$$(\underbrace{1, 0, 0, \ldots, 0}_{n}, \underbrace{\lambda, 0, 0, \ldots, 0}_{n}),$$

so the β-conjugate of ρ_A leaves invariant the rank 2 K-space

$$\overline{U} = \left\{ (\underbrace{x, 0, 0, \ldots, 0}_{n}, \underbrace{y, 0, 0, \ldots, 0}_{n}) \mid x, y \in K \right\},$$

and similarly ρ, which is its own β-conjugate, by eqn (12.3), also leaves $\overline{U}$ invariant. Thus the β-conjugate group $\beta G \beta^{-1}$ of G leaves invariant the 2-space W and clearly induces on it the group

$$\overline{G} =< \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} >

and now by Dickson's analysis, for $p > 3$[1], the subgroup of $H = SL(2, K)$ generated by $\overline{G}$ is $SL(2, L)$, where $L = Z(\lambda)$, Z the prime field of F. But as our hypothesis species that $Z = F$ and $K = F(\lambda)$, we conclude that $\overline{G} = SL(2, K)$.
Thus we have shown a β-conjugate of G induces $SL(2, K)$ on a rank 2 subspace of V_K. Hence the same must hold for G. ■
From now we adopt the hypothesis and notation of the proposition above: $K = F(\lambda)$, where $F = GF(p)$ and λ is any eigenvalue of A; U is a G-invariant two-dimensional K-subspace of V_K, and as remarked earlier U is a rank-two K-space that is also a Desarguesian spread; so we have seen that $\tilde{G} = G^U = SL(2, K)$. It follows that $\tilde{G}$ is transitive on the non-zero points of U.

Lemma 12.4.5 *The set of axes $\mathcal{E}$ of non-trivial elations in G are in natural 1-1 correspondence with the components of U, i.e., the map*

$$\eta \in \mathcal{E} \mapsto \eta_K \cap U,$$

is a bijection from $\mathcal{E}$ onto the one-spaces of U.

Proof: Since by remark 12.4.2X is in $\mathcal{E}$, the transtivity of $\tilde{G}$ on U^* implies that every one-space of U is of form $\eta_K \cap U$, for some component $\eta \in \mathcal{E}$. The converse that every member $\mathcal{E}$ meets U in a component, has been mentioned in lemma 12.4. ■
In order to count the conjugacy classes of p-elements in $SL(2, q)$ consider:

$$\begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x^{-1} & 0 \\ 0 & x \end{pmatrix} = \begin{pmatrix} 1 & ax^2 \\ 0 & 1 \end{pmatrix},$$

and so we have:

Remark 12.4.6 *Let P be a p-Sylow subgroup of $SL(2, q)$, q a power of the prime p. Then $N(P)$ has at most two non-trivial conjugacy classes in P and distinct classes have the same cardinality.*

Lemma 12.4.7 *There is an additive group of matrices $\mathcal{A} \cong (K, +)$ such that the identity $I \in \mathcal{A}$ and the subgroup of Y-shears in G is:*

$$\left\{ \begin{pmatrix} 1 & M \\ 0 & 1 \end{pmatrix} \mid M \in \mathcal{A} \right\}$$

[1]This explains Ostrom's constraint $p > 3$.

Proof: Let E be the elation subgroup of G associated with the Y-axis. This induces faithfully an elation group on U with axis Y, faithful, because elations of G extend to elations of G_K and hence cannot fix any points outside a component. So E may be identified with a *sub*group of η the full elation group in G^U with axis Y: it is conceivable that $G_{\{Y\}}$ contains a p-group $P > E$ such that P, although not itself an elation group, induces on U the full elation group η, of size $|K|$.
Consider the $G_{\{Y\}}$-conjugacy class of any non-trivial $a \in E$. We show that $E \cong (K, +)$ by showing that this class has $> |K|/2$ elements, and noting that any elation group in G has order $\leq |K|$, since it must faithfully induce an elation group of U.
Consider any non-trivial elation $a \in E$. Hence for any $t \in G_{\{Y\}}$, $\tilde{t}\tilde{a}\tilde{t}^{-1}$ agrees on U with the elation $tat^{-1} \in E$ and distinct $\tilde{t}\tilde{a}\tilde{t}^{-1}$ are 'induced' by distinct tat^{-1}, since they have distinct actions on U. So the number of elations $\nu \in E$ must exceed the number of elations of U fixing Y that lie in a conjugacy class of the stabilizer Y in $\tilde{G}$. So by remark 12.4.6, and the fact that E contains the identity, shows that $|E| > (q-1)/2$, so $|K| \geq |E| \geq q + 1/2 > q/2$ and this forces $E = K$, by Lagrange's theorem, and the fact that E^U may be identified with a subgroup of $(K, +)$. Since E consists of matrices of type ρ_M, where $y = xM$ is a component of $\mathcal{N}$ meeting U non-trivially, the desired result follows once we have noted $\mathbf{1} \in \mathcal{A}$. This holds because by remark 12.4.2.5 the unit line $y = x$ of $\mathcal{N}$ is in the G-orbit of Y and hence meets U non-trivially: so $\rho_1 \in G$ means that $\mathbf{1} \in \mathcal{A}$. ∎

Lemma 12.4.8 *The additive group $\mathcal{A} \cong (K, +)$ is also closed under inversion of its non-zero elements.*

Proof: Since $\mathbf{1} \in \mathcal{A}$, we have $-\mathbf{1} \in \mathcal{A}$, and the corresponding automorphism $\sigma \in G$. Hence by eqn 12.1 G contains:

$$\tau = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ -\mathbf{1} & \mathbf{0} \end{pmatrix}$$

and now any component $y = xM$ of $\mathcal{N}$ moves under τ to the component $y = x(-M^{-1}) \in \mathcal{N}$. So for, $M \in \mathcal{A}$, we must have $-M^{-1} \in \mathcal{A}$ since G, and hence τ, maps components meeting U into other components of the same type. But since τ is an additive group $M^{-1} \in \mathcal{A}$. ∎

Lemma 12.4.9 *G contains the map:*

$$\delta_A : (x, y) \mapsto (xA, yA^{-1}) >$$

Proof: Since $\mathcal{A}$ is closed under inversion we have $\rho_{A^{-1}} \in G$, since $A \in \mathcal{A}$. Hence G contains the map

$$\rho_A \tau^{-1} \rho_{A^{-1}} \tau \rho_A \tau^{-1},$$

which by a direct calculations is the matrix $Diag[A, A^{-1}]$ defining δ. ∎
The following result is essentially the theorem of Ostrom. It implies that if two elations with distinct axes fix a characteristic p partial spread $\mathcal{N}$, with $p > 3$ and $|\mathcal{N}| > 2$, then the group they generate a group $G \cong SL(2, q)$ and G leaves invariant a rational Desarguesian net contained in $\mathcal{N}$.

Theorem 12.4.10 (Ostrom's Elation Theorem.) *The spreadset $\mathcal{A}$ is a field $\cong K$, and $G = SL(2, \mathcal{A} \cong SL(2, K)$. Morover, the partial spread $\Delta_{\mathcal{A}}$ associated with $\mathcal{A}$ is a rational Desarguesian partial spread and G has the standard action on this partial spread, induced by its standard action on $\Delta_{\mathcal{F}}$, a Desarguesian spread associated with a field extension $\mathcal{F}$ of the field $\mathcal{A}$.*

Proof: We first establish that the polynomial ring $F[A]$ is a field $\cong K$. Since δ_A maps the component $y = xM$ onto $y = xA^{-1}MA^{-1}$, we have $A^{-1}MA^{-1} \in \mathcal{A}$, whenever $M \in \mathcal{A}$. Choosing M from $A, I \in \mathcal{A}$, we see that all odd and even powers of A^{-1}, and hence all powers of A lie in $\mathcal{A}$. But since $\mathcal{A}$ is an additive group it is also an F-module, over the prime field F. Thus the polynomial ring $F[A]$ is a subset of $\mathcal{A}$. But the non-zero elements of the algebra $< A >$ are invertible and, of course, closed under differences. Thus the algebra $< A >$ is also a partial spreadset of matrices, and hence, by proposition 12.2.3, the algebra is isomorphic to the field $F(\lambda) = K$.
But since, by lemma 12.4.8 $\mathcal{A} \cong (K, +)$, we now have $\mathcal{A} =< A >$ is a field of matrices $\cong K$. So, by lemma 12.4.7 we clearly obtain $< \rho, \rho_A >= SL(2, \mathcal{A}) \cong SL(2, K)$. Moreover, by lemma 12.4.7, the components of $\mathcal{N} - \{Y\}$, meeting U non-trivially are just those of form $y = xk$, $k \in \mathcal{A} \cong K$.
Next choose a matrix field $\mathcal{F} \supset \mathcal{A}$ such that $|\mathcal{F}| = |F|^n$, so the associated Desarguesian spread $\Delta_{\mathcal{F}}$ contains the partial spread associated with $\mathcal{A}$, that is the components of $\mathcal{N}$ that meet U non-trivially, or equivalently, the components of $\mathcal{N}$ that are the axis of non-trivial elations in the group G.

Thus $G \cong SL(2, K)$ leaves invariant a Desarguesian spread $\Delta_{\mathcal{F}}$ such that the components of $\mathcal{N}$ that are the non-trivial elation axes of elements in G, when G is regarded as acting on $\mathcal{N}$, form the slopeset of a subplane of $\Delta_{\mathcal{F}}$. [This may be esablished even without reference to 'U', since the only way that $SL(2, K)$ acts on a Desarguesian spread over a larger finite field $F \supset K$ is to leave invariant the subplane $K \oplus K$.] ∎

Ostrom's theorem needs to be slightly modified if we permit characteristic $p = 3$. We summarize without proof the situation when $p = 3$ is permitted in Ostrom's theorem.

Theorem 12.4.11 *Let π be a finite translation plane of odd order p^r. Let σ and τ denote two elations in the translation complement with distinct axes. Then one of the two following situations occur:*

1. *$\langle\sigma, \tau\rangle \cong SL(2, p^z)$ for some positive integer z and the elation net is a Desarguesian net which may be coordinatized by $GF(p^z)$.*

2. *$\langle\sigma, \tau\rangle \cong SL(2, 5)$ and $p = 3$ and the elation net is a Desarguesian net which may be coordinatized by $GF(9)$.*

Finally, it is noted that Ostrom's theorem is actually more general than considered above and can be more generally applied to collineation groups generated by Baer p-groups. Note that what needs to be considered is whether the group generated by the set of all elations is also isomorphic to $SL(2, p^w)$ for some positive integer w and what occurs when $p = 2$ or 3.

We also may observe that this result is generally valid over finite dimensional vector spaces of characteristic p. The proof given uses the above result to deal with the exceptional case when $p = 3$, but is otherwise self-contained although it largely follows the Ostrom argument described above.

Theorem 12.4.12 *Let π be a translation plane which is finite dimensional over its kernel and let K be a subfield of the kernel of characteristic not 2.*

Let σ and τ be affine elations with distinct axes in the translation complement and let $G = \langle\sigma, \tau\rangle$. Let N denote the net each of whose components are axes of elations in G.

If G is finite then the characteristic of π is $p < \infty$ and one of the two following situations occur:

1. *$G \cong SL(2, p^s)$ for some positive integer s. Furthermore, N is a Pappian net which may be coordinatized by $GF(p^s)$.*

2. $G \cong SL(2,5)$. *In this case, N is a Pappian net which may be coordinatized by $GF(9)$.*

Proof: Assume the dimension of π over K is $2k$. Represent σ by $(x,y) \longmapsto (x, Ax+y)$ and τ by $(x,y) \longmapsto (x+y,y)$ where A is a $k \times k$ matrix with elements in K.

Note that the order of σ is finite if and only if the characteristic is finite p.

The proof of the theorem now follows from the following sequence of lemmas.

Lemma 12.4.13 *Let λ be an eigenvalue of A in some extension field $K(\lambda)$. Then A and hence λ has finite order and*

$$F = GF(p)(\lambda) \simeq GF(p^s),$$

for some positive integer s.

Proof: Consider $\sigma\tau = \begin{bmatrix} I & A \\ 0 & I \end{bmatrix}\begin{bmatrix} I & 0 \\ I & I \end{bmatrix} = \begin{bmatrix} I+A & A \\ I & I \end{bmatrix}$. Now square $\sigma\tau$ to obtain $\begin{bmatrix} (I+A)^2+A & (I+A)A+A \\ 2I+A & I+A \end{bmatrix}$. Squaring this element, we note that the entries in the $(1,1)$ -position are always nontrivial polynomial in A over $GF(p)$. If this element has finite order, it follows that eventurally the element in the $(1,1)$-entry is a polynomial in A over $GF(p)$ which is equal to $(1,1)$ -entry of a previous element in $\langle\sigma\tau\rangle$. Hence, A satisfies a polynomial over $GF(p)$. Thus, the minimal polynomial for A has coefficients in $GF(p)$ so that every eigenvalue in an extension field does as well.

Consider the field $GF(p)(\lambda)$ within $K(\lambda)$. Let the minimal polynomial for A have degree n so that every element in $GF(p)(\lambda)$ may be written in the form $\sum_{i=0}^{n} \lambda^i \alpha_i$ for $\alpha_i \in GF(p)$. Hence, $GF(p)(\lambda) = F$ is a finite field isomorphic to $GF(p^s)$ for some positive integer s.■

Lemma 12.4.14 *Let V denote the underlying vector space over K and let U be a 1-dimensional λ-eigenvector in $V \otimes_K K(\lambda) = V^*$. Then $U \oplus U$ is G-invariant.*

Proof: Realize σ and τ as linear transformations over V^* and apply the form to conclude that $U \oplus U$ is G-invariant. ■

Lemma 12.4.15 *$U \oplus U$ defines a Pappian plane which contains a G-invariant Pappian subplane π_o coordinatized by F.*

Proof: Since $F = GF(p)(\lambda)$ is a subfield of $K(\lambda)$, there is a Pappian subplane π_o of $U \oplus U$.

Since the elements in G restricted to $U \oplus U$ are all in $End_{GF(p)}\pi_o$, it follows that G leaves π_o invariant. ■

Lemma 12.4.16 *If $G \mid \pi_o$ is G^{π_o} then either $G^{\pi_o} \simeq SL(2,p^s)$ or $p = 3$ and $G^{\pi_o} \simeq SL(2,5)$.*

Proof: Since π_o is a finite translation plane of odd order, the result follows from Ostrom's theorem ([34] and [35]). ■

Lemma 12.4.17 *There are exactly $1 + p^s$ elation axes in N when G^{π_o} is $SL(2,p^s)$ and 10 elation axes in N when G^{π_o} is $SL(2,5)$.*

Proof: It follows exactly as in the previous section that every elation axis of N is also an axis of π_o. Since the group generated by the elations is transitive on the components of π_o(even in the case that the group is $SL(2,5)$ where $F \simeq GF(9)$ and there are 10 elations in $SL(2,5)$), we have that every component of π_o is an elation axis of N. ■

Lemma 12.4.18 *When the group $G^{\pi_o} \simeq SL(2,p^s)$ then $(\mid A \mid, p) = 1$ and $x = 0, y = xM$ for all M in $GF(p)[A]$ is a partial spread. Hence, $GF(p)[A]$ is a field.*

Proof: The arguments of the previous section can be utilized in this case to conclude that $\left\langle \begin{bmatrix} A^{-1} & 0 \\ 0 & A \end{bmatrix} \right\rangle$ is a collineation group of the translation plane.

Hence, $y = x$ maps to $y = xA^{2i}$ under the group and $y = xA$ maps to $y = xA^{2i+1}$. It follows that $A^j - I$ is nonsingular or zero for each integer j. ■

Lemma 12.4.19 *When the group $G^{\pi_o} \simeq SL(2,5)$ then $(\mid A \mid, p = 3) = 1$ and $x = 0, y = xM$ for all M in $GF(3)[A]$ is a partial spread so that $GF(3)[A]$ is a field.*

Proof: Hence, we may conclude that the net N is $\{x = 0, y = x(xA\alpha + \beta I)$ for all α, β in $GF(3)\}$ assuming that the Ostrom theorem is proved for finite planes in this case. Moreover, although not a collineation necessarily of the plane, the net admits the group $\left\langle \begin{bmatrix} I & A\delta + \gamma I \\ 0 & I \end{bmatrix}; \delta, \gamma \in GF(p) \right\rangle$. Now again apply the arguments of the previous section again, we may conclude again that the group $\left\langle \begin{bmatrix} A^{-1} & 0 \\ 0 & A \end{bmatrix} \right\rangle$ acts on the elation net N so that by the above argument, $(|A|, p = 3) = 1$. ■

Lemma 12.4.20 *The elation net is a Pappian net and the group induced on $U \oplus U$ is faithful.*

Proof: Let $V = X \oplus X$. Then X is a semi-simple $K\langle A \rangle$-module $= \sum_{i=1}^{w} N_i$. Let F_i denote the restriction of $K\langle A \rangle$ to N_i. Then N_i is a 1-dimensional F_i -algebra. Moreover, $GF(p)[A]$ is a field which forms a partial spread set so that $GF(p)[A]$ acts faithfully on each N_i. Since one of these N_i's may be taken as U, it follows that $GF(p)[A]$ is isomorphic to $GF(p^s)$ or $GF(9)$ exactly when the induced group on $U \oplus U$ is $SL(2, p^s)$ or $GF(9)$. ■
The main result theorem 12.4.12 has now been established.■

12.5 Generalized Elations.

In this section, we present the preliminaries for the theorem of Foulser on Baer p-groups acting on translation planes of order p^r. When $p \neq 2$, Foulser showed that the Baer axes of two distinct Baer p-collineations in the translation complement are identical or share exactly the zero vector. In the previous section, Ostrom's theorem was presented. This theorem can be viewed as a theorem on partial spreads generated by certain automorphism groups called generalized elations. Once this is achieved, it is possible to show that Ostrom's Theorem may be applied to conclude that the groups generated by Baer p-elements are exactly those in the elation case. Using the extension of Ostrom's theorem, it is possible to extend Foulser's work to the finite dimensional case as well.

In this section, we follow Foulser's work in [11].

Definition 12.5.1 *Let V be a vector space of dimension n over $K \simeq GF(p^r)$. Let σ be a linear transformation of V . Let $Fix\sigma$ denote the set of vectors fixed by σ. Then σ is said to be a generalized elation of V of type t if and only if σ fixes $V/Fix\sigma$ pointwise and the dimension of $Fix\sigma = t$..*

The subspace $Fix\sigma$ is called the 'axis'of σ and $C(\sigma) = (\sigma - 1)V$ is called the 'center'of σ.

Remark 12.5.2 *We have seen that elations are generalized elations of type $n/2$. Consider a Baer collineation σ of order p. We shall show that σ a generalized elation also of type $n/2$.*

Note that σ is a generalized elation if and only if $(\sigma - 1)^2 = 0$.

Proposition 12.5.3 *Let σ be a generalized elation of V of type t. Then*

(1) The order of σ is p;

(2) $\dim C(\sigma) + \dim Fix\sigma = n$;

(3) $t \geq n/2$ *and*

(4) If W is a complement of $Fix\sigma$ then, with respect to $Fix\sigma \oplus W$, σ has the following matrix representation

$$\begin{bmatrix} I & A \\ 0 & I \end{bmatrix}$$

where A is a $t \times (n-t)$ matrix.

Exercise 12.5.4 *Prove (1).*

Exercise 12.5.5 *Prove (2) noting that $V/Fix\sigma \cong (\sigma - 1)V$.*

Exercise 12.5.6 *Use (2) to prove (3) noting that $C(\sigma) \subseteq Fix\sigma$.*

Exercise 12.5.7 *Prove (4).*

Corollary 12.5.8 *The group generated by a set of generalized elations with the same axis is elementary Abelian of order p^a for some positive integer a.*

We now specialize to the case when σ is a generalized elation which is a collineation of a translation plane π of order $q^{n/2}$ with associated vector space of dimension n over a field $K \simeq GF(q = p^r)$.

We recall that if π_o^+ is a projective subplane of order m of a projective plane π^+ of order w then $m \leq \sqrt{w}$. Hence, if τ is a planar collineation of a translation plane then $Fix\tau$ has dimension less than or equal to half the dimension of the underlying vector space.

Theorem 12.5.9 *A generalized elation acting as a collineation of a finite translation plane of order p^s is either an elation or a Baer p-element.*

Proof. Note that we must have that a generalized elation σ is of type s if the order of the plane is p^s since the dimension of the vector space is $2s$ over $GF(p)$. Hence, the cardinality of $Fix\sigma$ is also p^s.

Exercise 12.5.10 *Show that if a collineation σ of an affine plane of order k fixes exactly k points then $Fix\sigma$ is either a line or a Baer subplane.*

It remains to show that a Baer p-element is a generalized elation.

Choose any complement W of $Fix\sigma$ so that with respect to the decomposition $Fix\sigma \oplus W$, we have the following representation for σ

$$\begin{bmatrix} I & A \\ 0 & B \end{bmatrix}$$

It remains to show that $B = I$. Note that the order of σ is p so we must have $B^p = I$.

Suppose L and M are components intersecting $Fix\sigma$ in a $s/2$-dimensional subspace. Choose a basis for the intersections with $Fix\sigma$ and extend to a basis for L and M and hence for the translation plane. With the decomposition $L \oplus M$, we have a basis of $4(s/2)$-vectors and letting x_i, y_i be $(s/2)$-vectors, the representation is (x_1, x_2, y_1, y_2) where M is $x_1 = x_2 = 0$, L is $y_1 = y_2 = 0$ and $Fix\sigma$ is given by the equation $x_2 = 0 = y_2$. Without loss of generality, we assume that $y = x$ is a component of $Fix\sigma$.

Now consider the $p^{s/2} + 1$-components of the translation plane that lie on $Fix\sigma$. These have matrix equations as follows $x = 0, y = 0, y = x, y = x\begin{bmatrix} B_{1i} & B_{2i} \\ 0 & B_{4i} \end{bmatrix}$ where it may be noted that the components of $Fix\sigma$ are $x_2 = 0, y_2 = 0, y_2 = x_2$ and generally $y_2 = x_2 B_{4i}$ for $i = 1, 2, ..., p^{s/2} - 1$.

Since the collineation fixes $x = 0, y = 0, y = x$ it follows that the form for σ is now

$$\begin{bmatrix} I & E & 0 & 0 \\ 0 & D & 0 & 0 \\ 0 & 0 & I & E \\ 0 & 0 & 0 & D \end{bmatrix}$$

Note that comparing the previous decomposition, we have $\begin{bmatrix} E & 0 \\ 0 & E \end{bmatrix} = A$ and $\begin{bmatrix} D & 0 \\ 0 & D \end{bmatrix} = B$.

Since σ fixes each line of $Fix\sigma$ so that $\begin{bmatrix} I & -ED^{-1} \\ 0 & D^{-1} \end{bmatrix} \begin{bmatrix} B_{1i} & B_{2i} \\ 0 & B_{4i} \end{bmatrix} \begin{bmatrix} I & E \\ 0 & D \end{bmatrix}$ $= \begin{bmatrix} B_{1i} & B_{2i} \\ 0 & B_{4i} \end{bmatrix}$ which implies in particular that $D^{-1}B_{4i}D = B_{4i}$. Since $\{B_{4i}$ $i = 1, 2, ..., p^{s/2} - 1\}$ defines a spread set, and a spread set acts transitively on the non-zero vectors of the associated vector space $V_{s/2}$, it follows that D centralizes an irreducible set of linear transformations of $V_{s/2}$. By Schur's lemma, it follows that D belongs to a field (finite division ring) isomorphic to $GF(p^e)$. In any case, since $B^p = 1$ also $D^p = 1$ and hence $D = 1$ so that also $B = 1$.

We now may restate Ostrom's theorem for generalized elations of vector spaces provided the set of images of the fixed point subspaces is a partial spread. The previous proof may be reread to prove the following theorem.

Theorem 12.5.11 *Let V be a finite vector space of dimension $2k$ over $GF(p)$. Let σ and τ be generalized elations of V with distinct axes.*

Let $S = \{Fix\sigma \langle\sigma, \rho\rangle, Fix\tau \langle\sigma, \rho\rangle\}$.

Then the following are equivalent:

(1) $\langle\sigma, \rho\rangle \simeq SL(2, p^z)$ for some positive integer z.

(2) S is a partial spread of V.

(3) Representing $\langle\sigma, \rho\rangle = \left\langle \begin{bmatrix} I & I \\ 0 & I \end{bmatrix}, \begin{bmatrix} I & 0 \\ A & I \end{bmatrix} \right\rangle$ then $GF(p)[A]$ is a field isomorphic to $GF(p^z)$.

Furthermore, when the above conditions are satisfied then S is a Desarguesian partial spread coordinatizable by $GF(p^z)$ within the Desarguesian plane coordinatized by $GF(p^k)$ and the unique involution in $SL(2, p^z)$ is the kernel homology -1.

The questions now are whether it can be guaranteed that two Baer p-collineations always or ever have disjoint axes and if it is possible that, in the above theorem σ could be an elation while ρ is a Baer p-collineation. Both of these questions have been resolved by Foulser when $p > 3$. Recall that a Baer subplane of a finite projective plane of order n is a subplane of order $\sqrt{n}$.