Chapter 6

Central Collineations and Desarguesian Nets.

Central collineations have a strong bearing on the planes upon which they act. In this section we study central collineations using two parallel but distinct approaches: the quasifield approach and the spreadset approach.

The machinery developed provides useful characterizations of *rational* Desarguesian net, those nets that are isomorphic to the nets defined by the parallel classes of subplanes of a Desarguesian plane. Note that rational partial spreads were introduced in defintion 5.7.2 and the associated nets, particularly the associated rational Desarguesian nets will be further considered in 6.3.

6.1 Central Collineations in Standard Form.

In this section, $\pi(Q)$ is a translation plane coordinatized by a quasifield $(Q, +, \circ)$. So the associated spread on $Q \oplus Q$ has as its components $X = Q \oplus \mathbf{0}$ and $Y = \mathbf{0} \oplus Q$ and all subspaces $y = x \circ m$, $m \in Q$; thus, $m = 0$ corresponds to X.

We shall investigate affine affine central collineations when their axes and coaxes are chosen canonically. Specifically, when dealing with homologies, we assume that the axis and coaxis have been chosen from the two standard components, X and Y, and when dealing with affine elations we take Y as the axis.

Since all such collineations g are among the additive bijections of $Q \oplus Q$

and when this condition holds the shear g is the collineation:

$$\begin{aligned}(x,y) &\mapsto (x, x\circ a + y)\\ (m) &\mapsto (m+a)\end{aligned}$$

In particular, g maps the component $y = x \circ m$, for $m \in Q$, onto the component $y = x \circ (a+m)$.

6.1.2 When g is a Y-axis homology of $\pi(Q)$.

We consider the case when g is a homology of $\pi(Q)$ with axis Y and coaxis X. So g fixes Y elementwise and, since (0) is the center, g leaves the y-coordinate of all points unchanged. So the eqns (6.1) yield:

$$\boxed{\text{For all } x,y,m \in Q: \begin{aligned} g:(x,y) &\mapsto (A(x),y)\\ g:(m) &\mapsto (m^S)\\ g:(0) &\mapsto (0^S)=(0)\end{aligned}},$$

and now the point $(x, x\circ m)$ on the component $y = x\circ m$ gets mapped to the point $(A(x), x\circ m)$, and since this must lie on the component $y = x \circ m^S$, we have:

$$\forall x, m \in Q : A(x) \circ m^S = x \circ m$$

and writing $c^S := f$, where e is a right identity for $\circ$, yields $A = T_f^{-1}$, so the above equation becomes

$$\forall x, m \in Q : (x)T_f^{-1} \circ m^S = x \circ m$$

hence:

$$\forall x, m \in Q : (x) \circ m^S = xT_f \circ m$$

so

$$(x)T_{m^S} = xT_f T_m$$

and $x = e$ yields

$$\forall m \in Q : m^S = f \circ m,$$

so both A and S have been determined in terms of f, where $y = x \circ f$ is the g-image of the unit line $y = x \circ e$. Thus g is the map:

$$\begin{aligned} g:(x,y) &\mapsto ((x)T_f^{-1}, y)\\ g:(m) &\mapsto (f\circ m)\end{aligned}$$

and so

$$(x, x \circ m) \mapsto (xT_f{}^{-1}, x \circ m),$$

and the image can lies on the component $y = x \circ (f \circ m)$ only if

$$\begin{aligned} xT_f{}^{-1} \circ (f \circ m) &= x \circ m, \\ \Rightarrow x \circ (f \circ m) = (xT_f) \circ m &= (x \circ f) \circ m, \end{aligned}$$

yielding:

Theorem 6.1.2 *Let $\pi(Q)$ be the translation plane associated with a quasi-field $(Q, +, \circ)$, with multiplicative identity e. Let G be the group of affine homologies of $\pi(Q)$ with axis Y and coaxis X. Then the G-orbit of the unit line $y = x \circ e$ consists of all components of type $y = x \circ f$, where $f \in N_m(Q)^*$, and now the unique $g \in G$ that maps the unit line onto $y = x \circ f$, for $f \in N_m(Q)^*$, is the collineation:*

$$\begin{aligned} g : (x, y) &\mapsto ((x)T_f^{-1}, y) \\ g : (m) &\mapsto (f \circ m), \end{aligned}$$

where $T_f : x \mapsto x \circ f$ is the slope of f. Moreover, the component $y = x \circ m$, $m \in Q^*$, is mapped by g onto the component $y = x \circ (f \circ m)$.

6.1.3 When g is an X-axis homology of $\pi(Q)$.

We consider the case when g is a homology of $\pi(Q)$ with axis X and coaxis Y. So g fixes X elementwise and, since (∞) is the center, g leaves the x-coordinate of all points unchanged. So the eqns (6.1) yield:

$$\boxed{\forall x, y, m \in Q : \begin{array}{rcc} g : (x, y) & \mapsto & (x, B(y)) \\ g : (m) & \mapsto & (m^S) \\ g : (0) & \mapsto & (0^S) = (0) \end{array}},$$

and now the point $(x, x \circ m)$ on the component $y = x \circ m$ gets mapped onto the point of $(x, B(x \circ m))$ and this must lie on the component $y = x \circ m^S$, thus:

$$\forall x, m \in Q : x \circ m^S = B(x \circ m)$$

and writing $e^S := f$, where e is the identity for $\circ$, yields $B = S$ and so the above equation becomes:

$$\forall x, m \in Q : x \circ m^S = (x \circ m)^S$$

and $m = e$ gives $x \circ f = (x)^S$ so

$$\forall x, m \in Q : x \circ (m \circ f) = (x \circ m) \circ f$$

and $f \in N_r^*$, yielding:

Theorem 6.1.3 *Let $\pi(Q)$ be the translation plane associated with a quasifield $(Q, +, \circ)$, with multiplicative identity e. Let G be the group of affine homologies of $\pi(Q)$ with axis X and coaxis Y. Then the G-orbit of the unit line $y = x \circ e$ consists of all components of type $y = x \circ f$, where $f \in N_r(Q)^*$, and now the unique $g \in G$ that maps the unit line onto $y = x \circ f$, for $f \in N_r(Q)^*$, is the collineation:*

$$\begin{aligned} g : (x, y) &\mapsto (x, y \circ f) \\ g : (m) &\mapsto (m \circ f), \end{aligned}$$

where $T_f : x \mapsto x \circ f$ is the slope of f. Moreover, g maps the component $y = x \circ m$, $m \in Q^*$,onto the component $y = x \circ (m \circ f)$.

6.2 Central Collineations In Matrix Form.

We have just seen how the properties of a quasifield Q are related to certain 'standard' affine central collineation groups of $\pi(Q)$. We now repeat the analysis for *spreadsets* coordinatizing a spread π. One way to proceed would be to express the results of the last section in spread-theoretic terms. But we prefer to directly establish these results so as to introduce the reader to matrix-based techniques that are indispensible in translation plane theory. For example, transposing the matrices of a spreadset, sometimes leads to a new translation plane with distinct geometric properties: this method of getting new-planes-from-old is not available without stepping back from quasifields, and even translation planes, to spreadsets.

However, working with spreadsets of matrices becomes very messy when dealing with translation planes that are infinite-dimensional over their kerns. Thus, we shall only consider spreads that are finite dimensional over a field K, and leave it to the determined reader to consider more general situations. Hence, by *the basis decomposition theorem* we are entitled to focus on the concrete case, when the spreads are constructed on the ambient space $K^n \oplus K^n$, and all K-linear automorphism and spreadset elements are K-matrices.

Throughout the section, τ denotes a spreadset of $n \times n$ matrices, that includes zero, over a field K, and $\pi_\tau = (W \oplus W, \Gamma_\tau)$ is an associated spread, where $W = K^n$, and the members of Γ_τ are $y = xT$, $T \in \tau$, along with $x = 0$: so the subspaces $X = W \oplus \mathbf{O}$ and $Y = \mathbf{O} \oplus W$ are among the components. Now any K-linear automorphism g of the spread π_τ may will be regarded as a 2×2 block matrix, where each block is an $n \times n$ matrix over K.

Exercise 6.2.1 *Any central collineation of the K-spread π_τ (so the origin is fixed by convention) is a K-linear map and hence may be represented by a 2×2 block matrix.*

Suppose E is the elation group of π_τ with axis Y. We shall describe E in terms of the matrices in τ.

Lemma 6.2.2 *Supppose A is a matrix such that $A + \tau \subseteq \tau$ (or equivalently $A + \tau = \tau$); so the matrix $A \in \tau$, and the additive matrix group $< A >$ partitions the set of matrices τ into a union of cosets of $< A >$.*

Now the block matrix

$$g_A := \begin{pmatrix} \mathbf{1} & A \\ \mathbf{O} & \mathbf{1} \end{pmatrix}$$

is a an elation with axis Y that maps $y = xT$, $T \in \tau$, to $y = x(T + A)$. Hence the orbit under g_A, of any component $y = xT$, $T \in \tau$, consists of the the components $y = xC$ where C ranges over the additive coset $T+ < A >$.

Proof: For $T \in \tau$, we have:

$$(x, xT)g_A = (x, xT)\begin{pmatrix} \mathbf{1} & A \\ \mathbf{O} & \mathbf{1} \end{pmatrix} = (x, x(A + T)).$$

But since by hypothesis $A + T \in \tau$, the mapping g_A is an automorphism of the spread τ that leaves Y elementwise fixed, and cannot be a homology as it is semiregular on the other components. The lemma follows easily. ∎

We now verify the converse of the lemma: all elations with axis Y have form g_A. Assume g is any elation with axis Y. Thus g fixes Y identically so its matrix on the standard basis has form

$$\begin{pmatrix} * & * \\ \mathbf{O} & \mathbf{1} \end{pmatrix},$$

and to determine the two upper blocks we note that g leaves the X-component of any $x \oplus y \in W \oplus W$ unchanged because the lines of form $x = C$ pass through the center of g. Thus g fixes identically the first n elements of the canonical basis of $K^n \oplus K^n$:

$$e_i \oplus \mathbf{0}, \quad i = 1, \ldots n,$$

so g can now be written as

$$g_A = \begin{pmatrix} \mathbf{1} & A \\ \mathbf{O} & \mathbf{1} \end{pmatrix},$$

and this matrix maps the component $y = xM$ onto $y = x(M + A)$, so $M + A$ must be in τ, in order that g preserves the spread. Thus τ is closed under addition by A, and, by lemma 6.2.2 above, this is sufficient for π_τ to admit g_A as an elation. Thus all the Y-axis elations are of form g_A, where A runs over the largest subset $\alpha \subseteq \tau$ such that $\alpha + \tau \subseteq \tau$. Now α is clearly an additive group of matrices and the map $E \in \alpha \mapsto g_E$ is an isomorphism from α onto the group of all Y-axis elations of τ. Hence we have obtained the following description of the group of Y-axis elations in matrix terms.

Theorem 6.2.3 *Let τ be a spreadset of matrices, that incudes zero, and let π_τ be the associated standard spread. Let*

$$E = \{A \in \tau \mid A + \tau \subseteq \tau\},$$

and define for each $A \in E$ the block matrix (all blocks with same order):

$$g_A := \begin{pmatrix} \mathbf{1} & A \\ \mathbf{O} & \mathbf{1} \end{pmatrix}.$$

Then

1. *E is an additive group and τ is the union of a set of additive E-cosets, including E.*

2. *A collineation g of π_τ is an elation with axis Y if and only if $g = g_A$, for some A in E; g_A maps X onto the component $y = xA$.*

3. *The map $A \mapsto g_A$ defines an isomorphism from the additive group of matrices E onto the full group of Y-elations of π_τ.*

4. *Let $S \leq E$ be an additive subgroup of E and g_S be the corresponding elation group, defined by $A \mapsto g_A$. Then the component orbits of g_S,*

other than Y, are in natural one-one correspondence with the additive cosets of S in τ, that union to τ: thus if $t \in \tau$ then the coset $t + S$ defines the components of the g_S-orbit of the component $y = xt$ to be the set of all components $y = xu$, $u \in t + S$.

Corollary 6.2.4 *A translation plane admits a transitive group of affine elations iff it is isomorphic to the translation plane associated with a spread π_τ, where τ is a matrix spreadset closed under addition.*

The following exercise considers the extension of the above to the infinite-dimensional case.

Exercise 6.2.5 *Let V be a finite-dimensional vector space over any [skew] field K. Define a spreadset to be a sharply one-transitive set Suppose $\tau \in GL(V, K)$ be a sharply one-transitive set of linear bijections of V: this means that for any $x, y \in V^*$ there is a unique $t \in \tau$ such that $x^t = y$. Determine the elation subgroup of the associated π_τ, in terms of τ, by generalising the above. Hence prove corollary 6.2.4for this case. Are there any problems in proving this corollary when the finite-dimensional restriction is removed? What happens if K is commutative but the vector space V is infinite-dimensional over K?*

Now we turn to the full group of *homologies* of π_τ with axis Y and with coaxis X. We follow the procedure for the elation case, but we shall insist that τ contains the multiplicative identity (to substitute for the additive identity in the elation case).

Lemma 6.2.6 *Assume the spread-set τ contins the identity matrix. Suppose A is a non-zero matrix such that $A\tau \subseteq \tau$; thus $A \in \tau$, A is non-singular and hence $A\tau = \tau$.*

Now the block matrix

$$h_A := \begin{pmatrix} A^{-1} & \mathbf{O} \\ \mathbf{O} & \mathbf{1} \end{pmatrix}$$

is a homology with axis Y and coaxis X. Hence the orbit under h_A, of any non-zero component $y = xT$, $T \in \tau^*$, consists of the components $y = xC$ where C ranges over the multiplicative coset $T < A >$, of the multiplicative group $< A >$.

Proof: The map h_A sends the component $y = xT$ onto the subspace $y = xAT$, so $AT \in \tau$. Now continue arguing as in lemma 6.2.2, to get the desired result. ∎

Conversely suppose that h is any homology with axis Y and coaxis X. Thus h has matrix $Diag(H, \mathbf{1})$, for some non-singular H corresponding to $h|X$. Now the component $y = xM$ maps to the subspace $y = xH^{-1}M$, so $H = A^{-1}M$. Now, *if τ contains the matrix* $\mathbf{1}$, then $H \in \tau^{-1}$. Now repeating the argument used in the elation case we get an analogue of the theorem above.

Theorem 6.2.7 *Let τ be a spreadset of matrices, that includes zero and the identity matrix. Let π_τ be the associated standard spread; so $Z = \{(w, w) \mid w \in K^n\}$, the unit line is in π_τ. Let:*

$$M^* = \{A \in \tau^* \mid A\tau^* \subseteq \tau^*\},$$

and define for each $A \in M^*$ the block matrix (all blocks with same order):

$$h_A := \begin{pmatrix} A^{-1} & \mathbf{O} \\ \mathbf{O} & \mathbf{1} \end{pmatrix}.$$

Then

1. *M^* is a multiplicative group of matrices such that τ is the union of a set of right multiplicative M^*-cosets, including M^*.*

2. *A collineation h of π_τ is a homology with axis Y and coaxis X if and only if $h = h_A$, for some A in M; h_A maps X onto the component $y = xA$.*

3. *The map $A \mapsto h_A$ defines an isomorphism from the multiplicative group of matrices M^* onto the full group of homologies, of π_τ, with axis Y and coaxis X.*

4. *Let $S \leq M^*$ be a multiplicative subgroup of M^* and h_S the corresponding homology group, defined by $A \mapsto h_A$. Then the component orbits of h_S, other than X and Y, are in natural one-one correspondence with the left multiplicative cosets of S in τ, that union to τ: thus if $t\tau$ then the left coset tS defines the components of the h_S-orbit of the component $y = xt$ to be the set of all components $y = xu$, $u \in tS$.*

Corollary 6.2.8 *A translation plane admits a transitive group of affine homologies sharing the same axis and coaxis iff it is isomorphic to the translation plane associated with a spread π_τ, where τ is a matrix spreadset such that τ^* is a multiplicative group.*

Next consider the situation when X is the axis and Y the coaxis of the homology group. Using a slightly 'dualised' version of the above analysis we get results similar to the above. For example, the general form the homologies being considered are matrices of type $Diag(1, A)$ and this maps a component $y = xT$ onto $y = xTA$, so τ is closed under multiplication by A from the right. Continuing in this way we obtain:

Theorem 6.2.9 *Let τ be a spreadset of matrices, that includes zero and the identity matrix. Let π_τ be the associated standard spread; so $Z = \{(w, w) \mid w \in K^n\}$, the unit line is in π_τ. Let:*

$$L^* = \{A \in \tau^* \mid \tau^* * A \subseteq \tau^*\},$$

and define for each $A \in L^*$ the block matrix (all blocks with same order):

$$h_A := \begin{pmatrix} \mathbf{1} & \mathbf{O} \\ \mathbf{O} & A \end{pmatrix}.$$

Then

1. *L^* is a multiplicative group of matrices such that τ is the union of a set of right multiplicative L^*-cosets, including L^*.*

2. *A collineation h of π_τ is a homology with axis X and coaxis Y if and only if $h = h_A$, for some A in M; h_A maps I onto the component $y = xA$.*

3. *The map $A \mapsto h_A$ defines an isomorphism from the multiplicative group of matrices L^* onto the full group of homologies, of π_τ, with axis Y and coaxis X.*

4. *Let $S \leq L^*$ be a multiplicative subgroup of L^* and h_S the corresponding homology group, defined by $A \mapsto h_A$. Then the component orbits of h_S, other than X and Y, are in natural one-one correspondence with the multiplicative right cosets of S in τ, that union to τ: thus if $t\tau$ then the right coset tS defines the components of the h_S-orbit of the component $y = xt$ to be the set of all components $y = xu$, $u \in St$.*

Corollary 6.2.10 *A translation plane admits a transitive group of affine homologies sharing the same axis and coaxis iff it is isomorphic to the translation plane associated with a spread π_τ, where τ is a matrix spreadset such that τ^* is a multiplicative group.*

Corollaries 6.2.8 and 6.2.10 are each equivalent to asserting that the non-zero elements of a spreadset form a multiplicative group. Hence the spread π_τ admits a Y-axis-X-coaxis transitive homology group iff it admits an X-axis-Y-coaxis transitive homology group. So if a translation plane of order n admits an affine homology group of order $n-1$ then it admits another with axis and coaxis reversed! Thus we have:

Corollary 6.2.11 *A translation plane admits a transitive group M of affine homologies with axis Y and coaxis X iff it admits another transitive homology group L with with axis X and coaxis Y.*

6.3 Rational Desarguesian Partial Spreads.

We have already encountered rational partial spreads in section 5.7. The point being made there was that rational partial subspreads (and hence their nets) are just those arising from a subquasifield of a coordinatising quasifield. In this section we focus on rational *Desarguesian* partial spreads, and the point we make is that partial spreads defined by a Desarguesian subplane need not be Desarguesian: that is, a partial spread with a Desarguesian plane across it need not be embedable in a *Desarguesian* spread.

In view of the importance of this fact, we have kept this section independent of our earlier treatment in section 5.7. The notation here also differs slightly from our earlier notation: there is as yet no standard notation in this area.

Definition 6.3.1 *Let $\pi_\Delta := (V, \Delta)$ be a partial spread, or a spread, on $(V, +)$, the additive group of a vector space. Suppose W is any non-trivial additive subgroup of V, such that $W \neq V$. Then the components of π_Δ* DETERMINED BY *W, or the components* ACROSS *W is the subset of the component set Δ given by*

$$W^\Delta := \{D \in \Delta \mid D \cap W \neq \mathbf{O}\},$$

and the corresponding INDUCED STRUCTURE *on W is $\pi_W := (W, W_\Delta)$ where W^Δ consists of the non-trivial intersections of the components of π_Δ with W:*

$$W_\Delta := \left\{ d \cap W \mid d \in W^\Delta \right\}.$$

The subspace W is called a SUBSPREAD *of π_Δ if the structure π_W induced on it, is a spread in the usual sense, that is, every pair of distinct members of W_Δ direct-sum to W.*

To get used to this terminology we observe:

Remark 6.3.2 *A subspace W of a spread $\pi = (V, \Delta)$ is a subspread of π_Δ iff the components W^Δ across W induce a spread on it.*

Note that the spread induced on W depends only on the set of components across it, viz. W^Δ, and not on any larger [partial] spread $\delta \supset \Delta$. Such partial spreads, defined by the components of a subspread of a [partial] spread are called rational partial spreads.

Definition 6.3.3 *A partial spread (V, Λ) of a [partial] spread $\pi_\Theta = (V, \Theta)$ is a* RATIONAL *partial spread if an additive subspace W, of $(V, +)$, is such that: (1) W is a subspread of π_Θ; and (2) the components of π_Θ meeting W non-trivially are precisely the members of the partial spread Λ.*

A rational partial spread (V, Λ) is said to be a rational DESARGUESIAN *partial spread if Λ is a subset of a Desarguesian spread Δ on V.*

Thus a rational Desarguesian partial spread is a partial spread obtained from a Desarguesian spread π by taking as its components all the components of some subsplane π_0 of π. We shall usually follow the common practice of calling a Desarguesian partial spread a *Desarguesian net*; thus rational Desarguesian nets will mean the partial spread determined by a rational Desarguesian partial spread, according to our convention, and will also mean the net, in the strict sense of the word, determined by this partial spread.

If a subspace W of a partial spread defines a rational Desarguesian net of a partial spread or a spread (V, Δ), then W is Desarguesian as it lies in a Desarguesian plane. However, the converse is false: this will emerge from the following exercise.

Exercise 6.3.4 *In the following exercise assume all spreads etc. are finite.*

1. *Let $\pi_\Delta := (V, \Delta)$ be a finite spread, two-dimensional over a kern field K. Then a K-subspace W, of V, is either a component Δ or a Desarguesian Baer subplane of π_Δ.*

2. *Let $\mathcal{F}$ be a subspreadset of a spreadset $\mathcal{T}$, such that $\mathbf{O}, vecI \in \mathcal{F}$. Then the partial spread defined by $\mathcal{F}$ is a rational Desarguesian spread iff $\mathcal{F}$ is a field under matrix operations.*

3. *If Q is a right quasifield then $\pi(F)$ is a rational Desarguesian net iff Q is a right vector space over F.*

4. *Let Q be a quasifield and K a kern field. Show that $\pi(K)$ need not define a rational Desarguesian net.*

5. *Show that a spread (V, Γ) can contain a Desarguesian subplane W such that the partial spread defined by W, viz., W^Γ, need not be Desarguesian.*