Chapter 1

André's Theory Of Spreads.

André's theory of spreads is arguably one of the most important events in finite geometry: hardly any finite projective planes were known before André seminal 1954-paper, [2]. André's paper is ultimately responsible for the explosive growth in the discovery of finite non-Desarguesian planes during the last thirty years. Moreover, the theory of spreads, which reduces the study of translation planes to structures that live on vector spaces, has meant that all the machinery of linear algebra, and hence also group repreentation theory, can be brought to bear on the study of translation planes.

The lectures in this chapter will mainly be concerned with developing the André theory of spreads and its computational aspect — spreadsets of matrices. In the next chapter, the associated theory of spreads as structures that live in *projective* spaces will be emphasized.

1.1 Affine Planes with a Transitive Translation Groups.

In this first lecture, we begin our study of projective and affine planes. With the exception of three infinite families of projective planes called the planes of Hughes, Figueroa, and Coulter-Matthews, all finite projective planes are related to a class called 'translation planes.'

In this lecture, we consider a fundamental representation of a translation plane. This is the classical description of translation planes using vector spaces due to André. In a later lecture, we shall consider the Bruck-Bose approach using projective spaces.

Less well known but of increasing importance are what might be called coordinate methods. These include the study of quasifields, spread sets and Oyama coordinates. Professor Jha will be lecturing on some of these topics in the algebraic tract.

We begin with the definition of an affine plane, which we state in terms of an INCIDENCE STRUCTURE $(\mathcal{P}, \mathcal{L}, \mathcal{I})$. This means that $\mathcal{P}$ and $\mathcal{L}$ are disjoint sets of objects called POINTS and LINES resp. and $\mathcal{I} \subset \mathcal{P} \times \mathcal{L}$. To facilitate discussion we make extensive use of geometric terminology: any set of points incident with the same line is said to be collinear, two lines are DISJOINT if they are not incident with any common point. Similarly we use notation based on geometry and set theory: we write $P \in p$, or say the point P LIES ON the line P, if $(P, p) \in \mathcal{I}$, and if $P, Q \in \mathcal{P}$ are distinct points that share exactly one line we write PQ to denote the unique line that they share.

Definition 1.1.1 *An affine plane π is an incidence structure $(\mathcal{P}, \mathcal{L}, \mathcal{I})$ with the following properties:*

1. *Given two distinct points $P, Q \in \mathcal{P}$, there exists a unique line p such that (P, p) and $(Q, p) \in \mathcal{I}$; thus $PQ = p$.*

2. *Given a point P and a line p such that P is not incident with p, there exists a unique line q disjoint from p such that $P \in q$.'*

3. *There exists at least three noncollinear points.*

Two lines of an affine plane are said to be PARALLEL, if they are disjoint, and the notation $p \parallel q$ means that lines p and q are parallel when $p \neq q$. However, in order to force $\parallel$ to be an equivalence relation, *we continue to write $p \parallel q$ even when $p = q$.*

Remark 1.1.2 *Let $\pi = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ be an affine plane.Then $\parallel$ is an equivalence relation on the set of lines. The equivalence classes are called 'parallel classes'.*

Proof: Routine exercise. ∎

We shall often use variations of the above terminology that often arise in the literature. For example the parallel classes of an affine plane is often called its SLOPESET, or its set of 'infinte points' or its 'ideal points'. Similarly, the class of any line is its SLOPE, or its 'point at infinity', etc.

We shall encounter many incidence structures related to affine planes: projective planes, Desarguesian affine and projective spaces, nets, etc. We therefore give a general definition of an isomorphism from one incidence structure to another.

Definition 1.1.3 *Let $\pi_i = (\mathcal{P}_i, \mathcal{L}_i, \mathcal{I}_i)$, $i = 1, 2$, be incidence structures. Then an isomorphism from π_1 onto π_2 is an ordered pair of bijections*

$$(\rho : \mathcal{P}_1 \to \mathcal{P}_2, \lambda :: \mathcal{L}_1 \to \mathcal{L}_2),$$

from the points and lines of π_1 onto the points and lines of π_2 (respectively), such that incidence is preserved in both directions:

$$(p, \ell) \in \mathcal{I}_1 \Longleftrightarrow (\rho(p), \lambda(\ell)) \in \mathcal{I}_2.$$

An isomorphism from an incidence structure π to itself is = an AUTOMORPHISM, *and the group of automorphism of π is usually denoted by $Aut(\pi)$.*

An automorphism of an affine plane is completely specified by its action on the points: this is because two points determine a unique line and every line lies on at least two points. Thus we have

Remark 1.1.4 *Let π be an affine plane. Show that if (σ, τ) and (σ, ρ) are collineations of π then $\tau = \rho$.*

The above remark justifies the usage of only the *point*-bijection to refer to the automorphism. This applies to any incidence structure where the *incidence is set-theoretic*: this means that lines may be viewed as sets of points and distinct lines are associated with distinct sets of points. All the incidence structures we encounter may be regarded as being set-theoretic incidence structures. This allows us to freely use set-theoretic language rather than the more cumbersome terminology associated with incidence.

Thus, in any set-theoretic incidence structure, an automorphism $(\phi : P \to P, \psi : L \to L)$ is full determined by the action of the associated point-bijection $\phi : P \to P$; the action on the lines correspond to the usual action induced by ϕ on the powerset 2^P. We shall refer to ϕ as a *collineation*: thus a collineation is the action on the points corresponding to an automorphism of a set-theoretic incidence structure. In particular:

Definition 1.1.5 *A collineation of a set-theoretic incidence structure π is a bijection of its points that extends to an automorphism of π. $Aut\pi$ will be used to denote the collinetaion group of π and also its automorphism group: both groups are of course isomorphic.*

Thus, the collineation group in the above sense is the faithful representation of the automorphism group on the points. Accordingly, we shall not attempt to seriously distinguish between the two concepts.

Exercise 1.1.6 *Let ϕ be a bijection from the points of an affine plane A onto the points of an incidence structure B such that ϕ maps collinear sets of points onto pairwise incidence sets of points. Is it true in general that ϕ induces an isomorphism from A onto B? Show that ϕ does induce an isomorphism when B is also an affine plane.*

Definition 1.1.7 *A* TRANSLATION *of an affine plane is a collineation which leaves each parallel class invariant and fixes each line of some parallel class.*

Our goal is to verify that the translations of an affine plane form a group and this group acts semiregularly on the affine points, that is, the points other than the parallel classes. The first step is to note that all non-trivial translations are semiregular:

Lemma 1.1.8 *A translation of an affine plane which fixes a point is the identity.*

Proof: Exercise. ∎
The following remark may be taken as an alternative definition of a translation, equivalent to definition 1.1.7 above.

Remark 1.1.9 *Let σ be a non-trivial collineation of an affine plane $\mathcal{A}$.*
Then σ is a translation iff it fixes every parallel classes of $\mathcal{A}$ and does not fix any affine point.

Proof: $\Rightarrow$ follows from lemma 1.1.8 above. Conversely assume σ leaves invariant every parallel class but does not fix any affine point. So choosing any affine point A, we have $B := A\sigma$ is distinct from A, and let m be the parallel class of AB. Let ℓ be any other affine line in the parallel class m. It is sufficient to show that such ℓ are σ-invariant. Choose an affine point $C \in \ell$. By hypothesis $D = C\sigma \neq C$. So CD is in the parallel class of m and, like ℓ, contains C. Hence both ℓ and CD are lines in the class m that contain C, so they coincide. Hence $\ell\sigma = CD = \ell$, since the image of any line is completely determined by the image of any one of itss affine points and the image of its parallel class. Thus all lines in the parallel class m are fixed by σ. ∎
We now consider collineations of the above type that might not fix any parallel class.

Definition 1.1.10 *A collineation fixing all the parallel classes of an affine plane is called a* DILATATION. *Dilations that are not translations are called kern homologies.*

So by remark 1.1.9 above, dilations other than translations fix at least one affine point point. If they fix more than one affine point then the set of fixed points form a subaffine plane which actually coincides with the parent plane. Thus, if π is any affine plane then the dilations other than translations, that is, the kern homologies, fix exactly one affine point Z, called its *center*. Moreover, remark 1.1.9 further implies that a non-trivial translation fixes all the lines of *exactly* one parallel class. This class is called the *center* of the translation. We summarize all this.

Remark 1.1.11 *Every non-trivial dilation of an affine plane is either a translation or a kern homology. Every non-trivial translation fixes all the lines of exactly one parallel class, called its* CENTER, *and no other affine lines or points, while every non-trivial kern homology fixes exactly one affine point, called its* CENTER, *and the other affine line that it fixes are just the lines through its center.*

Thus the set of all dilations of an affine plane form a group: the DILATION group, and it has as subgroups: the TRANSLATION group and the KERN HOMOLOGY group. To discuss these further we recall some standard definitions from permutation groups.

Definition 1.1.12 *Let G denote a permutation group acting on a set Ω. Then the G-orbit of $x \in \Omega$ is denoted by*

$$Orb_G(x) := \{x^g \mid g \in G\},$$

and the STABILIZER *of $x \in \Omega$ in G is denoted by:*

$$G_x = \{g \in G \mid x^g = x\}.$$

In particular, G is transitive on Ω if it has only one orbit, or equivalently:

$$x, y \in \Omega \Rightarrow \exists g \in G \ni x^g = y.$$

G is REGULAR *if additionally G_a is trivial for all $a \in \Omega$. More generally, a permutation group G on Ω is* SEMIREGULAR *if only the identity of G fixes any element in Ω.*

Using the above terminology, remark 1.1.11, yields:

Proposition 1.1.13 *Let $\mathcal{A}$ be an affine plane and G its dilation group. Then the translation subgroup T of G is normal in G and semiregular on all the affine points of $\mathcal{A}$. G is the union of T and all its maximal groups of kern homologies, and any two distinct groups in the union have trivial intersection.*

Proof: Exercise. ■
The above result is far from optimal, particularly in the finite case, where the finite case where theory of Frobenius may be applied. But the reader is warned that glib generalizations to the infinite case might be dangerous.

We may now define translation planes.

Definition 1.1.14 *A translation plane is an affine plane whose translation group acts transitively on the affine points.*

As an immediate consequence of remark 1.1.9 we have

Remark 1.1.15 *An affine plane is a translation plane iff its translation group is regular on the affine points.*

A Frobenius group is a transitive permutation group in which the stabilser of any two points is trivial. By proposition 1.1.13 we have:

Remark 1.1.16 *The dilation group of an affine plane acts, faithfully, as a Frobenius group on its affine points.*

The point being made is that there is a deep and powerful theory for *finite* Frobenius groups that has been exploited in *finite* translation plane theory.

We now describe a simple construction for translation planes, and eventually we shall demonstrate that the construction is generic. The method is based on the notion of a *spread*, the most important concept in translation plane theory. *A spread is a partition of the non-zero points of a vector space by a collection of subspaces that pairwise direct-sum to the whole space.*

The lines through the origin in the real plane $\Re^2$ is the most familiar example of a spread: the real translation plane consists of the cosets of the components of the spread.

Viewing $\Re^2$ as a vector space over the rational field $\mathcal{Q}$, we have a $\mathcal{Q}$-spread with the same components as before — the lines through the origin — but now these components are infinite-dimensional subspaces. One can of course generalize all this: start with a rank two vector space over a a skewfield F, then the one-spaces form a spread, and if F is an extension of a subskewfield K then the '1-dimensional spread' F-spread becomes a d-dimensional K-spread, when $dim_K F = d$, and the additive cosets define a translation plane.

Of course, the translation planes described above are the familiar Desarguesian planes, and indeed one could regard this construction as a definition of a Desarguesian spread: thus *a Desarguesian plane is the affine plane consisting of the cosets of the components of a one-dimensional spread over a skewfield F.*

We summarize our terminology for spreads and related items:

Definition 1.1.17 *Let V be a vector space, and let $\mathcal{S}$ be a collection of mutually disjoint additive subgroups of $(V,+)$ such that $V = \cup\mathcal{S}$ and the sum of each distinct pair of additive subgoups of $\mathcal{S}$ is V. Then $\mathcal{S}$ is called a* SPREAD *on V, or with* AMBIENT SPACE *V, and the subspaces on V are its* COMPONENTS. *The associated incidence structure is defined to be*

$$\Pi_{\mathcal{S}} := (V, \mathcal{C}),$$

with pointset V, lineset

$$\mathcal{C} := \{x + S \mid S \in \mathcal{S}, x \in V\},$$

and with set-theoretic incidence.

*If V is a vector space over a specified skewfield K, such that all the components of $\mathcal{S}$ are themselves K-subspaces of V, then $\mathcal{S}$ is called a K-*SPREAD*; this spread is called a d-*DIMENSIONAL *K-spread if each component is K-dimensional as a K-vector space.*

Remarks 1.1.18

1. *It will often be useful to draw attention to the ambient space V, associated with a spread $\mathcal{S}$, by referring to the pair $\pi = (V, \mathcal{S})$ as a spread. Thus,* π is viewed as being synonymous with $\mathcal{S}$.

2. *Every spread on V is a K-spread when K is chosen to be the prime subfield of the skewfields over which V is a vector space.*

3. *The direct-sum condition forces all components of a K-spread to have the same dimension d over K; d has sometimes been called the Ostrom dimension of the spread, to distinguish it from the dimension of the ambient space V which is $2d$, for finite d.*

We now note that the incidence structure of a spread is always a translation plane, and later we shall establish that all translation planes arise in this manner.

Theorem 1.1.19 *Let $\mathcal{S}$ be a spread on an ambient vector space V. Then the associated set-theoretic incidence structure, $\Pi(V,\mathcal{S})$, definition 1.1.17, is a translation plane. The full group of translations of $\Pi(V,\mathcal{S}$ is just the group of translations of V regarded as a vector space:*

$$\Theta := T = \{T_a : x \mapsto x + a \mid a \in V\}.$$

Moreover, if V is a vector space over a skewfield K such that the components are K-subspaces, that is $(V,\mathcal{S})$ is a K-spread, then the scalar action of K^* on V is a group of kern homologies of $\Pi(V,\mathcal{S})$; thus, the group of bijections on V

$$\widehat{K^*} := \{\forall x \in V : x \mapsto (x)k \mid k \in K^*\},$$

where $(x)k$ denotes the image of x under $k \in K$, is a group of kern-homologies, c.f., definition 1.1.10 of the translation plane $\Pi(V,\mathcal{S})$.

Proof: Straightforward exercise. ∎

Of particular importance are the maximal skewfields K over which the components are K-spaces. It will turn out that there is a unique maximal skewfield with this property. This will become clear as we develop the theory more fully.

Exercise 1.1.20 *Let $(V,\mathcal{S})$ be a spread and let T be the full translation group of the associated translation plane. To each component $\sigma \in \mathcal{S}$ assign the subgroup $T_{\{\sigma\}}$, the global stabilser of σ in T. Show that*

$$\cup_{\sigma \in \mathcal{S}} T_{\{\sigma\}} = T,$$

and that $T_\mu \cap T_\nu$ is the trivial group, whenever $\mu, \nu \in \mathcal{S}$ are distinct components.

Thus the full translation group T of a translation plane admits a partition by subgroups and thus appears to be analogous to the ambient space of a spread on a vector space. Our study of group partitions, in the next lecture, will show that such group partitions may be identified with spreads, and, in particular, that any translation group T may be taken to be the additive group of a vector space.

1.2 Group Partitions and André Theory.

In this lecture we develop André's fundamental theory relating translation planes to spreads. Our starting point is concerned with group partitions: a collection of pairwise disjoint subgroups of a group G that union to G.

A partition of a vector space by its set of one-spaces is an example of a group partition. A less trivial example arises when a field F is r-dimensional over a subfield K for then the additive group of F is partitioned by its r-dimensional K-spaces. Note, only the case $r = 2$ corresponds to examples of spreads in the sense of definition 1.1.17.

Definition 1.2.1 *(Group Partition.) Let G be a group. A* PARTITION *of G is a set N of nontrivial pairwise disjoint proper subgroups of G such that $G = \cup N$; the members of N are the* COMPONENTS *of the partition and if all the components in N are normal in G, then N is a* NORMAL PARTITION *of G.*

We have already noted that many normal partitions do not yield spreads, in the sense of definition 1.1.17. However, if the ambient group G of a normal partition is generated by any two of its elements then this is the only possibility, by the following fundamental characterization:

Theorem 1.2.2 *Let G be a group and N a normal partition of G such that*

$$G = < N_1, N_2 > \forall N_1, N_2 \in N, N_1 \neq N_2.$$

Then each of the following is valid:
(1) G is a direct product of any two distinct subgroups of N.
(2) each two distinct subgroups of N are isomorphic and
(3) G is Abelian.

Proof: (1) This is elementary as the elements of disjoint normal subgroups commute.
(2) A group cannot be expressed as the disjoint union of two distinct subgroups. Hence N contains at least three members. So given distinct $N_1, N_2 \in N$, we may choose a third $N_0 \in N$, and now $N = N_1 \oplus N_0$ and also $N = N_2 \oplus N_0$. Hence

$$N_1 \cong \frac{N}{N_0} \cong N_2,$$

as required.
(3) Since G is the direct sum of any two distinct members of N, we see that

elements from distinct subgroups of N commute. So assume $x, y \in A \in N$ and choose a nonidentity $b \in B \in N - \{A\}$, and observe

$$\begin{aligned} xyb &= xby \\ &\Rightarrow xyb = byx \text{ since } by \notin A \\ &\Rightarrow xyb = yxb \\ &\Rightarrow xy = yx, \text{ as required.} \end{aligned}$$

■

In view of the above it is desirable to call a normal partition N of a group G a *generating* normal partition if G is generated by every pair of distinct components $N_1, N_2 \in N$ generate G as a group.

Theorem 1.2.3 *Let N be the components of a spread on a group G. Then the set-theoretic incidence structure whose pointset is G and whose lines are the cosets of the elements of N is an affine translation plane whose translation group consists of the bijections of G, for every $a \in G$ of type:*

$$G \to G$$
$$g \mapsto ga$$

Proof: A straightforward consequence of the theorem above.■

Theorem 1.2.4 *Let G be a group and N a generating normal partition of G. Let $\mathcal{K}$, denote the set of group endomorphisms which leave each component invariant.*

Then $\mathcal{K}$ is a skewfield and G is a vector space over $\mathcal{K}$.

The elements of the skewfield $\mathcal{K}$ are called the "kernel endomorphisms" of the partition. The skewfield $\mathcal{K}$ is called the "kernel of the spread."

Proof: Since G is abelian by the previous result, the endomorphisms in $\mathcal{K}$ clearly form a ring. Hence it is clearly sufficient to show that all the non-zero maps $\phi \in \mathcal{K}$ are bijective. Suppose $a^\phi = 0$ for $a \neq 0$. Now we force $\phi = 0$ by demonstrating that ϕ vanishes on every component $B \neq A$, where $a \in A$. We note that this is more than sufficient to force $\phi = 0$ since any two components of N generate G. As $0 = a^\phi = (a+b)^\phi + (-b)^\phi$ then $(a+b)$ and b are on C and B respectively which are distinct components so that

$b^\phi \in B \cap C = 0$ whenever b is in any component $B \neq A$. Thus all members $\phi \in \mathcal{K}$ are injective homomorphisms of G.

Next we check the elements $\phi \in \mathcal{K}$ are surjective.

Given nonzero $v \in G$, we require $w \in G$ such that $v = w^\phi \exists \phi \in \mathcal{K}$. Let V denote the component containing v and let $u \in U$ be a nonzero element in some other component of N, and define a third component Z that contains $u^\phi - v$. Now we claim that the required w is the unique point in the set $(Z + u) \cap X$. Note that the intersection is unique since it is the intersection of two lines of the affine point associated with the spread N.

It is now sufficient to show that $v - w^\phi = 0$, and we demonstrate this by showing that $v - w^\phi \in V \cap Z$. Since $w \in V$, $v - w^\phi$ certainly lies in V. Thus, it is sufficient to verify that $v - w^\phi \in Z$. But, by definition, $u^\phi - v \in Z$, so it is sufficient to verify that$(v - w^\phi) - (u^\phi - v) = (w - u)^\phi \in Z$. This condition holds because $w \in u + Z$ means that $(w - u) \in Z$ and Z is ϕ-invariant. Thus, ϕ is surjective. ∎

The following standard notation concerning linear groups will be used thoughout our lectures.

Definition 1.2.5 *Let V be a left vector space over a skewfield K. Let σ be an additive mapping on V. We shall say that σ is K-semi-linear if and only if for all α in K and for all x in V then $\sigma(\alpha x) = \alpha^\rho \sigma(x)$ where ρ is an automorphism of K. We shall say that σ is K-linear if and only if $\rho = 1$.*

The group $\Gamma L(V, K)$ of all bijective K-semi-linear mappings is called the general semi-linear group. The subgroup $GL(V, K)$ of linear mappings is called the general linear group.

Let F denote the prime field of K. Then $\Gamma L(V, F) = GL(V, F)$. Since any additive bijection is in $GL(V, F)$, the notation $GL(V, +)$ is always used.

In 1954, André provided the foundation for the theory of translation planes by proving that any translation plane may be identified with a normal partition of a group which actually turns out to be a vector space over a skewfield:

Theorem 1.2.6 *(The Fundamental Theorem Of Translation Planes.)*
Let π be a translation plane with translation group T and let $\mathcal{P}$ denote the set of parallel classes of π.
Let T_p denote the subgroup of T fixing all the lines of p, for $p \in \mathcal{P}$. Then all the following hold.

1. *$\Gamma = \cup\{T_p \mid p \in \mathcal{P}\}$ is a spread on T and hence T is a vector space over the associated kernel $\mathcal{K}$.*

2. *π is isomorphic to π_Γ, the translation plane constructed from the spread of T.*

3. *The full collineation group G of π_Γ is TG_O where G_O is the full subgroup of the group $GL(T,+)$, that permutes the members of Γ among themselves.*

4. *The full collineation group G of π_Γ is TG_O where G_O is the full subgroup of the group $\Gamma L(T,\mathcal{K})$ that permutes the members of Γ among themselves.*

Proof: (1) T_p is the subgroup of T fixing individually all the lines through p, hence it is trivially normalized by T since T fixes p. Since every translation in T has a unique center, T gets partitioned by its normal subgroups of type T_p. It remains to show that $T = T_p \oplus T_q$ whenever p and q are distinct points on the translation axis. Let $t \in T$, and suppose $t : a \mapsto b$, where a is any affine point, and assume $b \neq a$, to avoid trivialities. Since T_p and T_q are normal and disjoint, it is sufficient to verify that $t \in < T_p, T_q >$. Let $pa \cap qb = x$. Since T_p has as its non-trivial orbits all the affine on each line through p, there is a $g \in T_p$ such that $g : a \mapsto x$ and, similarly, there is an $h \in T_q$ such that $h : x \mapsto b$. Now clearly $a^{gh} = b$. But the regularity of T now forces $t = gh$. Thus T is generated by any two distinct T_p and T_q.

(2) Fix an affine point O of π, and to each affine point a of π assign the translation $\tau_a \in T$ that maps O onto a. Consider the bijection $\Theta : a \mapsto \tau_a$, from the affine points of π onto the points of the vector space T.

Consider the affine point $a \in A$, where A is any affine line of π. Let A_m be the unique line parallel to A through O with slope m. Clearly, T_m has A_m as its O-orbit, so $(A_m)\Theta = T_m$.

Next note that the points of A may be expressed as $O^{\tau_a T_m}$, as the group T_m acts transitively on the affine points of each line through m. Now the image $\left(O^{\tau_a T_m}\right)\Theta = \tau_a T_m$, i.e., a coset of T_m. Thus we have shown the bijection Θ maps the lines of π to cosets of the spreads associated with T, which means Θ is a bijection from the affine plane π onto the affine plane associated with the spread on T that sends lines onto lines. Hence, Θ is an isomorphism between the planes.

(3) The translation subgroup of the full collineation group G of π_Γ may, of course, be identified with T itself. Let $H = G_O$, so $G = HT$, by the transitivity of T , and by its regularity we further have $H \cap T = \{1\}$ (the identity element). We next verify that H is in $GL(T, +)$.

We define addition in π as follows: $a + x = \tau_a(x)$. It follows that this makes $(\pi, +)$ isomorphic to T.

Since T is normal in the translation plane π_Γ, we have for every $a \in T$ a unique $a' \in T$ such that

$$h\tau_a = \tau_{a'}h$$

so

$$h\tau_a(x) = \tau_{a'}h(x)\forall x \in T$$

hence

$$h(a + x) = a' + h(x)\forall x \in T.$$

Putting $x = O$, we observe that $a' = h(a)$ and so the above identity yields

$$h(a + x) = h(a) + h(x)$$

so h is additive and hence lies in $GL(T, +)$, and permutes the members of Γ. Conversely, any map with these two properties also permutes the cosets of the components of Γ, and is thus a collineation of π_Γ. Thus (3) is established.

(4) By (3), H is the largest subgroup of $GL(T, +)$ that permutes the members of Γ among themselves, and the kernel of this representation of H on Γ is thus normal in H and coincides with $\mathcal{K}^*$ by definition. The normality of $\mathcal{K}^*$ now forces H to be semilinear over $\mathcal{K}$.

This completes the proof of the theorem. ∎

Since the translation group of any spread $(V, \mathcal{S})$, associated with a translation plane π, is additively isomorphic as an additive group to the translation group of π, all such spreads $(V, \mathcal{S})$ have isomorphic additive groups $(V, +)$. The non-zero kernel endomorphisms of such spreads are permutation isomorphic to the the kern homologies, acting on the plane. This suggests that all such spreads, associated with a fixed translation plane, are related by a spread isomorphism semilinear over their kern, and more generally that any collineation the of the planes associated with the spreads that sends zero to zero must be a semilinear map of the type indicated. This is indeed the case as we shall now verify.

The main problem is to verify that such collineations are additive; we shall verify this directly rather than attempting to derive it from part (3) of the fundamental theorem above.

Theorem 1.2.7 *Let $(V, \mathcal{S})$ and $(W, \mathcal{T})$ be spreads defining isomorphic translation planes, and suppose that $\Psi : W \to V$ is any isomorphism from the translation plane $\Pi_{(W,\mathcal{T})}$ to the translation plane $\Pi_{(V,\mathcal{S})}$ such that $\mathbf{0} \mapsto \mathbf{0}$; ψ exists since the planes admit point-transitive translation groups. Let K and L be respectively the skewfields of kernel endomorphisms of the spreads $(V, \mathcal{S})$ and $(W, \mathcal{T})$. Then there is bijective ring isomorphism $\psi : L \to K$ such that there is a K−L-semilinear bijection $\Psi : V \to W$, satisfying $\Psi(aw) = a^{\psi}(w)$, for all $w \in W$, $a \in K$.*

Proof: Since the translation groups of the two planes are isomorphic, V and W are isomorphic additive groups, so $(V, +)$ can be made into a K-vector space such that a K-linear bijection from V to W exists ad this bijection identifies the spread $\mathcal{T}$ with a spread on V, such that the components of $\mathcal{T}$ are K-spaces, and that K is still the full ring of kernel endomorphisms.

Thus we consider $\mathcal{S}$ and $\mathcal{T}$ to be spreads on the same vector space $(V, +)$, over K, such that K is the largest ring leaving the components of $\mathcal{T}$ invariant. Since Ψ is a collineation of the associated planes it must map the components of $\mathcal{T}$ onto the components of $\mathcal{S}$. Since the non-zero kernel endomorphisms of the spreads are subgroups of $GL(V, +)$ that leave its components invariant it is clear that the planar isomorphism Ψ must conjugate the kernel endomorphisms of $\mathcal{T}$ to $\mathcal{S}$, and since the planes are simorphic under Ψ we actualy have a field isomorphism $\psi : K \to L$, $K \mapsto k^{\psi}$, such that $\Psi(av) = a^{\psi}\Psi(v)$, for $a \in K$, $v \in V$, and in particular that $\Psi(-x) = -\Psi(x)$ for all $x \in V$.

It remains to show that Ψ is bijective. It preserves, in the associated affine plane, the parallelogram $\mathbf{0}, a, b, a + b$, whenever a and b are in different components of $\mathcal{T}$, hence in such cases $\Psi(a + b) = \Psi(a) + \Psi(b)$. If they are in same component W then we esatblish this by choosing $u \notin W$ and noting that:

$$\begin{aligned}
\Psi(a+u+b) = \Psi(a+u) + \Psi(b) &= \Psi(a) + \Psi(u) + \Psi(b) \\
\Psi(a+u+b) - \Psi(u) = \Psi(a+u) + \Psi(b) &= \Psi(a) + \Psi(b) \\
\Psi(a+u+b) - \Psi(+u) = \Psi(a+u) + \Psi(b) &= \Psi(a) + \Psi(b) \\
\Psi(a+b) + \Psi(u) - \Psi(+u) = \Psi(a+u) + \Psi(b) &= \Psi(a) + \Psi(b) \\
\Psi(a+b) + \Psi(u) - \Psi(u) = \Psi(a+u) + \Psi(b) &= \Psi(a) + \Psi(b)
\end{aligned}$$

$$\Psi(a+b) = \Psi(a+u) + \Psi(b) \quad = \quad \Psi(a) + \Psi(b)$$

provided u is further restricted not to lie in the component containing $a+b$.
■
As an immediate consequence we have:

Theorem 1.2.8 *Let $(V, \mathcal{S})$ and $(W, \mathcal{T})$ be spreads, with associated translation planes $\pi_{\mathcal{S}}$ and $\pi_(W, \mathcal{T})$. Let L and T denote the kernel endomorphism rings of $(V, \mathcal{S})$ and $(W, \mathcal{T})$ respectively.*
Let $\Psi : W \to V$ be an additive bijection. Then the following are equivalent:

1. *Ψ is an isomorphism from the spread $(W, \mathcal{T})$ onto the spread $(V, \mathcal{T})$.*

2. *There is a bijective kern isomorphism $\psi : K \to L$ such that Ψ is a K-L semilinear isomorphism, with companion isomorphism ψ, that induces a spread isomorphism from $(W, \mathcal{T})$ onto the spread $(V, \mathcal{T})$.*

3. *Ψ is an isomorphism from the plane $\pi_(W, \mathcal{T})$ onto the plane $\pi_(W, \mathcal{T})$.*

In view of the importance of the above reformulate as follows:

Theorem 1.2.9 *(Isomorphism Theorem For Translation Planes.) Let $\Phi := \Pi_{(V,\mathcal{S})}$ be a translation plane defined by a spread $(V, \mathcal{S})$, where the components of $\mathcal{S}$ are K-subspaces of the K-vector space V, where K is any skewfield. Suppose that there is an affine-plane isomorphism:*

$$f : \Phi \longrightarrow \Psi$$

from the translation plane Φ to a translation plane $\Psi := \Pi_{(W,\mathcal{R})}$, defined by a spread $(W, \mathcal{R})$, where the components of $\mathcal{R}$ are L-subspaces of the L-vector space W, where L is any skewfield.

1. *Then L and K are isomorphic skewfields and ϕ may be considered a semi-linear mapping from W onto V.*

2. *If $\Phi = \Psi$ then ϕ is an element of the group $\Gamma L(V, K)$.*

3. *The full automorphism group G of the translation plane π is a semi-direct product of the translation group T by the subgroup G_0 of $\Gamma L(V, K)$ which permutes the components of the spread S.*

The subgroup G_0 of $\Gamma L(V, K)$ is called the 'translation complement' of G or π. $G_0 \cap GL(V, K)$ is called the 'linear translation complement.'

Proof: See above. ∎

We now make some conventions regarding the kernel of a spread, or its *kern*, as we shall usually call it. These relates to the fact that the components of a spread $\mathcal{S}$ may be regarded as being subspaces of the ambient vector space V, over *any* subfield F in the kern of $\mathcal{S}$, in the sense of theorem 1.2.6 above: so in general there is a multitude of dimensions associated with a spread — depending on the field or skewfield over which we choose to *represent* it. If the 'chosen field' is F, in the kern K of the spread $\mathcal{S}$, then we shall sometimes call F the 'chosen kern', the 'component kern' or the 'intended kern'.

Definition 1.2.10 *Let V be a vector space over a skewfield F that contains a spread $\mathcal{S}$), consisting of F-subpaces; so F is the component kern. The* RANK OVER F *of $(V, \mathcal{S})$ is the common dimension of the members of $\mathcal{S}$: so an n-dimensional F-spread $\mathcal{S}$ has ambient space V with* dimension $2n$; *now* $\pi := (V, \mathcal{S})$ IS REGARDED AS BEING AN F-SPREAD OF F-RANK n. *The* RANK *of $\mathcal{S}$ is its rank over K, the kern of $\mathcal{S}$.*

Since any rank two vector space, over an arbitrary skewfield K, partitions into a collection of rank one spaces, we conclude that *one dimensional spreads exist over every sfield*! But, as indicated earlier, we may now regard these spreads as being F-spreads of rank > 1 whenever F is a subfield of K. Thus F-spreads of F-rank n exist in abundance. This raises a problem — not too hard but certainly non-trivial — how do we know whether any spread that we construct is not a rank-one spread in disguise? Putting it somewhat more provocatively:

$$\boxed{\text{Are ALL spreads rank ONE!?}} \tag{1.1}$$

So we need to first of all describe all rank one spreads, that is, spreads that are rank one over their *full* kern. We begin by officially adopting the definition:

Definition 1.2.11 *A rank one spread is called a* DESARGUESIAN SPREAD.

A rank one spread is isomorphic to a spread δ on the vector space $V = K^2$, where

1. K is a skewfield acting wlog from the left in the standard way:

$$\forall k, k_1, k_2 \in K : k(k_1, k_2) = (kk_1, kk_2);$$

2. The components of δ are the subspaces of type 'y=xm', $m \in K$, and $x = 0$, as in coordinate geometry.

The proof follows from the fact that any rank two vector space may be regarded as some K^2, with K acting form the left, and *all* the rank-one spaces must be components. The associated affine plane consists of all cosets of the spread components and hence the lines are of form $y = xm + b$ and $y = c$. Thus rank-one spreads correspond to precisely the high-school interpretation of the term. Hence we have justified our terminology by showing that:

Remark 1.2.12 (Desarguesian Spreads.) *The following are equivalent for a spread $\mathcal{S}$:*

1. *$\mathcal{S}$ is rank one over its kern;*

2. *The affine plane $\pi_{\mathcal{S}}$, associated with $\mathcal{S}$, is a Desarguesian plane.*

Note that we have now described all one-dimensional spreads over any skewfield K! In the finite case all finite skewfields are Galois fields, so *all rank one spreads are* REALLY! *known.* So the obvious next step is:

$$\boxed{\text{INVESTIGATE THE RANK TWO SPREADS OVER A GALOIS FIELD!}} \qquad (1.2)$$

During the last twenty years a great deal of attention has been given to this project; there are also associations with other areas of finite geometries, particularly flocks and generalized quadrangles. Note that the existence of rank two spreads obviously settles as a by-product the 'first question' for spreads, see (1.1). The principal tool for such investigations involve *spreadsets*, the main concern of the next lecture.

1.3 Spreadsets and Partial Spreads.

In the previous lecture, we saw that by the fundamental theorem of translation planes, theorem 1.2.6, translation planes may be identified with spreads. Here we introduce tools and concepts that arise inevitably in the study of spreads. The concept of a *partial* spread describes collections of subspaces of a vector space that putatively extend to a spread. The other concept that we introduce aims at 'coordinatizing' spreads and partial spreads by sets of matrices (in the finite-dimensional case), exploiting the fact that spreads (and hence translation planes) are always associated with some vector space. These sets of matrices, or linear maps in the general case, are called [partial] spreadsets: they provide the most important computational tool in the study of translation planes.

In the motivating case, *a spreadset is a set of* q^n *matrices* $\mathcal{M} \subset GL(n,q) \cup \{O\}$ *such that any two members of* $\mathcal{M}$ *differ by a non-singular matrix and* $O \in \mathcal{M}$. Such a set yields a spread $\pi_{\mathcal{M}}$ in $V = GF(q)^n \oplus GF(q)^n$: the components are $y = xM$, $M \in \mathcal{M}$ and $x = 0$, mimicking the the construction of elementary coordinate geometry. The spread $\pi_{\mathcal{M}}$ actually turns out to be a generic form for any $GF(q)$-spread on V: so spreads may be computationally investigated via their spreadsets of matrices.

The complete definition of a spreadset is a routine generalization of the above, assigning to *any* spread a spreadset of linear maps that represents it. As in the finite case, this association enables all the major tools of linear algebra to be brought to bear on the study of spreads. When the underlying field $GF(q)$ is generalized to an arbitrary skewfield K the cardinality and dimensionality condition implicit in $|\mathcal{M}| = |K|^n$ needs to be reformulated. This will be achieved by defining the familiar concepts of semiregularity and transitivity from permutation group theory so as to apply to SETS of possibly infinite bijections.

Accordingly, we begin by explaining what transitivity and regularity mean in the context of a set of permutations on Ω, where Ω may be an infinite set. The definitions here generalize the corresponding definitions for permutation *groups* listed in definition 1.3.1.

Definition 1.3.1 *Let G denote a set of bijections of a set Ω. Then the G-orbit* AT $x \in \Omega$ *is*

$$Orb_G(x) := \{x^g \mid g \in G\}.$$

G is called a TRANSITIVE *set of maps on Ω if $Orb_G(x) = \Omega$ for all $x \in \Omega$. The set G is called semi-regular if:*

$$(x,y) \in \Omega \times \Omega \Longrightarrow \exists! g \in G \ni xg = y,$$

and G is a REGULAR *set of bijections of Ω if it semiregular and transitive on Ω.*

For finite sets, it is straightforward to check that all of the above concepts coincide provided G and Ω have the same size:

Remarks 1.3.2 *If G is a set of bijections on a finite set Ω and $|G| = |\Omega|$ then the following are equivalent:*

(a) G is semiregular;

(b) G is regular;

(c) G is transitive.

Note that condition (b) above implies $|G| = |K|$, even in the infinite case, and hence we shall use it as the basis of our general definition of a spread. However, we begin by introducing spreadsets, not in their most general form, but rather in the form that they are most frequently encountered: as sets of q^n matrices in $\overline{GL(n,q)} := GL(n,q) \cup \{\mathbf{O}\}$ that act regularly on $GF(q)^n - \{\mathbf{0}\}$.

Definition 1.3.3 *An $n \times n$ SPREADSET OF MATRICES over $GF(q)$ is a set of matrices*

$$\{\mathbf{O}\} \subset \mathcal{M} \subset \overline{GL(n,q)}$$

such that (1) $|\mathcal{M}| = q^n$; (2) Any two distinct member of $\mathcal{M}$ differ by a non-singular matrix.

It is immediate that the action of the above $\mathcal{M}^* := \mathcal{M} - \{\mathbf{O}\}$ on $GF(q)^n - \{\mathbf{0}\}$ is regular and that the regularity of $\mathcal{M}^*$ is actually equivalent to the definition of a finite spreadsets. Thus the concept of a spreadset, as indicated earlier, can be generalized to arbitrary vector spaces over any skewfield as follows:

Definition 1.3.4 *Let K be any skewfield, and V a vector space over K. A K-SPREADSET of V is a set $\mathcal{M}$ of linear maps:*

$$\{\mathbf{O}\} \subset \mathcal{M} \subset \overline{GL(V,K)}$$

such that $\mathcal{M}^*$ acts as a regular set of maps on V^*.

Thus, a finite set of matrices over $K = GF(q)$, is a spreadset of matrices in the sense of defintion 1.3.3 iff it is a spreadset of linear maps in the sense of definition 1.3.4 above: just apply remark 1.3.2 above.

It is important to realise that the non-singularity-of-difference condition, in the definition of finite matrix spreadsets, definition 1.3.3, may be used in characterising general spreadsets:

Remark 1.3.5 *Let K be any skewfield, and V a vector space over K. A set $\mathcal{M}$ of linear maps of V satisfying:*

$$\{\mathbf{O}\} \subset \mathcal{M} \subset \overline{GL(V,K)}$$

is a spreadset iff:

1. *$A, B \in \mathcal{M}$ are distinct then $A - B \in GL(V,K)$;*
2. *If $(x,y) \in V^* \times V^*$ then there is an element $M \in \mathcal{M}$ such that $xM = y$.*

In particular, a set $\mathcal{M}$ of $n \times n$ matrices over $GF(q)$ is a spreadset iff they form a matrix spreadset in the sense of definition 1.3.3, that is, $\mathcal{M}$ has q^n elements, including zero, any two of which differ by a non-singular matrix or zero.

Proof: The second condition means that $\mathcal{M}^*$ is transitive on V^*, and the first condition means that $\mathcal{M}^*$ is semiregular on V^*, since otherwise $x(A-B)$ would be zero for some $x \in V^*$. ■

With every spreadset we shall associate a collection of subspaces which turn out to be spreads. The notation that we use here is suggested by elementary coordinate geometry, and similar notation will be used throughout these notes, sometimes without explicit definition.

Definition 1.3.6 *Let W be a vector space over a skewfield K and let $\mathcal{M}$ be a K-spreadset on W. Then $\pi_{\mathcal{M}}$ is a collection of subsets of $V = W \oplus W$ defined by*

$$\pi_{\mathcal{M}} := \{Y\} \cup \{y = xM \mid M \in \mathcal{M}\},$$

where $Y = \mathbf{O} \oplus W$ and $y = xM$, $m \in \mathcal{M}$, denotes the subset $\{(w, wM) \mid w \in W\}$ of V — so $y = \mathbf{O}$, also called X, is in $\pi_{\mathcal{M}}$. The collection $\pi_{\mathcal{M}}$ is called the SPREAD ASSOCIATED WITH $\mathcal{M}$.

We now justify the terminology by verifying that $\pi_{\mathcal{M}}$ is a genuine K-spread:

Remark 1.3.7 *Let W be a vector space over a skewfield K. Let $\mathcal{M}$ is a K-spreadset on W. Then its associated spread $\pi_{\mathcal{M}}$, definition 1.3.6 above, is a collection of K-subspaces of V that form a K-spread, in the standard sense, with ambient space $V = W \oplus W$.*

Proof: The linearity of M over K ensures that $\{(x, xM) \mid x \in W\}$ is a K-subspace of $V = W \oplus W$ — the linearity means that the K-action on W commutes with the M-action. Next we note that $y = xM$ and $y = xN$, where $M, N \in \mathcal{M}$, are *disjoint* K-subspaces of V, for $M \neq N$: for otherwise $M - N$ would be singular, contradicting $M - N \in GL(n, W)$, c.f., remark 1.3.5. Given $(a, b) \in W^* \oplus W^*$, there is an $M \in \mathcal{M}$ such that $b = aM$, by the transitivity condition on $\mathcal{M}$. Hence, it easily follows that the subspaces in the structure $\pi_{\{}\mathcal{M}\}$ form a pairwise disjoint cover of V^*. It remains to check that V is a direct sum of any two of the 'components' in $\pi_{\{}\mathcal{M}\}$. (This is obvious if W is finite dimensional over K, in particular if $\mathcal{M}$ is finite.) The main case is when the components are $y = xM$ and $y = xN$, where $\mathbf{O} \neq M \neq N \neq \mathbf{O}$, and here we need to show that any $(a, b) \in V$ lies in the sum of $y = xM$ and $y = xN$. Thus, we need to show that

$$(a, b) = (u, uM) + (v, vN) \exists u, v \in W,$$

or, equivalently, for some $u, v \in W$:

$$\begin{aligned} a &= u + v \\ b &= uM + vN \end{aligned}$$

and this means $b - aN = u(M - N)$, which can be solved for u by the non-singularity condition on $M - N$, remark 1.3.5, and the desired result follows easily. ∎

Thus to find a spread, and hence a translation plane of order q^n, it is sufficient to find a set of $q^n - 1$ matrices in $GL(n, q)$ such that any two of them differ by a non-singular matrix. This follows from the above, also c.f. defintion 1.3.3. We illustrate this with an important example, discovered first by Donald Knuth.

Example 1.3.8 *(Knuth's γ-spreads.)* Let $K \cong GF(q)$ be a finite field, where $q = p^r > p$ is odd. Let γ be a fixed nonsquare in K, and $\sigma \in Gal(K)^*$ Then

$$\mathcal{M} = \left\{ \begin{bmatrix} u & \gamma t^\sigma \\ t & u \end{bmatrix} \forall u, t \in K \right\}.$$

Proof: Because γ is non-square, the determinant $u^2 - \gamma t^(\sigma + 1)$ cannot be zero unless $u = t = 0$. Thus we have an additive group of matrices whose non-zero elements are non-singular. This means that the difference between

any two distinct members of $\mathcal{M}$ are non-singular, and since we have $q^n = q^2$ such matrices $\mathcal{M}$ is a spreadset by remark 1.3.5. ■
This is the first spread of rank 2 that we have displayed, although we have not yet shown that it is not Desarguesian, i.e. a rank-one spread in disguise. Once we have developed some more machinery this will become immediately obvious. At this stage more compuational effort is required: as an exercise the reader is invited to verify that the group of kern-homologies is *not* transitive, as a group of homolgies: this means the spread cannot be Desarguesian and hence must be rank two — thereby answering the 'first question' (1.1), and also contributing to (1.2).

Note also that the argument used in example 1.3.8 above yields a more general result: the proof is left as an exercise, and involves recalling the connection between spreads and translation planes:

Proposition 1.3.9 *An additive group $\mathcal{M}$ of $n \times n$ matrices over $GF(q)$ is a spreadset iff the group has order q^n and its non-zero elements are all nonsingular. Moreover, the associated spread $\pi_{\mathcal{M}}$ corresponds to a translation plane that admits a group of kern homologies of order $q-1$.*

The spreadsets of the above type are called *additive* spreadsets, and will be treated in detail later on. They form a major branch of translation plane theory with their own methodology, related to non-associative divison ring theory.

We now turn to the converse of remark 1.3.7. The eventual goal is to show that every *spread* is associated with a *spreadset.* But we first take the opportunity to work from more general premises, by introducing *partial* spreads and the partial spreadsets that coordinatize them.

Definition 1.3.10 *Let $\mathcal{T}$ be a non-empty collection of subspaces of a vector space V over a skewfield K.Then $\mathcal{T}$ is a* PARTIAL SPREAD *on V, and its members are its* COMPONENTS *if $V = A \oplus B$ for every pair of distinct $A, B \in \mathcal{T}$, and if $|\mathcal{T}| \leq 2$ assume explicitly that $V/A \cong A$ for $A \in \mathcal{T}$.*

Of course, $V/A \cong A$ applies automatically if $\mathcal{T}$ has at least three components. Note also that although subsets of spreads are always partial spreads, there are many partial spreads that cannot be extended to spreads: thus, there are maximal partial spreads that are *not* spreads.

To construct partial spreads, we generalize, in obvious ways, the notation and concepts that relate spreadsets to spreads in definition 1.3.6. We continue with our convention of applying the language of coordinate geometry to any direct sum $V = W \oplus W$, c.f. definiton 1.3.6.

Definition 1.3.11 *Let W be a vector space over a skewfield K. Then a non-empty set $\tau \subset GL(V,K) \cup \{\mathbf{O}\}$ is a* PARTIAL K-SPREADSET *if*

$$T_1, T_2 \in \tau \Longrightarrow T_1 - T_2 \in \overline{GL(V,K)}.$$

The associated structure of τ is the collection of subspaces of $V = W \oplus W$ given by:

$$\pi_\tau := \{y = xT \mid T \in \tau\} \cup \{Y\}.$$

In general, $\tau \subset GL(V,K) \cup \{\mathbf{O}\}$ is a SPREAD SET *if τ is a K-spread where K is the prime field over which V is a vector space.*

Note that we have included $\{Y\}$, as our earlier convention requires us to do this if τ is a spreadset, c.f., definition 1.3.6. Stating the obvious:

Remark 1.3.12 *If τ is a partial spreadset on a vector space W then π_τ is a partial spread on $V = W \oplus W$, and π_τ is a spread iff τ is a spreadset. Hence π_τ is called the* PARTIAL SPREAD ASSOCIATED WITH THE PARTIAL SPREADSET τ.

It is worth restressing that the above remark assumes that the spread on $W \oplus W$ by a spreadset τ of W always includes $Y := \mathbf{O} \oplus W$, unless the contrary is indicated: without this assumption π_τ fails to be a spread when τ is a spread.

The following easy exercise emphasizes that in the finite case a partial spreadset is just a set of non-singular matrices, possibly augmented by $\mathbf{O}$, such that any two differ by a non-singular matrix.

Remark 1.3.13 *Let V be a vector space over a skewfield K. A non-empty set $\tau \subset \overline{GL(V,K)}$ is a partial spread iff τ^* is semiregular on V^*.*

In particular, if $\tau \subset \overline{GL(n,q)}$ is a non-empty set of matrices then τ is a partial spreadset iff and the difference between any two distinct matrices in τ is non-singular.

Proof: Exercise. ∎

We now introduce the notion of isomorphic partial spreads, generalising the corresponding notion for a spread.

Definition 1.3.14 *Let $\pi_i = (V_i, \tau_i)$, $i = 1, 2$, be partial spreads, where V_1 and V_2 are the underlying vector spaces over a common skewfield K. Then a K-linear bijection $\Psi : V_1 \to V_2$ is* A K-LINEAR ISOMORPHISM *from π_1 to π_2, or τ_1 to τ_2, iff it bijectively maps the components τ_1 onto those of τ_2.*

More generally, an ISOMORPHISM *from π_1 onto π_2 is an additive isomorphism from V_1 onto V_2 that maps components onto components.*

There are of course a number of equivalent ways of defining isomorphisms among partial spreads, for example an additive isomorphism from V_1 onto V_2 is an isomorphism of the associated spreads iff it maps components onto components. The usual terminology associated with isomorphism, automorphism etc. will be used without further comment.

The following theorem implies that all spreads arise from spreadsets: there is an isomorphism from any K-spread (or partial spread) to the spread arising from a spreadset (or partial spreadset). This is one of the most important connections in translation plane theory.

Theorem 1.3.15 (Equivalence Of (Partial) Spreads and Spreadsets *Let V be a vector space over a skewfield K, and let $\mathcal{T}$ a partial spread of subspaces, with at least three components X, Y, $W, \ldots$. Choose a K-linear bijection* IDENTIFYING *Y with X:*

$$\Psi : Y \longrightarrow X.$$

Then relative to (X, Y, Ψ):

1. *For every $W \in \mathcal{T} \setminus \{Y\}$ the map $\tau_W : X \to Y$ specified by:*

$$\begin{array}{rcl} \tau_W : X & \longrightarrow & Y \\ x & \longmapsto & y \Leftrightarrow x \oplus y \in W \end{array}$$

   is a linear bijection from X onto Y when $W \neq X$ ($\tau_X = \mathbf{O}$) and hence $\Psi\tau_W : X \to X$, WRITTEN *σ_W, is an element of $GL(V, K)$; σ_W is called the* SLOPE MAP, *or the* SLOPE ENDOMORPHISM, *of W, relative to* AXES *(X, Y) (via the identification $\Psi : Y \to X$).*

2. *For fixed X and Y and any choice of $W \in \mathcal{T} \setminus \{X, Y\}$, Ψ can be chosen so that $\sigma_W = \mathbf{1}$; in fact $\Psi = {\tau_W}^{-1}$.*

3. *The set of all endomorphisms of* $\mathcal{T}$:

$$\sigma_{\mathcal{T}} := \{\sigma_W \mid W \in \mathcal{T} \setminus \{Y\}\}$$

corresponds, after deleting the zero map, to a semiregular subset of $GL(X, K)$ *on* X^*.

4. *The partial spread determined by* $\sigma_{\mathcal{T}}$, *viz.* $\pi_{\sigma_{\mathcal{T}}}$, *c.f., definition 1.3.11 and remark 1.3.12, is isomorphic to the given spread* $\mathcal{T}$. *In fact, the linear bijection* $\mathbf{1}_X \oplus \Psi^{-1}$:

$$\begin{array}{ccc} X \oplus X & \longrightarrow & X \oplus Y \\ a \oplus b & \longmapsto & a \oplus (b)\Psi^{-1} \end{array}$$

is a linear isomorphism from the [partial] spread $\pi_{\sigma_{\mathcal{T}}}$ *onto the [partial] spread* $\mathcal{T}$ *that maps* $X \oplus \mathbf{O}$ *and* $\mathbf{O} \oplus X$ *onto* X *and* Y *respectively, that is, the isomorphism can be chosen so that the* X *and* Y *'axes' are preserved.*

Moreover, if the axes-identifying' linear bijection $\Psi : X \to Y$ *is specified by* $\Psi := \tau_W{}^{-1}$, *where* $\tau_W : X \to Y$ *is the linear bijection associated with* $W \in \tau \setminus \{X, Y\}$, *then the 'unit component'* $Z := \{(x, x) \mid x \in X\}$ *is assigned, by the partial spread isomorphism* $\mathbf{1}_X \oplus \Psi^{-1}$, *to the chosen component* $W \in \mathcal{T}$.

To summarize, $\mathcal{T}$ *may be identified, via a linear bijection* $\Lambda : V \to X \oplus X$, *with a partial spread* π_τ *on* $X \oplus X$, *corresponding to a spreadset* τ *on* X, *such that the identification sends respectively the components* X *and* Y *of* $\mathcal{T}$ *onto respectively the* x*-axis, i.e.* $X \oplus \mathbf{O}$, *and the* y*-axis, i.e.* $\mathbf{O} \oplus X$. *Morever, the map* $\Phi : Y \to X$ *that* Λ *induces naturally from* Y *to* X, *defined by restrictiing it to* Y:

$$\Phi := \Lambda | Y \to \mathbf{O} \oplus X \stackrel{\text{natural}}{\longrightarrow} X$$

can be chosen, for appropriate $\Lambda := \Lambda_\Phi$, *so that* $\Phi = \Psi^{-1}$, *where* $\Psi : Y \to X$ *is the given identification; and if now* Ψ *is taken as* $\tau_W{}^{-1}$ *then* Λ *additionally maps the component* W, *distinct from* X, Y, *onto the unit line* Z *defined above.*

5. *If* $\mathcal{T}$ *is a spread then the following are equivalent:*

(a) *The set of slope endomorphisms* σ_T *is a spreadset on* X.

(b) σ_T^* *is regular on* X^*.

(c) σ_T^* *is transitive on* X^*.

Proof: (1) We first show that τ_W is a map. Consider $\tau_W(x)$. If y_1 and y_2 are distinct elements of Y such that $x + y_1 \in W$ and $x + y_2 \in W$, then $y_1 - y_2 \in W$, and this is a contradiction because the components of a partial spread do not overlap. Since $X \oplus Y$ is the whole space we certainly have $x + y \in W$, for some $y \in Y$. Hence $\tau_W : X \to Y$ is a map, and it is equally straightforward to check that this map is linear and injective, for $W \notin \{X, Y\}$.

To verify that τ_W is bijective, for W distinct from X and Y, consider $y \in Y$. If $y \neq \tau_W(u)$ for all $u \in X$ then $u + y \notin W$ for all $u \in X$, so $y \notin X \oplus W$, contradicting the fact that any two components must direct-sum to the whole space V. Hence (1) holds, since it is trivial that $\tau_X = \mathbf{O}$.
(2) This case is immediate.
(3) Now consider τ_A and τ_B, where A and B are distinct components, other than X and Y. If $\tau_A - \tau_B(x) = \mathbf{0}$, for $x \neq \mathbf{0}$, then $x \oplus \tau_A(x) \in A \cap B$, contradicting the fact that distinct components do not over lap. Thus $\Psi\tau_A(x) \neq \Psi\tau_B(x)$, for $x \neq \mathbf{0}$, which means $\sigma_A(x) \neq \sigma_B(x)$, and hence σ_T is a semiregular spreadset in $GL(V, K)$.
(4) The partial spread π_{σ_T} associated with σ_T, in the sense of remark 1.3.12, has components $\{(x, x\tau_W\Psi) \mid x \in X\}$, for $W \in \tau$ The linear bijection $\mathbf{1}_X \oplus \Psi^{-1}$ defined by

$$\begin{array}{rcl} X \oplus X & \longrightarrow & X \oplus Y \\ a \oplus b & \longmapsto & a \oplus (b)\Psi^{-1} \end{array}$$

maps $(x, x\sigma_W) = (x, x\tau_W\Psi)$ onto the component $(x, x\tau_W)$ and $\mathbf{O} \oplus X$ onto $\mathbf{O} \oplus Y$.

The 'summary' is just a restatement of the facts established about $\mathbf{1}_X \oplus \Psi^{-1} : X \oplus X \to X \oplus Y$, in terms of its inverse map $\Lambda : X \oplus Y \to X \oplus X$.
(5) The equivalence of the conditions follows from remark 1.3.2, giving the corresponding equivalences for arbitrary sets of permutations, together with the fact that a partial spreadset is a spread iff it is regular on X^*, c.f. definition 1.3.4. ∎
Thus, the fundamental identification of partial spreads with partial spreadsets corresponds to a generalization of the situation in elementary coordinate

geometry: sets of lines through the origin are identifed with the set of their gradients, the subspace y=xm being identified with its slope m. Moreover, we have shown, as in elementary geometry, that any two lines may be taken as the x and y axis, and that by rescaling (recall the identification $\Phi : Y \to X$) on the y axis we can further force any chosen third line through the origin to be the unit line.

Note however that in our case the 'points' of the x-axis are used as coordinate values, whereas in elementary geometry a distinct set, viz. the reals, are used as coordinate values. It is often convenient to mimic this setup in our situation by allowing the chosen components, X and Y, to be coordinatized by an arbitrary vector space R, isomorphic to the components of the given spread.

For example, the natural choice for R, when the components are n-dimensional over a field K, is to take $R = K^n$, and now X and Y are identified with W by specifying bases $(e_1, e_2, \ldots, e_n$ and $(f_1, f_2, \ldots, f_n$ respectively; in this setup the 'axes-identifying' linear bijection $\Psi : Y \to X$ is tacitly taken to be the linear map sending $f_i \mapsto e_i$, for $1 \le i \le n$. Now the associated [partial] spreadset becomes a set of matrices $\mathcal{M}$ and the 'canonical' form of the given [partial] spread is in $K^n \oplus K^n$, and the components are $y = xM$, $M \in \mathcal{M}$, plus the Y-axis.

Recall that to also force a component W, of the given spread, to become the unit line under the chosen coordinatization, it becomes necessary to fix the axes identifier map $\Psi : Y \to X$ — $\Psi = \tau_W{}^{-1}$, in the sense of the theorem. However, since by our convention Ψ is *fixed* by the chosen basis of X and Y we can specify the required Ψ by taking an appropriate basis $(f_1, f_2, \ldots, f_n)$ of B so that the unique linear bijection specified by the basis image $e_i \mapsto f_i$, for all i, coincides with Ψ.

The above, analysis can be repeated for arbitrary vector spaces over a skewfield K. The basis for X and Y are then families $(e_i)_{i\in\lambda}$ and $(f_i)_{i\in\lambda}$ (respectively), indexed by a possibly infinite set λ. As before, a component W can be forced to be the identity by choosing an *appropriate* $(f_i)_{i\in\lambda}$. Note that if K is a non-commutative skewfield and the chosen space R is taken to be the space K^λ, the 'λ-tuples' over K, then it might be necessary to specify whether K^λ is regarded as a left a right K-space.

We summarize our conclusions as follows:

Corollary 1.3.16 (Basis Decomposition Theorem.) *Let V be a vector space over a skewfield K, and suppose $\mathcal{T}$ is a partial spread on V with at* *least three distinct components $X, Y, W \ldots$. Let Z be any vector space that is isomorphic to the components of $\mathcal{T}$. Then*

1. *There is a partial spreadset τ on Z that contains the identity map $\mathbf{1}_Z$ and a K-linear isomorphism*

$$\Lambda : V \Longrightarrow Z \oplus Z$$

such that Λ is a K-linear partial spread isomorphism from $\mathcal{T}$ to π_τ satisfying:

$$\Lambda(X) = Z \oplus \mathbf{O}, \quad \Lambda(Y) = \mathbf{O} \oplus Z \quad \text{and } \Lambda(W) = \{(z \oplus z \mid z \in Z\}.$$

In fact, to each K-linear bijection $\alpha : X \to Z$ there corresponds a K-linear bijection $\beta : Y \to Z$ such that

$$\Lambda = \alpha \oplus \beta : V \Longrightarrow Z \oplus Z.$$

2. *Let $B_X := (e_i)_{i \in \lambda}$ be a basis of X and for any basis $B_Y := (f_i)_{i \in \lambda}$; so the juxtapostion $B_V := (B_X; B_Y)$ is a basis of V. Define the canonical K-linear isomorphism $\beta_X : X \to K^\lambda$, $\beta_Y : Y \to K^\lambda$, and $\beta_X \oplus \beta_Y \to X \oplus Y \to K^\lambda \oplus K^\lambda$. (N.B. If K is non-commutative, K^λ is made into a left or a right vector space, depending on whichever guarantees the required K-linear isomorphisms with X and Y.)*

Then there is a partial spreadset τ on K^λ such that the K-linear bijection

$$\beta_X \oplus \beta_Y : V \to X \oplus Y \to K^\lambda \oplus K^\lambda$$

defines an isomorphism from $\mathcal{T}$ to π_τ, the partial spread on $K^\lambda \oplus K^\lambda$ associated with τ.

Moreover, any component $W \in \mathcal{T} \setminus \{X, Y\}$ can be mapped to the unit line $x = y$ of $K^\lambda \oplus K^\lambda$, thus ensuring $\mathbf{1} \in \tau$, for any choice of the basis B_X, and for some choice of B_Y (depending on the B_X selected.

Proof: By the preceding remarks. ∎

For emphasis we restate what this means for finite-dimensional spreads.

Proposition 1.3.17 *Let V be a vector space of dimension $2n$, n a positive integer, over a field K, and that τ is a partial spread of K-subspaces of V with at least three distinct components $X, Y, Z \ldots$. Choose a K-basis $B_X :=$*

$(e_1, e_2, \ldots, e_n)$ *of* X *and* K*-basis* $B_Y := (f_1, f_2, \ldots, f_n)$ *of* Y*, and let* $B_V :=$ $[B_X, B_Y]$ *denote the associated* K*-basis of* V*, obtained by juxtaposition, thus:*

$$B_V =< B_X, B_Y >:= (e_1, e_2, \ldots, e_n, f_1, f_2, \ldots, f_n).$$

Then there is a basis B_Y *of* Y *such that relative to the basis* $[B_X, B_Y]$ *of* V *the canonical linear bijection:*

$$\beta : V \longrightarrow K^n \oplus K^n,$$

maps X *onto* $K^n \oplus \mathbf{O}$*,* Y *onto* $\mathbf{O} \oplus K^n$*, and* Z *onto the* UNIT LINE

$$\{(x, x) \mid x \in K^n\}.$$

Proof: The proposition is a special case of the result above, corollary 1.3.16. ∎

We conclude with a basic isomorphism result.

Theorem 1.3.18 *Let* π *be a translation plane with spread* S_π *of* $X \oplus X = V$ *where* X *is a left* K*-vector space and let* ρ *be a translation plane with spread* S_ρ *of* $Y \oplus Y = W$ *where* Y *is a left* L*-vector space. Assume that* K *and* L *are the component kernels of* π *and* ρ *respectively.*

Let ρ *and* π *be isomorphic by a bijective incidence presering mapping* ϕ*.*

(1) Then L *and* K *are isomorphic skewfields and* ϕ *may be considered a semi-linear mapping from* W *onto* K*.*

(2) If $\pi = \rho$ *then* ϕ *is an element of the group* $\Gamma L(V, K)$*.*

Furthermore, the full automorphism group G *of the translation plane* π *is a semi-direct product of the translation group* T *by the subgroup* G_0 *of* $\Gamma L(V, K)$ *which permutes the components of the spread* S*.*

The subgroup G_0 *of* $\Gamma L(V, K)$ *is called the 'translation complement' of* G *or* π*.* $G_0 \cap GL(V, K)$ *is called the 'linear translation complement.'*

Proof: We have seen (2) previously. We note that if g is in the kernel endomorphism skewfield $\mathcal{K}$ of π then $g^{-1}\phi g$ is in the kernel endomorphism skewfield $\mathcal{L}$ of ρ. Hence,

$$K \cong \mathcal{K} \cong \mathcal{L} \cong \mathcal{L}.$$

1.4 Tutorial On Spreadsets.

This tutorial discusses important aspects of the above theory: low rank spreads; reguli. The latter suggests the need for introducing a projective-space version of the theory of spreads and partial spreads. This Bruck-Bose theory will be systemaically introduced later on. The focus in the tutorial is on the motivating cases rather than the general case. The reader is invited to tidy up the sketchy treatment presented and to anticipate developments.

Rank-Two Spreads.

We have mentioned on several occasions that all rank-one spreads have been described. It is thus natural to turn to rank two spreads. The literature concerned with this area of translation planes is enormous; part of the interest stems form its connection with the theory of flocks, generalized quadrangles and packing problems that are themselves associated with highly interesting higher rank spreads.

By specialising the above we can reduce the study of rank two spreads to spreadsets indicated in the following theorem. This theorem underpins the enormous literature concerning two-dimensional spreads; the theorem also provides a pathway to the theory of flocks and certain types of generalized quadrangles.

Theorem 1.4.1 *Let $\pi := (V, \mathcal{S})$ be a spread of rank ≤ 2 over a skewfield K. Then there are functions g and f from $K \times K$ to K such that*

$$\mathcal{M}_{(g,f)} \begin{bmatrix} g(t,u) & f(t,u) \\ t & u \end{bmatrix} \forall t, u \ in \ K$$

is a spreadset, and there is a K-linear spread isomorphism Ψ from π onto the spread $\pi_{\mathcal{M}_{(g,f)}}$, vewed as a K-spread such that any ordered triple (X, Y, Z), consisting of three distinct components of π, get mapped under Ψ onto the triple $(y = 0, x = 0, y = x)$: that is, the image under Ψ of X, Y and Z are resp. the x-axis, the y-axis and the the unit line of Ψ.

Proof: By the above we know that an isomorphism form π to $\pi_{\mathcal{M}}$ exists for some two-dimensional spreadset. So the only question is whether it has the given form. Since the difference between distinct members in $\mathcal{M}$ are to be non-singular, distinct members of $\mathcal{M}$ have different first rows and

also distinct second rows. (For *skew*fields consider the image of $(1,0)$ under distinct members of $\mathcal{M}$ to get distinct first rows, and similarly use $(0,1)$ for the second row). Moreover, the regularity condition on a spreadset means that the image of $(0,1)$ must range over K^2, so the second row ranges over all of K^2. Moreover, for any given value of the second row $(u,v) \in K^2$ we must have unique values $g(u,v)$ and $f(u,v)$ in positions $(1,1)$ and $(1,2)$ resp., for otherwise the fact that distinct components have distinct *second* rows gets violated. Hence g and f are single-valued, which is the desired result. ∎
The identification above may be expressed by interchanging the two rows of $\mathcal{M}$. One way to establish this is to appropriately modify the proof of the above. This is left as exercise. Note that the 'new' spreadset is the *same* one as before but expressed differently.

Remark 1.4.2 *The spreadset $\mathcal{M}$, for the given (X,Y,Z), can be alternatively writen as $\mathcal{M}$*

$$\mathcal{M}_{(g,f)} = \begin{bmatrix} t & u \\ g(t,u) & f(t,u) \end{bmatrix} \forall t, u \text{ in } K$$

We end with some simple, but important, exercises on *finite* rank two spreads, or rather on spreads that have a rank two representation — so as not exclude the Desarguesian case. The reader is encouraged to consider how far the results generalize: (1) to finite spreads of arbitrary rank; (2) spreads of rank two over commuative fields and skewfields, etc.

Exercise 1.4.3 *Let $K = GF(q)$, $q = p^r$. Let $\mathcal{M}$ be a 2×2 spreadset with entries in K. Then:*

1. *Let A and B be non-singular matrices in $GL(2,q)$. Then $\mathcal{N} := A^{-1}\mathcal{M}B$ is a spreadset and there is a K-linear spread-isomorphism from $\pi_{\mathcal{M}}$ to $\pi_{\mathcal{N}}$. In fact the mapping*

$$A \oplus B : K^2 \oplus K^2 \longrightarrow K^2 \oplus K^2$$

is the required isomorphism.

2. *Suppose $\mathcal{M}$ and $\mathcal{N}$ are spreadsets such that one is obtained from the other by a sequence of row and/or column transformations (so each transform θ in the sequence must be applied to every member of the spreadset being considered). Then there is a K-linear spread isomorphism from $\pi_{\mathcal{M}}$ to $\pi_{\mathcal{N}}$ such that the x-axis and the y-axis are both preserved.*

3. *If $\mathcal{M}$ is a spreadset then so is $\mathcal{M}^t$, obtaned by transposing every member of $\mathcal{M}$.*

The Regulus

In the following exercises on partial spreads and partial spreadsets, we introduce the regulus. They provide one of the most important tools for the construction and analysis of spreads, and hence translation planes. A systematic treatment of reguli will follow later, based on the projective space approach to [partial] spreads. The treatment provided here clearly indicates the desirability for introducing projective language instead of always working directly with vector spaces. This approach, the Bruck-Bose version of André's theory, will be introduced systematically in section 2.2.

Exercise 1.4.4 *Let $\mathcal{K}$ denote the scalar regulus in $K^n \oplus K^n$, K a field; thus $\mathcal{K}$ has the scalar field $K \leq GL(n, K)$ as its partial spreadset; $\mathcal{K} = \pi_K$. Here K is identifed with the $n \times n$ scalar matrix field with entries in K.*

1. *Show that for $A \in GL(n, K)$, $\{kA \mid k \in K\}$ is the partial spreadset of a regulus $\mathcal{R}_A$ that contains $y = xA$, and shares $x = 0$ and $y = 0$ with the scalar regulus $\mathcal{K}$. Conversely, every regulus in $K^n \oplus K^n$, that contains the x-axis and y-axis, is of the form $\mathcal{R}_A$, for some $A \in GL(n, K)$. (Apply the linear bijection $Diag\,[\mathbf{1}, A]$ to the scalar regulus; also remember that a regulus is determined by any three of its components.)*

2. *For A, B non-singular,*
$$\mathcal{R}_A \cap \mathcal{R}_B = \{x = 0, y = 0\} \quad or \quad \mathcal{R}_A = \mathcal{R}_B$$

3. *In $PG(2n-1, K)$, let $\mathcal{R}_{X,Y}$ be tthe set of all reguli $\mathcal{R}_{X,Y}$ that share two fixed components, X and Y. Then $\mathcal{R}_{X,Y}$ induces a partition on all the subspaces of $PG(2n-1, K)$, that have projective dimension $n-1$, and are distinct from X and Y, and the subgroup G of $PGL(n, K)$ that fixes X identically and leaves Y invariant induces a transitive group on $\mathcal{R}_{X,Y}$, and the global stabilizer in G of any $R \in \mathcal{R}_{X,Y}$ acts sharply transitively [i.e. regularly] on $R \setminus \{X, Y\}$. (Interpret the earlier parts projectively; observe that G is sharply transitive on $\mathcal{K} \setminus \{X, Y\}$.)*

We can now establish that our definition of regulus coincides with the classical definition, used in finite geometry.

Exercise 1.4.5 *A regulus in $PG(2n-1,q)$ is a partial spread with $q+1$ components such that a line meeting three of the components meets all of them.*

We note in passing that when $n=1$, then the regulus coincides with a ruling class of a hyperbolic quadric.

Exercise 1.4.6 *A spread $\mathcal{S}$ is called regular iff $\mathcal{R} \subset \mathcal{S}$, whenever $\mathcal{R}$ is the regulus containing three distinct components of $\mathcal{S}$. In $PG(2n-1,2)$ every spread is regular.*

Reguli In Projective Spaces.

Any vector space V over a skewfield K may be viewed as projective spcae $PG(V,K)$ whose points are the rank one K subspaces of V and whose lines are the rank two subspaces; in general the projective dimension of a rank k-subspace W of V is $k-1$ by definition. Using this terminology the fundamental theorem of spreads and partial spreads may be expressed in terms of projective spaces, which is the Bruck-Bose model. All this will be developed in the next section on the basis of a systematic review of projective spaces.

The goal here is to consider certain aspects of partial spreads called reguli: these are the most important partial spreads arising in translation plane theory.

Exercise 1.4.7 *A regulus in $PG(2n-1,K)$, K a field, is a partial spread $\mathcal{S}$, of the associated vector space V, such the set of projective lines meeting three distinct components of $\mathcal{S}$ cover the same projective points as are covered by the members of $\mathcal{S}$. Show that when $V = X \oplus X$ then $y = xk$, $k \in K$, together with $x = 0$, form a regulus called the* scalar regulus *on $X \oplus X$.*

What if K is a non-commutative skewfield?

Proof: The rank two space ℓ_u, $u \in K$, spanned by $\{u \oplus 0, 0 \oplus u\}$ meets ever component in a rank one space, and the totality of points covered are all the projective point of type $[(u,uk)]$, $u,k \in K$ and the points on the y-axis. If K is not commutative then $y = xk$ is additive but not a K-space if K operates from the right as (xa, xka) is not on $y = xk$ if a is not centralized by K. So, although the covering is there and the spread $y = xk$ are both there, the components of the spreads are not always K-spaces: they are spaces over fields in the center of K. ■

Thus the scalar regulus is a genuine regulus iff the scalar field K is a *commutative* fields!

Now consider any regulus $\mathcal{S}$ in $PG(2n-1, K)$, the underlying vector space being V, K any field. So we have a K-linear isomorphism Ψ onto a regulus in $K^n \oplus K^n$ such that a triad of distinct components (X, Y, Z) of $\mathcal{S}$ get mapped onto the triad $(y = 0, x = 0, y = x)$; also a line cover of $\mathcal{S}$ gets mapped onto a line cover of the image $\Psi(\mathcal{S})$. But any line meeting all three members of the triad $(y = 0, x = 0, y = x)$ must meet every set $y = xk$, for $k \in K$, and lies in the totality of such subspaces. Thus the regulus $\Psi(\mathcal{S})$ must concide with the scalar regulus. Hence we have established several facts: (1) every regulus over a field may be viewed as a scalar regulus and three components of a partail spread over a field lie in a unique regulus (which may not be in the partial spread).

Thus we have established

Remark 1.4.8 *In $PG(2n-1, K)$, for K a commutative field, there is a linear bijection from any regulus onto the scalar regulus and this bijection can be chosen so that any three components may be mapped respectively onto the x-axis, the y-axis, and the unit inc of the scalar regulus. Moreover, three components of a partial spread lie in a unique regulus and hence the subgroup of $PGL(2n-1, K)$ fixing a regulus is triply transitive on its components.*

We shall eventually deal with the most general case associated with the above result: K any skewfield with infinite dimensions allowed. This is essentially a repeat of the above but with more attention to some details.