CHAPTER 1

A SHORT INTRODUCTION TO BANACH LATTICES AND POSITIVE OPERATORS

In this chapter we give a brief introduction to Banach lattices and positive operators. Most results of this chapter can be found, e.g., in [26], [1] or [21].

1.1 BANACH LATTICES

A non empty set M with a relation $\leq$ is said to be an *ordered set* if the following conditions are satisfied.

i) $x \leq x$ for every $x \in M$,

ii) $x \leq y$ and $y \leq x$ implies $x = y$, and

iii) $x \leq y$ and $y \leq z$ implies $x \leq z$.

Let A be a subset of an ordered set M. The element $x \in M$ (resp. $z \in M$) is called *an upper bound* (*lower bound* resp.) of A if $y \leq x$ for all $y \in A$ (resp. $z \leq y$ for all $y \in A$). Moreover, if there is an upper bound (resp. lower bound) of A, then A is said *bounded from above* (*bounded from below* resp.). If A is bounded from above and from below, then A is called *order bounded*. Let $x,y \in M$ such that $x \leq y$. We denote by

$$[x,y] := \{z \in M : x \leq z \leq y\}$$

the *order interval* between x and y. It is obvious that a subset A is order bounded if and only if it is contained in some order interval.

Definition 1.1.1 *A real vector space E which is ordered by some order relation $\leq$ is called a vector lattice if any two elements $x, y \in E$ have a least upper bound denoted by $x \vee y = \sup(x,y)$ and a greatest lower bound denoted by $x \wedge y = \inf(x,y)$ and the following properties are satisfied.*

(L1) $x \leq y$ *implies* $x+z \leq y+z$ *for all* $x,y,z \in E$,

(L2) $0 \leq x$ *implies* $0 \leq tx$ *for all* $x \in E$ *and* $t \in \mathbb{R}_+$.

Let E be a vector lattice. We denote by $E_+ := \{x \in E : 0 \leq x\}$ the *positive cone* of E. For $x \in E$ let

$$x^+ := x \vee 0,\, x^- := (-x) \vee 0, \text{ and } |x| := x \vee (-x)$$

be the *positive part*, the *negative part*, and the *absolute value* of x, respectively. Two elements $x, y \in E$ are called *orthogonal* (or *lattice disjoint*) (denoted by $x \perp y$) if $|x| \wedge |y| = 0$.

For a vector lattice E we have the following properties (cf. [26, Proposition II.1.4, Corollary II.1.1 and II.1.2] or [21, Theorem 1.1.1]).

Proposition 1.1.2 *For all $x,y,z \in E$ and $a \in \mathbb{R}$ the following assertions are satisfied.*

(i) $x+y = (x \vee y) + (x \wedge y)$,
$x \vee y = -(-x) \wedge (-y)$,
$(x \vee y) + z = (x+z) \vee (y+z)$,
and $(x \wedge y) + z = (x+z) \wedge (y+z)$.

(ii) $x = x^+ - x^-$.

(iii) $|x| = x^+ + x^-$, $|ax| = |a||x|$, *and* $|x+y| \leq |x| + |y|$.

(iv) $x^+ \perp x^-$ *and the decomposition of x into the difference of two orthogonal positive elements in unique.*

(v) $x \leq y$ *is equivalent to* $x^+ \leq y^+$ *and* $y^- \leq x^-$.

(vi) $x \perp y$ *is equivalent to* $|x| \vee |y| = |x| + |y|$. *In this case we have* $|x+y| = |x| + |y|$.

(vii) $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$ *and* $(x \wedge y) \vee z = (x \vee z) \wedge (y \vee z)$.

(viii) *For all* $x,y,z \in E_+$ *we have* $(x+y) \wedge z \leq (x \wedge z) + (y \wedge z)$.

(ix) $|x-y| = (x \vee y) - (x \wedge y)$, *and* $|x-y| = |(x \vee z) - (y \vee z)| + |(x \wedge z) - (y \wedge z)|$.

A norm on a vector lattice E is called a *lattice norm* if

$$|x| \leq |y| \text{ implies } \|x\| \leq \|y\| \quad \text{for } x, y \in E.$$

Definition 1.1.3 *A Banach lattice is a real Banach space E endowed with an ordering $\leq$ such that $(E, \leq)$ is a vector lattice and the norm on E is a lattice norm.*

For a Banach lattice E the following properties hold (cf. [26, Proposition II.5.2] or [21, Proposition 1.1.6]).

Proposition 1.1.4 *Let E be a Banach lattice. Then,*

(a) the lattice operations are continuous,

(b) the positive cone E_+ is closed, and

(c) order intervals are closed and bounded.

• Sublattices, solids, bands and ideals

A vector subspace F of a vector lattice E is a *vector sublattice* if and only if the following are satisfied.

(1) $|x| \in F$ for all $x \in F$,

(2) $x^+ \in F$ or $x^- \in F$ for all $x \in F$.

A subset S of a vector lattice E is called *solid* if $x \in S$, $|y| \le |x|$ implies $y \in S$. Thus a norm on a vector lattice is a lattice norm if and only if its unit ball is solid. A solid linear subspace is called an *ideal*. Ideals are automatically vector sublattices since $|x \vee y| \le |x| + |y|$. One can see that a subspace I of a Banach lattice E is an ideal if and only if

$$x \in I \text{ implies } |x| \in I \text{ and } 0 \le y \le x \in I \text{ implies } y \in I.$$

Consequently, a vector sublattice F is an ideal in E if $x \in F$ and $0 \le y \le x$ imply $y \in F$. A subspace $B \subseteq E$ is a *band* in E if B is an ideal in E and $\sup(M)$ is contained in B whenever M is contained in B and has an upper bound (supremum) in E. Since the notion of sublattice, ideal, band are invariant under the formation of arbitrary intersections, there exists, for any subset M of E, a uniquely determined smallest sublattice (ideal, band) of E containing M. This will be called the sublattice (ideal, band) generated by M.

Next, we summarize all properties which we will need in the sequel (cf. [21, Proposition 1.1.5, 1.2.3 and 1.2.5]).

Proposition 1.1.5 *If E is a Banach lattice, then the following properties hold.*

(i) If I_1, I_2 are ideals of E, then $I_1 + I_2$ is an ideal and if furthermore I_1 and I_2 are closed, then $I_1 + I_2$ is also a closed ideal.

(ii) The closure of every solid subset of E is solid.

(iii) The closure of every sublattice of E is a sublattice.

(iv) The closure of every ideal of E is an ideal.

(v) Every band in E is closed.

(vi) *For every non-empty subset $A \subset E$, the ideal generated by A is given by*

$$I(A) = \bigcup \{n[-y,y] : n \in \mathbb{N},\, y = |x_1| \vee \ldots \vee |x_r|,\, x_1,\ldots,x_r \in A\}.$$

(vii) *For every $x \in E_+$, the ideal generated by $\{x\}$ is*

$$E_x = \bigcup \{n[-x,x] : n \in \mathbb{N}\}.$$

Example 1.1.6 **1.** *If $E = L^p(\Omega,\mu)$, $1 \le p < \infty$, where μ is σ-finite, then the closed ideals in E are characterized as follows: A subspace I of E is a closed ideal if and only if there exists a measurable subset Y of Ω such that*

$$I = \{\psi \in E : \psi(x) = 0 \text{ a.e. } x \in Y\}.$$

2. *If $E = C_0(X)$, where X is a locally compact topological space, then a subspace J of E is a closed ideal if and only if there is a closed subset A of X such that*

$$J = \{\varphi \in E : \varphi(x) = 0 \text{ for all } x \in A\}.$$

Let E be a Banach lattice. If $E_e = E$ holds for some $e \in E_+$, then e is called an *order unit*. If $\overline{E_e} = E$, then $e \in E_+$ is called a *quasi interior point* of E_+.
It follows that e is an order unit of E if and only if e is an interior point of E_+. Quasi interior points of the positive cone exist, for example, in every separable Banach lattice.

Example 1.1.7 **1.** *If $E = C(K)$, K compact, then the function constant $1\!\mathrm{I}_K$ equal to 1 is an order unit. In fact, for every $f \in E$, there is $n \in \mathbb{N}$ such that $\|f\|_\infty \le n$. Hence, $|f(s)| \le n 1\!\mathrm{I}_K(s)$ for all $s \in K$. This implies $f \in n[-1\!\mathrm{I}_K, 1\!\mathrm{I}_K]$.*

2. *If $E = L^p(\mu)$ with σ-finite measure μ and $1 \le p < \infty$, then the quasi interior points of E_+ coincide with the μ–a.e. strictly positive functions, while E_+ does not contain any interior point.*

• Spaces with order continuous norm

If the norm on E satisfies

$$\|x \vee y\| = \sup(\|x\|, \|y\|) \text{ for } x,y \in E_+$$

then E is called an *AM-space*. The above condition implies that the dual norm satisfies

$$\|x^* + y^*\| = \|x^*\| + \|y^*\| \text{ for } x^*, y^* \in E_+^*.$$

Such spaces are called *AL-spaces*.

Definition 1.1.8 *The norm of a Banach lattice E is called order continuous if every monotone order bounded sequence of E is convergent.*

One can prove the following result (cf. [21, Theorem 2.4.2]).

Proposition 1.1.9 *A Banach lattice E has order continuous norm if and only if every order interval of E is weakly compact.*

As a consequence one obtains the following examples.

Example 1.1.10 *Every reflexive Banach lattice and every L^1-space has order continuous norm.*

The Banach space dual E^* of a Banach lattice E is a Banach lattice with respect to the ordering $\leq$ defined by

$$0 \leq x^* \text{ if and only if } \langle x, x^* \rangle \geq 0 \text{ for all } x \in E_+.$$

A linear form $x^* \in E^*$ is called *strictly positive* if $\langle x, x^* \rangle > 0$ (notation: $x^* > 0$) for all $0 \lneqq x$ (means $0 \leq x$ and $x \neq 0$). The absolute value of $x^* \in E^*$ being given by

$$\langle x, |x^*| \rangle = \sup\{\langle y, x^* \rangle : |y| \leq x\}, \quad x \in E_+.$$

• Hahn-Banach's theorem

The following results are consequences of the Hahn-Banach theorem.

Proposition 1.1.11 *Let E be a Banach lattice . Then $0 \leq x$ is equivalent to $\langle x, x^* \rangle \geq 0$ for all $x^* \in E^*_+$.*

Proposition 1.1.12 *Let E be a Banach lattice. For each $0 \lneqq x \in E$ there exists $x^* \in E^*_+$ such that $\|x^*\| = 1$ and $\langle x, x^* \rangle = \|x\|$.*

Proposition 1.1.13 *In a Banach lattice E every weakly convergent increasing sequence (x_n) is norm-convergent.*

Proof: Let $A := \{\sum_{i=1}^n a_i x_i : n \in \mathbb{N},\, a_i \geq 0,\, a_1 + \ldots + a_n = 1\}$ be the convex hull of $\{x_n : n \in \mathbb{N}\}$. By the Hahn-Banach theorem, the norm-closure of A coincide with the weak closure. This implies that $x \in \overline{A}$, where $x := \text{weak} - \lim_{n\to\infty} x_n$. Thus, for $\varepsilon > 0$ there exist

$$y = a_1 x_1 + \ldots + a_n x_n \in A,\, a_1, \ldots, a_n \geq 0,\, a_1 + \ldots + a_n = 1,$$

such that $\|y - x\| < \varepsilon$. Since $x_k \leq x$, it follows that $\|x - x_k\| \leq \|x - y\| < \varepsilon$ for all $k \geq n$. □

The following lemma will be useful in the proof of Proposition 2.5.3.

Lemma 1.1.14 *Let E be a totally ordered (this means $x \in E \Rightarrow 0 \leq x$ or $x \leq 0$) real Banach lattice. Then $\dim E \leq 1$.*

Proof: Let $e \in E_+$ and $x \in E$. We consider the closed subsets $C_+ := \{\alpha \in \mathbb{R} : \alpha e \geq x\}$ and $C_- := \{\alpha \in \mathbb{R} : \alpha e \leq x\}$ of $\mathbb{R}$. It is obvious that $C_+ \cup C_- = \mathbb{R}$. Since $\mathbb{R}$ is connected, it follows that $C_+ \cap C_- \neq \emptyset$. Hence there is $\alpha \in \mathbb{R}$ such that $x = \alpha e$. □

• **Complexification of real Banach lattices** (cf. [26, II.11])

It is often necessary to consider complex vector spaces (for instance in spectral theory). Therefore, we introduce the concept of a complex Banach lattice.

The complexification of a real Banach lattice E is the complex Banach space $E_{\mathbb{C}}$ whose elements are pairs $(x,y) \in E \times E$, with addition and scalar multiplication defined by $(x_0,y_0)+(x_1,y_1) := (x_0+x_1, y_0+y_1)$ and $(a+ib)(x,y) := (ax-by, ay+bx)$, and norm

$$\|(x,y)\| := \| \sup_{0\leq\theta\leq 2\pi} (x\sin\theta + y\cos\theta)\|.$$

One can show that the above supremum exists in E (cf. [26], p. 134). By identifying $(x,0) \in E_{\mathbb{C}}$ with $x \in E$, E is isometrically isomorphic to a real linear subspace of $E_{\mathbb{C}}$. We write $0 \leq x \in E_{\mathbb{C}}$ if and only if $x \in E_+$.

A complex Banach lattice is an ordered complex Banach space $(E_{\mathbb{C}}, \leq)$ that arises as the complexification of a real Banach lattice E. The underlying real Banach lattice E is called the real part of $E_{\mathbb{C}}$ and is uniquely determined as the closed linear span of all $x \in (E_{\mathbb{C}})_+$.

Instead of the notation (x,y) for elements of $E_{\mathbb{C}}$, we usually write $x+iy$. The complex conjugate of an element $z = x+iy \in E_{\mathbb{C}}$ is the element $\overline{z} = x - iy$. we use also the notation $\Re(z) := x$ for $z = x+iy \in E_{\mathbb{C}}$. The modulus $|\cdot|$ in E extends to $E_{\mathbb{C}}$ by

$$|x+iy| := \sup_{0\leq\theta\leq 2\pi} (x\sin\theta + y\cos\theta).$$

All concepts first introduced for real Banach lattices have a natural extension to complex Banach lattices. A complex Banach lattice has order continuous norm if its real part has.

1.2 Positive operators

This section is concerned with positive operators and their properties. Let E,F be two complex Banach lattices. A linear operator T from E into F is called *positive* (notation: $T \geq 0$) if $TE_+ \subset F_+$, which is equivalent to

$$|Tx| \leq T|x| \quad \text{for all } x \in E.$$

Every positive linear operator $T : E \to F$ is continuous (cf. [21, Proposition 1.3.5]). Furthermore,

$$\|T\| = \sup\{\|Tx\| : x \in E_+, \|x\| \leq 1\}.$$

We denote by $\mathcal{L}(E,F)_+$ the set of all positive linear operators from E into F. For positive operators one can prove the following properties.

Proposition 1.2.1 *Let $T \in \mathcal{L}(E,F)_+$. Then the following properties hold.*

(i) $(Tx)^+ \leq Tx^+$ *and* $(Tx)^- \leq Tx^-$ *for all* $x \in E_{\mathbb{R}}$.

(ii) If $S \in \mathcal{L}(E,F)$ *such that* $0 \leq S \leq T$ *(this means that* $0 \leq Sx \leq Tx$ *for all* $x \in E_+$*), then* $\|S\| \leq \|T\|$.

Let $(A, D(A))$ be a linear operator on a Banach lattice E. It is a *resolvent positive* operator if there is $\omega \in \mathbb{R}$ such that $(\omega, \infty) \subseteq \rho(A)$ and $0 \leq R(\lambda, A)$ for all $\lambda > \omega$. A C_0-semigroup on E is called positive if $0 \leq T(t)$ for all $t \geq 0$. Since

$$R(\lambda, A) = \int_0^\infty e^{\lambda t} T(t)\, dt \text{ for } \lambda > \omega_0(A) \text{ and}$$

$$T(t)x = \lim_{n\to\infty} (\frac{n}{t} R(\frac{n}{t}, A))^n x$$

for all $x \in E$ and $t \geq 0$ (cf. [2, Corollary 3.3.6]), it follows that a C_0-semigroup on a Banach lattice E is positive if and only if its generator is resolvent positive operator.
For resolvent positive operators one has the following result (see [2, Theorem 3.11.8]).

Theorem 1.2.2 *Let E be a Banach lattice with order continuous norm. If A is a resolvent positive operator, then $\overline{D(A)}$ is an ideal in E.*

Proof: Since E is the complexification of a real Banach lattice $E_{\mathbb{R}}$ and $R(\lambda, A)E_{\mathbb{R}} \subseteq E_{\mathbb{R}}$, $\lambda > \omega$, we have $\Re(z) \in \overline{D(A)}$ for $z \in \overline{D(A)}$. Remark that if I is a closed ideal of $E_{\mathbb{R}}$, then $I \oplus iI$ is a closed ideal of E. Therefore we can suppose, without loss of generality, that E is a real Banach lattice. Moreover, we assume $s(A) < 0$, by considering $A - \omega$ instead of A otherwise.

a) Let $0 \leq y \leq R(0,A)x$, $x \in E_+$. We claim that $y \in \overline{D(A)}$. In fact, for $\lambda > 0$ we have

$$0 \leq \lambda R(\lambda, A) y \leq \lambda R(\lambda, A) R(0,A)x = R(0,A)x - R(\lambda, A)x \leq R(0,A)x.$$

From Proposition 1.1.3 it follows that $[0, R(0,A)x]$ is weakly compact. Hence, there is $z \in E$ such that $z = \text{weak} - \lim_{\lambda\to\infty} \lambda R(\lambda, A)y$. In particular, $z \in \overline{D(A)}$ (because $\overline{D(A)} = \overline{D(A)}^{\text{weak}}$). Therefore,

$$\begin{aligned} \text{weak} - \lim_{\lambda\to\infty} (R(0,A)y - R(\lambda,A)y) &= \text{weak} - \lim_{\lambda\to\infty} \lambda R(\lambda,A)R(0,A)y \\ &= R(0,A)z. \end{aligned}$$

Since $0 \leq R(\lambda, A)y \leq \frac{1}{\lambda} R(0,A)x$, we have $R(0,A)y = R(0,A)z$ and hence $y = z$.

b) Let $y \in \overline{D(A)}$. Then there is $(y_n) \subseteq D(A)$ such that $\lim_{n\to\infty} y_n = y$. Moreover, there exists $x_n \in E$ with $y_n = R(0,A)x_n$ and then $0 \leq |y_n| \leq R(0,A)|x_n|$. Now **a)** implies that $|y_n| \in \overline{D(A)}$ and hence $|y| \in \overline{D(A)}$.

c) Let $0 \le y \le x \in \overline{D(A)}$. Let $(x_n) \in D(A)$ with $\lim_{n\to\infty} x_n = x$. From **b)** we have $|x_n| \in \overline{D(A)}$. On the other hand,

$$y \wedge |x_n| \le |x_n| = |R(0,A)Ax_n| \le R(0,A)|Ax_n|$$

and **a)** implies that $y \wedge |x_n| \in \overline{D(A)}$. Hence,

$$y = \lim_{n\to\infty} y \wedge |x_n| \in \overline{D(A)}.$$

□

Positive operators on $C(K)$ with $T1\!\!1_K = 1\!\!1_K$ are contraction operators (cf. [22, B.III. Lemma 2.1]).

Lemma 1.2.3 *Suppose that K is compact and $T : C(K) \to C(K)$ is a linear operator satisfying $T1\!\!1_K = 1\!\!1_K$. Then $0 \le T$ if and only if $\|T\| \le 1$.*

Proof: If $0 \le T$, then

$$|Tf| \le T|f| \le T(\|f\|_\infty 1\!\!1_K) = \|f\|_\infty 1\!\!1_K.$$

Hence $\|T\| \le 1$.
To prove the converse, we first observe that

$$-1\!\!1_K \le f \le 1\!\!1_K \Leftrightarrow \|f - ir1\!\!1_K\|_\infty \le \rho_r := \sqrt{1+r^2} \text{ for all } r \in \mathbb{R}. \tag{1.1}$$

Let $f \in C(K)$ with $0 \le f \le 21\!\!1_K$. Then $-1\!\!1_K \le f - 1\!\!1_K \le 1\!\!1_K$. By (1.1) we have $\|f - 1\!\!1_K - ir1\!\!1_K\|_\infty \le \rho_r$ for all $r \in \mathbb{R}$. Since $T1\!\!1_k = 1\!\!1_K$ and $\|T\| \le 1$, $\|Tf - 1\!\!1_K - ir1\!\!1_K\|_\infty \le \rho_r$ for all $r \in \mathbb{R}$. So by (1.1) we obtain $-1\!\!1_K \le Tf - 1\!\!1_K \le 1\!\!1_K$. This implies $0 \le Tf \le 21\!\!1_K$. □

• Lattice homomorphism and signum operators

Let E, F be two Banach lattices and $T \in \mathcal{L}(E,F)$. It is called *lattice homomorphism* if one of the following equivalent conditions is satisfied (cf. [21, Proposition 1.3.11]).

(a) $T(x \vee y) = Tx \vee Ty$ and $T(x \wedge y) = Tx \wedge Ty$ for all $x, y \in E$.

(b) $|Tx| = T|x|$, $x \in E$.

(c) $Tx^+ \wedge Tx^- = 0$, $x \in E$.

The following result, due to Kakutani, shows that for every $e \in E_+$ the generated ideal satisfies $E_e \cong C(K)$ for some compact K. Here, E_e is equipped with the norm $\|x\|_e := \inf\{\lambda > 0 : x \in \lambda[-e,e]\}$, $x \in E_e$ (cf. [21, Theorem 2.1.3]).

Theorem 1.2.4 *Let $e \in E_+$ and take E_e the ideal generated by e. Let $B := \{x^* \in (E_e)^*_+ : \langle e, x^*\rangle = 1\}$ and $K = \mathrm{ex}(B)$ the set of all extreme points of B. Then K is $\sigma(E^*,E)$-compact and the mapping $U_e : E_e \ni x \mapsto f_x \in C(K)$; $f_x(x^*) = \langle x, x^*\rangle$, $x^* \in K$, is an isometric lattice isomorphism.*

If $|h|$ is a quasi interior point of E_+, then $E_{|h|}$ is a dense subspace of E isomorphic to a space $C(K)$. Consider the lattice isomorphism $U_{|h|}$ from Kakutani's theorem. Let $\widetilde{h} := U_{|h|}h$. Then, $|\widetilde{h}| = U_{|h|}|h| = 1\!\mathrm{I}_K$. Consider the operator

$$\widetilde{S}_0 : C(K) \to C(K);\ f \mapsto (\mathrm{sign}\widetilde{h})f := \frac{\widetilde{h}}{|\widetilde{h}|} f = \widetilde{h} f,$$

and put $S_h := U_{|h|}^{-1}\widetilde{S}_0 U_{|h|}$. Then S_h is a linear mapping from $E_{|h|}$ into itself satisfying

(i) $S_h\overline{h} = |h|$,

(ii) $|S_h x| \leq |x|$ for every $x \in E_{|h|}$,

(iii) $S_h x = 0$ for every $x \in E_{|h|}$ orthogonal to h.

Since (ii) implies the continuity of S_h for the norm induced by E and $\overline{E_{|h|}} = E$, S_h can be uniquely extended to E. This extension will be also denoted by S_h and is called *signum operator* with respect to h.
We now give the following auxiliary result which we need in Section 2.5. See [22, B.III. Lemma 2.3] for a similar result.

Lemma 1.2.5 *Let $T, R \in \mathcal{L}(E)$ and assume that $|h|$ is a quasi interior point of E_+. Suppose we have $Rh = h$, $T|h| = |h|$, and $|Rx| \leq T|x|$ for all $x \in E$. Then $T = S_h^{-1}RS_h$.*

Proof: It follows from $|Rx| \leq T|x|$, $x \in E$, that T is a positive operator. Since $T|h| = |h|$, $E_{|h|}$ is T–and R–invariant. Consider the operators $\widetilde{T} := U_{|h|}TU_{|h|}^{-1}$, $\widetilde{R} := U_{|h|}RU_{|h|}^{-1}$, and put $\widetilde{h} := U_{|h|}h$. We then have

$$\widetilde{R}\widetilde{h} = \widetilde{h},\ \widetilde{T}1\!\mathrm{I}_K = 1\!\mathrm{I}_K,\ |\widetilde{R}f| \leq \widetilde{T}|f| \text{ for all } f \in C(K). \tag{1.2}$$

Define $T_1 := M_{\widetilde{h}}^{-1}\widetilde{R}M_{\widetilde{h}}$, where $M_{\widetilde{h}}$ is the multiplication operator by $\widetilde{h}$ on $C(K)$. By (1.2) we have

$$\begin{aligned} T_1 1\!\mathrm{I}_K &= 1\!\mathrm{I}_K \quad \text{and} \\ |T_1 f| &= |M_{\widetilde{h}}^{-1}\widetilde{R}M_{\widetilde{h}}f| = |\widetilde{R}M_{\widetilde{h}}f| \leq \widetilde{T}|M_{\widetilde{h}}f| = \widetilde{T}|f| \end{aligned} \tag{1.3}$$

for all $f \in C(K)$. Hence $\|T_1\| \leq \|\widetilde{T}\| = \|\widetilde{T}1\!\mathrm{I}_K\|_\infty = 1$. So by Lemma 1.2.3, T_1 is a positive operator and (1.3) implies that $0 \leq T_1 \leq \widetilde{T}$. Therefore, $\|\widetilde{T} - T_1\| = \|(\widetilde{T} - T_1)1\!\mathrm{I}_K\|_\infty = 0$. Since $|\widetilde{h}| = |U_{|h|}h| = U_{|h|}|h| = 1\!\mathrm{I}_K$, it follows that $\widetilde{S}_0 = M_{\widetilde{h}}$. Thus, $S_h = U_{|h|}^{-1}M_{\widetilde{h}}U_{|h|}$ and $T_1 = \widetilde{T}$ implies that $T = S_h^{-1}RS_h$. □