CHAPTER 2

HEAT EQUATIONS IN HILBERT SPACES

In this chapter, H is a separable Hilbert space and $(e_n)_{n\in\mathbb{N}}$ is an orthonormal basis of H.
For $\varphi \in C_b(H)$, the space of continuous and bounded functions $\varphi : H \to \mathbb{R}$, we say that φ is differentiable in the direction $e_k,\ k \in \mathbb{N}$, if the limit

$$D_k\varphi(x) := \lim_{h\to 0} \frac{1}{h}\left(\varphi(x + he_k) - \varphi(x)\right), \qquad x \in H$$

exists in $C_b(H)$. The operator D_k will be considered as the linear operator in $C_b(H)$ defined by

$$D(D_k) := \left\{\varphi \in C_b(H) : \lim_{h\to 0} \frac{1}{h}\left(\varphi(\cdot + he_k) - \varphi(\cdot)\right) \text{ exists in } C_b(H)\right\}$$

and

$$D_k\varphi(x) = \lim_{h\to 0} \frac{1}{h}\left(\varphi(x + he_k) - \varphi(x)\right), \quad \varphi \in D(D_k),\ x \in H,\ h \in \mathbb{R}.$$

We start by showing that D_k is a closed operator on $C_b(H)$, for every $k \in \mathbb{N}$.
In fact, let $(\varphi_n)_{n\in\mathbb{N}} \subseteq D(D_k)$, and $\varphi, \psi \in C_b(H)$ such that

$$\varphi_n \longrightarrow \varphi \quad \text{and} \quad D_k\varphi_n \longrightarrow \psi \text{ in } C_b(H).$$

We consider $\phi_n, \phi \in C(C[-1,1], C_b(H))$ defined by

$$\phi(h)(x) := \varphi(x + he_k) \quad \text{and} \quad \phi_n(h)(x) := \varphi_n(x + he_k),$$

$$x \in H,\ h \in [-1,1] \text{ and } n \in \mathbb{N}.$$

Then ϕ_n is differentiable, as a function of the variable h, and

$$\frac{d}{dh}\phi_n(h)(x) = D_k\varphi_n(x+he_k).$$

So we have

$$\phi_n(h)-\phi_n(0) = \int_0^h \frac{d\phi_n}{dh}(s)\,ds$$

and by the assumption we obtain

$$\phi(h)-\phi(0) = \int_0^h \psi(\cdot + se_k)\,ds,$$

which implies that $\varphi \in D(D_k)$ and $D_k\varphi = \psi$.
In a similar way we can define partial derivatives of any order.
Now, we fix a sequence $(\lambda_n)_{n\in\mathbb{N}},\ \lambda_n > 0$ for $n \in \mathbb{N}$. In this chapter we are interested to solve the heat equation

$$(HE)\quad \begin{cases} \frac{\partial}{\partial t}u(t,x) = \frac{1}{2}\sum_{n=1}^{\infty}\lambda_n D_n^2 u(t,x), & t>0,\ x\in H,\\ u(0,x)=\varphi(x), & x\in H,\ \varphi\in C_b(H) \end{cases}$$

and to study the regularity of the solution u of (HE) in the case $\dim H = \infty$.
For this purpose, let consider its finite dimensional approximation

$$(HE)_n\ \begin{cases} \frac{\partial}{\partial t}u(t,x) = \frac{1}{2}\sum_{k=1}^{n}\lambda_k D_k^2 u(t,x), & t>0,\ x\in H,\\ u(0,x)=\varphi(x), & x\in H,\ \varphi\in C_b(H). \end{cases}$$

It is easy to see that, for all $\varphi \in C_b(H)$, $(HE)_n$ has a unique classical solution given by

$$\begin{cases} u_n(t,x) = (2\pi t)^{-\frac{n}{2}}(\lambda_1\ldots\lambda_n)^{-\frac{1}{2}}\int_{\mathbb{R}^n} e^{-\sum_{k=1}^{n}\frac{\xi_k^2}{2t\lambda_k}}\varphi(x-\sum_{k=1}^{n}\xi_k e_k)\,d\xi, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } t>0\\ u_n(0,x)=\varphi(x), \quad x\in H. \end{cases}$$

If we denote by

$$x_k := < x, e_k >,\ x\in H$$

and

$$B_n := \begin{pmatrix} \lambda_1\, 0 & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & \lambda_n \end{pmatrix}$$

then

$$u_n(t,x) = \int_{\mathbb{R}^n}\varphi\left(y+\sum_{k=n+1}^{\infty}x_k e_k\right)\mathcal{N}(x,tB_n)(dy),\ \ x\in H,\ t>0.$$

In the sequel we denote by

$$P_t^{(n)}\varphi(x) := u_n(t,x)$$

for $t \geq 0,\ x \in H,\ n \in \mathbb{N}$, and $\varphi \in C_b(H)$. By an easy computation one has, for all $n \in \mathbb{N}$, $(P_t^{(n)})_{t\geq 0}$ is a semigroup on $C_b(H)$. Moreover, on $C_b(H)$, $(P_t^{(n)})$ is not strongly continuous at 0. In order to have strong continuity at 0 we have to work, for example, in $BUC(H)$, the space of all bounded and uniformly continuous functions from H into $\mathbb{R}$. Now, it is well-known that $(P_t^{(n)})$ is an analytic semigroup on $BUC(H)$ and

$$\|P_t^{(n)}\varphi\|_\infty \leq \|\varphi\|_\infty$$

for $\varphi \in BUC(H),\ t \geq 0$, and $n \in \mathbb{N}$.
Now, one asks under which conditions the limit

$$\lim_{n\to\infty} u_n(t,x) \quad \text{exists in } BUC(H)$$

for all $\varphi \in BUC(H)$?
A necessary condition for the existence of the above limit is

$$\sum_{n=1}^{\infty} \lambda_n < \infty.$$

In fact, let $\varphi(x) :=$exp$\big(-\frac{1}{2}\|x\|^2\big)$. By applying Proposition 1.2.8 with $\alpha = -1,\ m = x$, and $B = tB_n$ one has

$$u_n(t,x) = \prod_{k=1}^{n}(1+\lambda_k t)^{-\frac{1}{2}}\exp\left(-\frac{1}{2}\sum_{k=1}^{n}\frac{x_k^2}{1+\lambda_k t} - \frac{1}{2}\sum_{k=n+1}^{\infty} x_k^2\right).$$

If $\lim_{n\to\infty} u_n(t,x)$ exists, then $\prod_{k=1}^{\infty}(1+t\lambda_k)^{-\frac{1}{2}}$ exists for $t > 0$. Hence,

$$\log\prod_{k=1}^{\infty}(1+t\lambda_k) = \sum_{k=1}^{\infty}\log(1+t\lambda_k),\ t > 0$$

exists. In particular, $\lim_{k\to\infty}\lambda_k = 0$. Set $M := \sup_n \lambda_n$. Then we have

$$mt\lambda_k \leq \log(1+t\lambda_k) \leq t\lambda_k,\ t > 0,\ k \in \mathbb{N},$$

where $m := \inf\{\frac{1}{\alpha}\log(1+\alpha),\ 0 < \alpha \leq M\}$. Therefore,

$$\sum_{k=1}^{\infty}\lambda_k < \infty$$

and

$$\lim_{n\to\infty} u_n(t,x) = u(t,x) = \prod_{k=1}^{\infty}(1+\lambda_k t)^{-\frac{1}{2}}e^{-\frac{1}{2}\sum_{k=1}^{\infty}\frac{x_k^2}{1+t\lambda_k}}, \quad t > 0,\ x \in H.$$

If $\sum_{k=1}^{\infty} \lambda_k = \infty$, then

$$\lim_{n\to\infty} u_n(t,x) = \begin{cases} 0 & \text{if } x=0,\ t\neq 0 \\ 1 & \text{if } x=0,\ t=0. \end{cases}$$

Hence, u_n does not converge to a continuous function.
Now, in the sequel we assume that $\sum_{k=1}^{\infty} \lambda_k < \infty$. Set $Bx := \sum_{k=1}^{\infty} \lambda_k x_k,\ x \in H$. Then $B \in \mathcal{L}_1^+(H)$, $\ker B = \{0\}$, and Equation (HE) can be written as follows:

$$(HE) \quad \begin{cases} \frac{\partial}{\partial t} u(t,x) = \frac{1}{2}\mathrm{Tr}[BD^2 u(t,x)], & t>0,\ x\in H, \\ u(0,x) = \varphi(x), & x \in H, \end{cases}$$

where $\varphi \in BUC(H)$.

Many results of this chapter can be found in the monographs [12] and [13].

2.1 Construction of the heat semigroup

In this section we are concerned with the construction of the solution of Equation (HE). To this purpose we suppose without loss of generality that $\lambda_k > 0$ for all $k \in \mathbb{N}$ and $\sum_{k=1}^{\infty} \lambda_k < \infty$. The semigroup $(P_t^{(n)})$ can be written as

$$P_t^{(n)}\varphi = \prod_{k=1}^{n} T_k(t)\varphi, \quad t \geq 0,\ \varphi \in BUC(H),$$

where

$$T_k(t)\varphi(x) := \begin{cases} (2\pi t\lambda_k)^{-\frac{1}{2}} \int_{\mathbb{R}} e^{-\frac{s^2}{2t\lambda_k}} \varphi(x - se_k)\, ds & \text{if } t>0 \\ \varphi(x), & \text{if } t=0 \end{cases}$$

for $x \in H$ and $\varphi \in BUC(H)$. Note that $T_k(\cdot)$ is a C_0-semigroup of contractions on $BUC(H)$ for $k \in \mathbb{N}$. Before proving the strong convergence of P_t^n, $t \geq 0$, on $BUC(H)$, we recall some definitions and fix some notations.

We denote by $BUC^1(H)$ the subspace of $BUC(H)$ of all functions $\varphi : H \to \mathbb{R}$ which are Fréchet differentiable on H and the Fréchet derivative $D\varphi : H \to H$ is uniformly continuous and bounded. For $\varphi \in BUC^1(H)$ we set

$$\|\varphi\|_1 := \|\varphi\|_\infty + \sup_{x\in H} \|D\varphi(x)\|.$$

In the sequel we need the subspace $BUC^{1,1}(H)$ of $BUC^1(H)$ consisting of all functions $\varphi \in BUC^1(H)$ such that $D\varphi : H \to H$ is Lipschitz continuous and, for $\varphi \in BUC^{1,1}(H)$, we set

$$\|\varphi\|_{1,1} := \|\varphi\|_1 + \sup_{x,y\in H, x\neq y} \frac{\|D\varphi(x) - D\varphi(y)\|}{\|x-y\|}.$$

Theorem 2.1.1 *For all $\varphi \in BUC(H)$, the limit*

$$P_t\varphi := \lim_{n\to\infty} P_t^n\varphi$$

exists in $BUC(H)$, uniformly in t on bounded subsets of $\mathbb{R}^+$. Moreover (P_t) is a C_0-semigroup on $BUC(H)$ and

$$\|P_t\varphi\|_\infty \le \|\varphi\|_\infty$$

for $t \ge 0$ and $\varphi \in BUC(H)$.

Proof: Let compute first

$$\begin{aligned} P_t^n\varphi - P_t^{n-1}\varphi &= \prod_{k=1}^{n} T_k(t)\varphi - \prod_{k=1}^{n-1} T_k(t)\varphi \\ &= \prod_{k=1}^{n-1} T_k(t)(T_n(t)\varphi - \varphi), \end{aligned}$$

and hence,

$$\|P_t^n\varphi - P_t^{n-1}\varphi\|_\infty \le \|T_n(t)\varphi - \varphi\|_\infty, \quad t \ge 0,\ \varphi \in BUC(H),\ n \in \mathbb{N}.$$

So, for $\varphi \in BUC^{1,1}(H)$, we have

$$\begin{aligned} (T_n(t)\varphi - \varphi)(x) &= (2\pi\lambda_n t)^{-\frac{1}{2}} \int_{\mathbb{R}} e^{-\frac{s^2}{2\lambda_n t}} \left(\varphi(x - se_n) - \varphi(x)\right) ds \\ &= (2\pi\lambda_n t)^{-\frac{1}{2}} \int_{\mathbb{R}} e^{-\frac{s^2}{2\lambda_n t}} \int_0^1 -\frac{\partial}{\partial\gamma}\varphi\left(x - s(1-\gamma)e_n\right) d\gamma\, ds \\ &= -(2\pi\lambda_n t)^{-\frac{1}{2}} \int_{\mathbb{R}} e^{-\frac{s^2}{2\lambda_n t}} \int_0^1 < D\varphi\left(x - s(1-\gamma)e_n\right), \\ &\quad se_n > d\gamma\, ds. \end{aligned}$$

Since,

$$\int_{\mathbb{R}} e^{-\frac{s^2}{2\lambda_n t}} < D\varphi(x), se_n >\, ds =< D\varphi(x), e_n > \int_{\mathbb{R}} e^{-\frac{s^2}{2\lambda_n t}} s\, ds = 0,$$

it follows that

$$T_n(t)\varphi(x) - \varphi(x) = -(2\pi\lambda_n t)^{-\frac{1}{2}} \int_{\mathbb{R}} e^{-\frac{s^2}{2\lambda_n t}} \int_0^1 < D\varphi(x - s(1-\gamma)e_n) - D\varphi(x),$$

$se_n > d\gamma\, ds.$
Thus,

$$|T_n(t)\varphi(x) - \varphi(x)| \le (2\pi\lambda_n t)^{\frac{1}{2}} \|\varphi\|_{1,1} \int_{\mathbb{R}} s^2 e^{-\frac{s^2}{2\lambda_n t}} ds = \lambda_n t \|\varphi\|_{1,1}.$$

Hence,

$$\|T_n(t)\varphi - \varphi\|_\infty \leq \lambda_n t \|\varphi\|_{1,1}$$

for $t \geq 0,\ \varphi \in BUC^{1,1}(H)$, and $n \in \mathbb{N}$. Therefore,

$$\begin{aligned}
& \|P_t^{n+p}\varphi - P_t^n\varphi\|_\infty \\
\leq\ & \|\prod_{k=1}^{n+p} T_k(t)\varphi - \prod_{k=1}^{n+p-1} T_k(t)\varphi\|_\infty + \cdots + \|\prod_{k=1}^{n+1} T_k(t)\varphi - \prod_{k=1}^{n+1} T_k(t)\varphi\|_\infty \\
\leq\ & \|T_{n+p}(t)\varphi - \varphi\|_\infty + \cdots + \|T_{n+1}(t)\varphi - \varphi\|_\infty \\
\leq\ & t\|\varphi\|_{1,1} \sum_{k=n+1}^{n+p} \lambda_k, \quad n, p \in \mathbb{N}.
\end{aligned}$$

Since $\sum_{n=1}^\infty \lambda_n < \infty$, it follows that $(P_t^n\varphi)_n$ is a Cauchy sequence in $BUC(H)$, uniformly for t in bounded subsets of $\mathbb{R}_+$. Thus, the limit exists in $BUC(H)$ for all $\varphi \in BUC^{1,1}(H)$. Since $BUC^{1,1}(H)$ is dense in $BUC(H)$ (see [28] or [23]) and $\|P_t^n\| \leq 1$ for all $n \in \mathbb{N}$ and $t \geq 0$, the limit exists for all $\varphi \in BUC(H)$ and will be denoted by

$$P_t\varphi := \lim_{n\to\infty} P_t^n\varphi, \qquad t \geq 0,\ \varphi \in BUC(H).$$

The family $(P_t)_{t\geq 0}$ satisfies $P_{t+s}\varphi = P_tP_s\varphi,\ P_0\varphi = \varphi$ for all $t, s \geq 0$. This follows from the estimates $\|P_t^n\| \leq 1$ and the fact that (P_t^n) is a semigroup on $BUC(H)$. The strong continuity of $(P_t)_{t\geq 0}$ follows from the uniform convergence of P_t^n on bounded subsets of $\mathbb{R}_+$, and the strong continuity of $(P_t^n)_{t\geq 0}$ for every $n \in \mathbb{N}$. □

Remark 2.1.2 *An other proof of Theorem 2.1.1, using the Mittag-Leffler theorem, can be found in [2]. In this work the authors find conditions implying the convergence of the infinite product of commuting C_0-semigroups.*

Let show now that the semigroup $(P_t)_{t\geq 0}$ is given by a Gaussian measure.

Theorem 2.1.3 *If we denote by $\mu := \mathcal{N}(x, tB)$ the Gaussian measure with means $x \in H$ and covariance operator tB, then*

$$(P_t\varphi)(x) = \int_H \varphi(y)\mathcal{N}(x, tB)(dy)$$

for $\varphi \in BUC(H)$, and $t > 0$, where $B = \mathrm{diag}(\lambda_1, \ldots, \lambda_n, \ldots)$.

Proof: For $n \in \mathbb{N}$, $\varphi \in BUC(H)$, $t > 0$, and $x \in B^{\frac{1}{2}}(H)$, it follows from the Cameron-Martin formula (see Corollary 1.3.5) that

$$\begin{aligned}
&\int_H \varphi\left(\sum_{k=1}^n y_k e_k + \sum_{k=n+1}^\infty x_k e_k\right) \mathcal{N}(x,tB)(dy)\\
= &\int_{\mathbb{R}^n} \varphi\left(\sum_{k=1}^n y_k e_k + \sum_{k=n+1}^\infty x_k e_k\right) \cdot\\
&\qquad \exp\left(-\frac{1}{2t}|B^{-\frac{1}{2}}x|^2 + \frac{1}{t}\langle B_n^{-\frac{1}{2}}y, B_n^{-\frac{1}{2}}x\rangle\right) \mathcal{N}(0,tB_n)(dy)\\
= &\int_{\mathbb{R}^n} \varphi\left(\sum_{k=1}^n y_k e_k + \sum_{k=n+1}^\infty x_k e_k\right)\\
&\qquad \exp\left(-\frac{1}{2t}\left(|B^{-\frac{1}{2}}x|^2 - |B_n^{-\frac{1}{2}}x|^2\right)\right) \mathcal{N}(x,tB_n)(dy)\\
= &\exp\left(-\frac{1}{2t}(|B^{-\frac{1}{2}}x|^2 - |B_n^{-\frac{1}{2}}x|^2)\right) (P_t^n\varphi)(x).
\end{aligned}$$

So it follows from Theorem 2.1.1 that

$$\lim_{n\to\infty} (P_t^n\varphi)(x) \exp\left(-\frac{1}{2}(|B^{-\frac{1}{2}}x|^2 - |B_n^{-\frac{1}{2}}x|^2)\right) = (P_t\varphi)(x).$$

So by the dominated convergence theorem and Lemma 1.2.7 we obtain

$$\begin{aligned}
(P_t\varphi)(x) &= \int_H \varphi(y)\mathcal{N}(x,tB)(dy)\\
&= \int_H \varphi(y+x)\mathcal{N}(0,tB)(dy), \quad x \in B^{\frac{1}{2}}(H).
\end{aligned}$$

Since $\overline{B^{\frac{1}{2}}(H)} = H$ (see Remark 1.3.2), it follows that

$$(P_t\varphi)(x) = \int_H \varphi(y+x)\mathcal{N}(0,tB)(dy), \quad x \in H,$$

and the theorem follows now from Lemma 1.2.7. □

2.2 Regularity of the heat semigroup

Let prove first the differentiability of $P_t\varphi$ in any direction e_k, $k \in \mathbb{N}$, for $t > 0$ and $\varphi \in BUC(H)$.

Proposition 2.2.1 *Let $\varphi \in BUC(H)$ and $t > 0$. Then $P_t\varphi \in D(D_k)$ for all $k \in \mathbb{N}$ and*

$$D_k P_t\varphi(x) = \frac{1}{\lambda_k t}\int_H y_k \varphi(x+y)\mathcal{N}(0,tB)(dy), \qquad x \in H.$$

Proof: By the Cameron-Martin formula (see Corollary 1.3.5) we know that

$$P_t\varphi(x) = \int_H \varphi(y)\exp\left(-\frac{1}{2t}|B^{-\frac{1}{2}}x|^2 + \frac{1}{t} < B^{-\frac{1}{2}}y, B^{-\frac{1}{2}}x >\right)\mathcal{N}(0,tB)(dy)$$

for $t > 0,\ x \in H$ and $\varphi \in BUC(H)$.
It is now easy to see that $P_t\varphi$ is differentiable in the direction e_k and by Lemma 1.2.7 we obtain

$$\begin{aligned} D_kP_t\varphi(x) &= \frac{1}{t\lambda_k}\int_H (y_k - x_k)\varphi(y)\mathcal{N}(x,tB)(dy) \\ &= \frac{1}{t\lambda_k}\int_H y_k\varphi(x+y)\mathcal{N}(0,tB)(dy). \end{aligned}$$

□

By applying the Cameron-Martin formula to the derivatives $D_kP_t\varphi$ obtained in Proposition 2.2.1 one obtains by similar arguments the following result.

Proposition 2.2.2 *For $\varphi \in BUC(H)$ and $t > 0$ we have $P_t\varphi \in D(D_lD_k)$ for all $l, k \in \mathbb{N}$, and*

$$D_lD_kP_t\varphi(x) = \frac{1}{\lambda_l\lambda_k t^2}\int_H y_l y_k\varphi(x+y)\mathcal{N}(0,tB)(dy) - \frac{\delta_{l,k}}{\lambda_l t}P_t\varphi(x), \quad x \in H,$$

where $\delta_{l,k} := \begin{cases} 1 & \text{if } l = k, \\ 0 & \text{if } l \neq k. \end{cases}$

Now, we are interested in global regularity properties of the semigroup (P_t) on $BUC(H)$. To this purpose we define two subspaces $BUC^1_B(H)$ and $BUC^2_B(H)$ of $BUC(H)$.

Definition 2.2.3 *We said that a function $\varphi \in BUC(H)$ is in $BUC^1_B(H)$ if*

(i) $\varphi \in \bigcap_{k=1}^\infty D(D_k)$;

(ii) $\sup_{x\in H}\sum_{k=1}^\infty \lambda_k|D_k\varphi(x)|^2 < \infty$;

(iii) *the mapping $D_B\varphi : H \to H;\ x \mapsto \sum_{k=1}^\infty \sqrt{\lambda_k}D_k\varphi(x)e_k$ is uniformly continuous.*

It is clear that $BUC^1(H) \subseteq BUC^1_B(H)$ and $D_B\varphi(x) = B^{\frac{1}{2}}D\varphi(x)$ for $x \in H$, and $\varphi \in BUC^1(H)$.

Definition 2.2.4 *A function $\varphi \in BUC(H)$ is in $BUC^2_B(H)$ if*

(i) $\varphi \in \bigcap_{l,k=1}^\infty D(D_lD_k)$;

(ii) $\sup_{x\in H}\sum_{l=1}^\infty \left(\sum_{k=1}^\infty \sqrt{\lambda_l\lambda_k}D_lD_k\varphi(x)y_k\right)^2 \leq C^2|y|^2$ *for all $y \in H$ and some constant $C > 0$;*

(iii) *the mapping* $D_B^2\varphi$ *defined by* $D_B^2\varphi(x) : H \to \mathcal{L}(H);\ x \mapsto D_B^2\varphi(x)$, *where*

$$\langle D_B^2\varphi(x)y, z\rangle := \sum_{l,k=1}^{\infty} \sqrt{\lambda_l\lambda_k} D_l D_k\varphi(x) y_l z_k, \quad y, z \in H,$$

is uniformly continuous.

We propose now to show some auxiliary results.

Lemma 2.2.5 *The linear operator*

$$D_B : BUC_B^1(H) \to BUC(H, H)$$

is closed.

Proof: Let $(\varphi_n) \subset BUC_B^1(H)$, $\varphi \in BUC(H)$, and $F \in BUC(H, H)$ are such that

$$\lim_{n\to\infty} \|\varphi_n - \varphi\|_\infty = 0, \text{ and } \lim_{n\to\infty} \|D_B\varphi - F\|_{BUC(H,H)} = 0.$$

For any $k \in \mathbb{N}$, we have

$$\begin{aligned}&\lim_{n\to\infty}\sup_{x\in H} |\langle D_B\varphi_n(x) - F(x), e_k\rangle| = \\ &= \lim_{n\to\infty}\sup_{x\in H} \left|\sqrt{\lambda_k} D_k\varphi_n(x) - \langle F(x), e_k\rangle\right| = 0.\end{aligned}$$

Thus,

$$\lim_{n\to\infty}\sup_{x\in H} \left|D_k\varphi(x) - \frac{1}{\sqrt{\lambda_k}}\langle F(x), e_k\rangle\right| = 0.$$

Since D_k is closed in $BUC(H)$, it follows that $\varphi \in D(D_k)$ and

$$D_k\varphi(x) = \frac{1}{\sqrt{\lambda_k}}\langle F(x), e_k\rangle, \quad k \in \mathbb{N}.$$

Hence,

$$\begin{aligned}\sum_{k=1}^{\infty} \lambda_k |D_k\varphi(x)|^2 &= \sum_{k=1}^{\infty} |\langle F(x), e_k\rangle|^2 \\ &= |F(x)|^2 \le \|F\|_\infty^2.\end{aligned}$$

Moreover,

$$\sum_{k=1}^{\infty} \sqrt{\lambda_k} D_k\varphi(x) e_k = \sum_{k=1}^{\infty} \langle F(x), e_k\rangle e_k = F(x)$$

is uniformly continuous. Therefore, $\varphi \in BUC_B^1(H)$ and $D_B\varphi = F$. □

Lemma 2.2.6 *For $\varphi \in \bigcap_{l,k=1}^{\infty} D(D_l D_k)$ and $x \in H$, we define $D^2_{B_n}\varphi(x)$ by*

$$\langle D^2_{B_n}\varphi(x)y, z\rangle = \sum_{l,k=1}^{n} \sqrt{\lambda_l \lambda_k} D_l D_k \varphi(x) y_l z_k, \quad y, z \in H.$$

Assume that

(i) *there is a constant $c > 0$ such that*

$$\left|\langle D^2_{B_n}\varphi(x)y, z\rangle\right| \leq c|y||z|, \quad \forall x, y, z \in H,\, n \in \mathbb{N};$$

(ii) *for all $y, z \in H$, the limit*

$$\lim_{n\to\infty} \langle D^2_{B_n}\varphi(x)y, z\rangle \text{ exists uniformly in } x \in H.$$

Then, $\varphi \in BUC^2_B(H)$ and

$$\lim_{n\to\infty} \sup_{x\in H} \left|\langle D^2_{B_n}\varphi(x)y, z\rangle - \langle D^2_B\varphi(x)y, z\rangle\right| = 0, \quad y, z \in H.$$

Proof: From the assumptions we have

(i) $\varphi \in \bigcap_{l,k=1}^{\infty} D(D_l D_k)$;

(ii) $\sup_{x\in H} \left|\sum_{l=1}^n \left(\sum_{k=1}^n \sqrt{\lambda_l\lambda_k} D_l D_k \varphi(x) y_k\right) z_l\right| \leq c|y||z|$ for all $n \in \mathbb{N}$ and $y, z \in H$. Thus,

$$\sup_{x\in H} \sum_{l=1}^{n} \left(\sum_{k=1}^{n} \sqrt{\lambda_l\lambda_k} D_l D_k \varphi(x) y_k\right)^2 \leq c^2|y|^2, \quad \forall n \in \mathbb{N}.$$

(iii) Since the limit $\lim_{n\to\infty}\langle D^2_{B_n}\varphi(x)y, z\rangle$ exists uniformly in $x \in H$, for all $y, z \in H$, it follows that the mapping

$$D^2_B\varphi : H \to \mathcal{L}(H);\; x \mapsto D^2_B\varphi(x)$$

is uniformly continuous.

Thus, $\varphi \in BUC^2_B(H)$. The last assertion follows easily from the definition of $D^2_{B_n}\varphi$. □

We are now able to show global regularity results for the heat semigroup (P_t).

Theorem 2.2.7 *Let $\varphi \in BUC(H)$ and $t > 0$. Then $P_t\varphi \in BUC^1_B(H)$ and*

$$\langle D_B P_t\varphi(x), z\rangle = \frac{1}{t}\int_H \langle z, B^{-\frac{1}{2}}y\rangle \varphi(x+y)\mathcal{N}(0, tB)(dy), \quad x, z \in H.$$

Moreover,

$$\|D_B P_t\varphi(x)\| \leq \frac{1}{\sqrt{t}}\|\varphi\|_\infty, \quad \forall x \in H.$$

Proof: By Proposition 2.2.1 we have, $P_t\varphi \in D(D_k)$ for all $k \in \mathbb{N}$, and

$$\sum_{k=1}^{n} \sqrt{\lambda_k} D_k P_t\varphi(x) z_k = \sum_{k=1}^{n} \frac{1}{t\sqrt{\lambda_k}} \int_H y_k z_k \varphi(x+y) \mathcal{N}(0, tB)(dy).$$

So by the Hölder inequality we obtain

$$\begin{aligned}
\left|\sum_{k=1}^{n} \sqrt{\lambda_k} D_k P_t\varphi(x) z_k\right|^2 &\leq \frac{\|\varphi\|_\infty^2}{t^2} \int_H \left(\sum_{k=1}^{n} \frac{y_k z_k}{\sqrt{\lambda_k}}\right)^2 \mathcal{N}(0, tB)(dy) \\
&= \frac{\|\varphi\|_\infty^2}{t^2} \sum_{l,k=1}^{n} \frac{z_l z_k}{\sqrt{\lambda_l \lambda_k}} \int_H y_l y_k \mathcal{N}(0, tB)(dy) \\
&= \frac{\|\varphi\|_\infty^2}{t^2} \sum_{k=1}^{n} \frac{z_k^2}{\lambda_k} \int_H y_k^2 \mathcal{N}(0, tB)(dy) \\
&= \frac{\|\varphi\|_\infty^2}{t^2} \sum_{k=1}^{n} \frac{z_k^2}{\lambda_k} \int_{\mathbb{R}} y_k^2 \mathcal{N}(0, t\lambda_k)(dy_k) \\
&= \frac{\|\varphi\|_\infty^2}{t} \sum_{k=1}^{n} z_k^2.
\end{aligned}$$

Hence,

$$\sum_{k=1}^{n} \lambda_k |D_k P_t\varphi(x)|^2 \leq \frac{\|\varphi\|_\infty^2}{t}, \quad \forall n \in \mathbb{N}.$$

It remains to prove that the mapping

$$D_B P_t\varphi : x \mapsto \sum_{k=1}^{\infty} \sqrt{\lambda_k} D_k P_t\varphi(x) e_k$$

is uniformly continuous. First, we note that, by the last estimate, the series

$$D_B P_t\varphi(x) = \sum_{k=1}^{\infty} \sqrt{\lambda_k} D_k P_t\varphi(x) e_k$$

converges and we have

$$\langle D_B P_t\varphi(x), z\rangle = \frac{1}{t} \int_H \langle z, B^{-\frac{1}{2}} y\rangle \varphi(x+y) \mathcal{N}(0, tB)(dy), \quad z \in H.$$

Now, we introduce the uniform continuity modulus of $\varphi \in BUC(H)$,

$$\omega_\varphi(t) := \sup\{|\varphi(x) - \varphi(y)| : x, y \in H,\, |x-y| \leq t\}, \quad t \geq 0.$$

Since φ is uniformly continuous, it is easy to see that ω_φ is continuous in $[0, \infty)$. Let $x, y \in H$. By Hölder's inequality and Proposition 1.3.1, we obtain

$$\begin{aligned}
& |\langle D_B P_t\varphi(x) - D_B P_t\varphi(y), z\rangle|^2 \\
= & \left| \frac{1}{t}\int_H \langle z, B^{-\frac{1}{2}}\alpha\rangle(\varphi(x+\alpha) - \varphi(y+\alpha))\mathcal{N}(0,tB)(d\alpha)\right|^2 \\
\le & \frac{\omega_\varphi(|x-y|)^2}{t^2}\int_H |\langle z, B^{-\frac{1}{2}}\alpha\rangle|^2\mathcal{N}(0,tB)(d\alpha) \\
= & \frac{\omega_\varphi(|x-y|)^2}{t}|z|^2.
\end{aligned}$$

Hence,

$$\|D_B P_t\varphi(x) - D_B P_t\varphi(y)\| \le \frac{1}{\sqrt{t}}\omega_\varphi(|x-y|).$$

Then, $P_t\varphi \in BUC^1_B(H)$ for all $\varphi \in BUC(H)$ and $t > 0$. Moreover, by the same computation as above, we obtain

$$\|D_B P_t\varphi(x)\| \le \frac{1}{\sqrt{t}}\|\varphi\|_\infty$$

for all $\varphi \in BUC(H)$, $t > 0$, and $x \in H$. □

More global regularity is given by the following theorem.

Theorem 2.2.8 *For $\varphi \in BUC(H)$ and $t > 0$, we have $P_t\varphi \in BUC^2_B(H)$ and*

$$\begin{aligned}
\langle D^2_B P_t\varphi(x)z_1, z_2\rangle &= \frac{1}{t^2}\int_H \langle z_1, B^{-\frac{1}{2}}y\rangle\langle z_2, B^{-\frac{1}{2}}y\rangle\varphi(x+y)\mathcal{N}(0,tB)(dy) \\
&\quad -\frac{1}{t}\langle z_1, z_2\rangle P_t\varphi(x)
\end{aligned}$$

for $z_1, z_2, x \in H$. If in addition $\varphi \in BUC^1_B(H)$, then

$$\langle D^2_B P_t\varphi(x)z_1, z_2\rangle = \frac{1}{t}\int_H \langle D_B\varphi(x+y), z_2\rangle\langle z_1, B^{-\frac{1}{2}}y\rangle\mathcal{N}(0,tB)(dy)$$

for $x, z_1, z_2 \in H$. Moreover, for all $x \in H$,

$$\begin{aligned}
\|D^2_B P_t\varphi(x)\|_{\mathcal{L}(H)} &\le \frac{\sqrt{2}}{t}\|\varphi\|_\infty \quad \text{for } \varphi \in BUC(H), \qquad (2.1)\\
\|D^2_B P_t\varphi(x)\|_{\mathcal{L}(H)} &\le \frac{1}{\sqrt{t}}\|D_B\varphi\|_{BUC(H,H)} \quad \text{for } \varphi \in BUC^1_B(H).
\end{aligned}$$

Proof: From Proposition 2.2.2 it follows that

$$\begin{aligned}
\langle D^2_{B_n} P_t\varphi(x)z_1, z_2\rangle &= \frac{1}{t^2}\int_H \langle z_1, B_n^{-\frac{1}{2}}y\rangle\langle z_2, B_n^{-\frac{1}{2}}y\rangle\varphi(x+y)\mathcal{N}(0,tB)(dy) \\
&\quad -\frac{1}{t}\langle z_1, z_2\rangle P_t\varphi(x), \quad z_1, z_2, x \in H.
\end{aligned}$$

It is easy to see that all the assumptions of Lemma 2.2.6 are satisfied. Thus, $P_t\varphi \in BUC_B^2(H)$ and

$$\begin{aligned}\langle D_B^2 P_t\varphi(x)z_1, z_2\rangle &= \frac{1}{t^2}\int_H \langle z_1, B^{-\frac{1}{2}}y\rangle\langle z_2, B^{-\frac{1}{2}}y\rangle\varphi(x+y)\mathcal{N}(0,tB)(dy)\\ &\quad -\frac{1}{t}\langle z_1, z_2\rangle P_t\varphi(x), \quad z_1, z_2, x \in H.\end{aligned}$$

Hence, by Hölder's inequality and Theorem 2.1.3, we obtain

$$\begin{aligned}&|\langle D_B^2 P_t\varphi(x)z, z\rangle|^2 =\\ &= \left|\frac{1}{t^2}\int_H |\langle z, B^{-\frac{1}{2}}y\rangle|^2\varphi(x+y)\mathcal{N}(0,tB)(dy) - \frac{1}{t}|z|^2 P_t\varphi(x)\right|^2\\ &= \frac{1}{t^4}\left|\int_H \left(|\langle z, B^{-\frac{1}{2}}y\rangle|^2 - t|z|^2\right)\varphi(x+y)\mathcal{N}(0,tB)(dy)\right|^2\\ &\le \frac{\|\varphi\|_\infty^2}{t^4}\int_H \left(|\langle z, B^{-\frac{1}{2}}y\rangle|^2 - t|z|^2\right)^2\mathcal{N}(0,tB)(dy).\end{aligned}$$

Since

$$\begin{aligned}\int_H |\langle z, B^{-\frac{1}{2}}y\rangle|^4\mathcal{N}(0,tB)(dy) &= 3t^2|z|^4 \text{ and}\\ \int_H |\langle z, B^{-\frac{1}{2}}y\rangle|^2\mathcal{N}(0,tB)(dy) &= t|z|^2 \quad \text{(see Proposition 1.3.1)},\end{aligned}$$

it follows that

$$|\langle D_B^2 P_t\varphi(x)z, z\rangle|^2 \le \frac{2}{t^2}|z|^4\|\varphi\|_\infty^2$$

for all $x, z \in H$. Consequently,

$$\|D_B^2 P_t\varphi(x)\|_{\mathcal{L}(H)} \le \frac{\sqrt{2}}{t}\|\varphi\|_\infty, \quad \forall x \in H.$$

The second equality can be obtained similarly, by using Theorem 2.2.7 and the last estimate is a consequence of Proposition 1.3.1. □

We propose now to prove an additional regularity result, which will be needed to solve (HE).
We start by the following auxiliary result, where the proof can be founded in [15, Lemma XI.9.14 (a), p. 1098].

Lemma 2.2.9 *Let $B \in \mathcal{L}(H)$ and suppose that there is a constant $c > 0$ such that, for all finite rank linear operator N in $\mathcal{L}(H)$, $|\mathrm{Tr}(NB)| \le c\|N\|$. Then B is a trace class operator on H and*

$$\mathrm{Tr}\, B \le c\,.$$

The following result was proved first by L. Gross [19] by using probabilistic methods.

Theorem 2.2.10 *For $\varphi \in BUC^1(H)$ and $t > 0$, we have $D_B^2 P_t\varphi(x)$ is a trace class operator on H for all $x \in H$, and*

$$\text{Tr } (D_B^2 P_t \varphi(x)) = \frac{1}{t}\int_H < D\varphi(x+y), y > \mathcal{N}(0,tB)(dy) \ , \qquad x \in H.$$

Moreover, $\text{Tr} D_B^2 P_t\varphi(\cdot) \in BUC(H)$ *and*

$$|\text{Tr} D_B^2 P_t \varphi(x)| \leq \frac{1}{\sqrt{t}} \|\varphi\|_1 (\text{Tr} B)^{\frac{1}{2}}.$$

Proof: Since $\varphi \in BUC^1(H)$, it follows that, for $z_1 \in H$,

$$\begin{aligned} < DP_t\varphi(x), B^{\frac{1}{2}} z_1 > &= \int_H < D\varphi(x+y), B^{\frac{1}{2}} z_1 > \mathcal{N}(0,tB)(dy) \\ &= P_t \psi(x), \end{aligned}$$

where $\psi(x) := < D\varphi(x), B^{\frac{1}{2}} z_1 >, \ x \in H$. From Theorem 2.2.7 we have

$$\begin{aligned} < D_B P_t \psi(x), z_2 > &= \frac{1}{t}\int_H < z_2, B^{-\frac{1}{2}} y > \psi(x+y) \mathcal{N}(0,tB)(dy) \\ &= \frac{1}{t}\int_H < z_2, B^{-\frac{1}{2}} y >< D\varphi(x+y), B^{\frac{1}{2}} z_1 \qquad (2.2) \\ &> \mathcal{N}(0,tB)(dy) \end{aligned}$$

for $z_2 \in H$. On the other hand, by an easy computation, one can see,

$$< D_B P_t \psi(x), z_2 > = < D_B^2 P_t \varphi(x) z_1, z_2 > .$$

Hence,

$$\begin{aligned} < D_B^2 P_t\varphi(x) z_1, z_2 > &= \\ = \ \tfrac{1}{t}\textstyle\int_H < D\varphi(x+y), B^{\frac{1}{2}} z_1 >< z_2, B^{-\frac{1}{2}} y > \mathcal{N}(0,tB)(dy). \end{aligned}$$

Now, take $N \in \mathcal{L}(H)$ a finite rank operator. We obtain

$$\begin{aligned} < N D_B^2 P_t\varphi(x) z_1, z_2 > &= \\ = \ \tfrac{1}{t}\textstyle\int_H < D\varphi(x+y), B^{\frac{1}{2}} z_1 >< N^* z_2, B^{-\frac{1}{2}} y > \mathcal{N}(0,tB)(dy). \end{aligned}$$

Hence,

$$\text{Tr}(N D_B^2 P_t \varphi(x)) = \frac{1}{t}\int_H < D\varphi(x+y), B^{\frac{1}{2}} N B^{-\frac{1}{2}} y > \mathcal{N}(0,tB)(dy),$$

and by Hölder's inequality, we obtain

$$\begin{aligned}|\mathrm{Tr}(ND_B^2P_t\varphi(x))|^2 &\leq \frac{\|\varphi\|_1^2}{t^2}\int_H |B^{\frac{1}{2}}NB^{-\frac{1}{2}}y|^2\mathcal{N}(0,tB)(dy)\\ &= \frac{\|\varphi\|_1^2}{t^2}t\ \mathrm{Tr}(B^{\frac{1}{2}}NN^*B^{\frac{1}{2}}) \quad \text{(see Example 1.2.9.(b))}\\ &= \frac{\|\varphi\|_1^2}{t}\mathrm{Tr}(NN^*B).\end{aligned}$$

Thus,

$$|\mathrm{Tr}(ND_B^2P_t\varphi(x))| \leq \frac{1}{\sqrt{t}}\|\varphi\|_1\|N\|(\mathrm{Tr}B)^{\frac{1}{2}},\ x\in H.$$

So, by Lemma 2.2.9, $\mathrm{Tr}(D_B^2P_t\varphi(x)) < \infty$ for all $x \in H$. Moreover,

$$\mathrm{Tr}(D_B^2P_t\varphi(x)) = \frac{1}{t}\int_H < D\varphi(x+y), y > \mathcal{N}(0,tB)(dy), \quad x\in H,$$

and

$$|\mathrm{Tr}(D_B^2P_t\varphi(x))| \leq \frac{1}{\sqrt{t}}\|\varphi\|_1(\mathrm{Tr}B)^{\frac{1}{2}}\ , \quad x\in H.$$

The uniform continuity of $\mathrm{Tr}(D_B^2P_t\varphi(\cdot))$ follows from the fact that $\varphi \in BUC^1(H)$. □

2.3 Solutions of (HE) and characterization of the generator of (P_t)

We denote by $(G, D(G))$ the generator of (P_t) on $BUC(H)$.
First, we propose to compare G with the following operator
$D(G_0) :=$

$$\left\{\varphi\in BUC_B^2(H),\ D_B^2\varphi(x)\in\mathcal{L}_1(H),\ \forall\, x\in H \text{ and } \mathrm{Tr}(D_B^2\varphi(\cdot))\in BUC(H)\right\},$$

$$G_0\varphi = \frac{1}{2}\mathrm{Tr}(D_B^2\varphi),$$

where $\mathcal{L}_1(H)$ denotes the set of $S \in \mathcal{L}(H)$ with $\mathrm{Tr}\, S < \infty$.

Proposition 2.3.1 *The following hold:*

(a) $\overline{D(G_0)} = BUC(H)$;

(b) $\overline{G_0} = G$.

Proof: (a) Let $\varphi \in BUC(H)$. Since $BUC^1(H)$ is dense in $BUC(H)$, it follows that, for any $\varepsilon > 0$ there is $\varphi_\varepsilon \in BUC^1(H)$ such that $\|\varphi - \varphi_\varepsilon\|_\infty < \frac{\varepsilon}{2}$. On the other hand, from the strong continuity of (P_t) we have, for any $\varepsilon > 0$ there exists $\delta > 0$ with

$$0 < t < \delta \Longrightarrow \|\varphi_\varepsilon - P_t\varphi_\varepsilon\|_\infty < \frac{\varepsilon}{2}.$$

Thus, for $0 < t < \delta$,

$$\|\varphi - P_t\varphi_\varepsilon\| < \varepsilon.$$

Now, (a) follows from Theorem 2.2.10.
(b) Let $\varphi \in D(G_0)$ and take $g(t) := P_t\varphi$ and $g_n(t) : P_t^n\varphi$.
It follows from Theorem 2.1.1 that

$$g_n \longrightarrow g \quad \text{in } C\left([0,1]; BUC(H)\right).$$

Moreover,

$$\frac{dg_n}{dt}(t) = \frac{1}{2}\sum_{k=1}^{n} \lambda_k D_k^2 g_n(t) = \frac{1}{2}\sum_{k=1}^{n} \lambda_k D_k^2 P_t^n\varphi = P_t^n\left(\frac{1}{2}\sum_{k=1}^{n} \lambda_k D_k^2\varphi\right).$$

Hence,

$$\frac{dg_n}{dt}(t) \longrightarrow P_t(G_0\varphi) \quad \text{in } C\left([0,1], BUC(H)\right).$$

Consequently, $\dfrac{dg}{dt}(t) = P_t(G_0\varphi)$ and by taking $t = 0$ we have $\varphi \in D(G)$ and $G\varphi = G_0\varphi$, i.e., $G_0 \subseteq G$. In particular G_0 is closable. Now, take $\varphi \in D(G)$, $\lambda > 0$ and set $\psi := \lambda\varphi - G\varphi$. We know that there is $(\psi_n)_{n\in\mathbb{N}} \subseteq BUC^1(H)$ such that $\psi_n \to \psi$ in $BUC(H)$. Since (P_t) is a semigroup of contractions on $BUC(H)$, we can define $\varphi_n := R(\lambda, G)\psi_n$. It is clear that $\varphi_n \to \varphi$ in $BUC(H)$. Since $\varphi_n = \displaystyle\int_0^\infty e^{-\lambda t} P_t\psi_n dt$, it follows from Theorem 2.2.10 that

$$\varphi_n \in D(G_0) \quad \text{and} \quad \|G_0\varphi_n\|_\infty \le \left(\int_0^\infty e^{-\lambda t}\frac{1}{\sqrt{t}}dt\right)(\mathrm{Tr}B)^{\frac{1}{2}}\|\psi_n\|_1 .$$

Moreover, since

$$G_0\varphi_n = G\varphi_n = \lambda R(\lambda, G)\psi_n - \psi_n,$$

it follows that

$$\lim_{n\to\infty} G_0\varphi_n = \lambda R(\lambda, G)\psi - \psi = GR(\lambda, G)\psi = G\varphi.$$

This proves that $\overline{G_0} = G$. □

We solve now the heat equation. Let $\varphi \in BUC^1(H)$ and set

$$u(t,x) = P_t\varphi(x), \quad t \ge 0,\ x \in H.$$

From Theorem 2.2.10 we know that $P_t\varphi \in D(G_0)$ for $t > 0$. Since $G_0 \subseteq G$ we obtain

$$\frac{d}{dt}P_t\varphi = GP_t\varphi = G_0P_t\varphi,\ t > 0.$$

Thus, we have the following result.

Theorem 2.3.2 *Let $\varphi \in BUC^1(H)$. Then the function*

$$u(t,x) = P_t\varphi(x), \quad t > 0,$$

is a classical solution of (HE) with $u(0,x) = \varphi(x),\ x \in H$.

An other characterization of the generator $(G, D(G))$ of the heat semi-group (P_t) on $BUC(H)$, which will play an important role in Section 2.4, is given by the following proposition.

Proposition 2.3.3 *The set*

$$\begin{aligned} D_0(G) \quad := \quad & \{\varphi \in BUC^{1,1}(H) : D_kD_l\varphi \in BUC(H), \\ & \textit{for all } k,l \in \mathbb{N},\ \sup_{k,l\in\mathbb{N}} \|D_kD_l\varphi\|_\infty < \infty\} \end{aligned}$$

is a P_t-invariant core for G. Moreover,

$$G\varphi = \sum_{k=1}^{\infty} \lambda_k D_k^2\varphi \quad \textit{for } \varphi \in D_0(G).$$

Proof: Let show first that, for $\varphi \in BUC^{1,1}(H)$,

$$\sup_{l,k\in\mathbb{N}} \|D_lD_kP_t\varphi\|_\infty \leq \|\varphi\|_{1,1}, \quad t > 0. \tag{2.3}$$

Let $\varphi \in BUC^{1,1}(H)$ and $k \in \mathbb{N}$. Since D_k is closed and $D_kP_t^n\varphi = P_t^nD_k\varphi$ for $t \geq 0$ and $n \in \mathbb{N}$, it follows from Theorem 2.1.1 that

$$D_kP_t\varphi = P_tD_k\varphi$$

for all $t \geq 0$. So by Proposition 2.2.1 we have

$$D_kP_t\varphi \in D(D_l) \quad \text{for all } t > 0, \text{ and } l \in \mathbb{N}.$$

Thus, by Theorem 2.1.3, we deduce that

$$\begin{aligned} & |D_lD_kP_t\varphi(x)| = \\ & = \quad |D_lP_tD_k\varphi(x)| \\ & = \quad \left|\lim_{h\to 0}\frac{1}{h}(P_tD_k\varphi(x+he_l) - P_tD_k\varphi(x)\right| \\ & = \quad \left|\lim_{h\to 0}\int_H \frac{1}{h}(D_k\varphi(x+y+he_l) - D_k\varphi(x+y)\mathcal{N}(0,tB)(dy)\right| \\ & \leq \quad \|\varphi\|_{1,1} \end{aligned}$$

for all $l, k \in \mathbb{N}$, and $x \in H$. This proves (2.3). So we obtain

$$P_t D_0(G) \subseteq D_0(G), \quad \forall t \geq 0.$$

From Proposition A.2.5, it suffices now to prove that $D_0(G)$ is dense in $BUC(H)$. This can be seen by using (2.3) and exactly the same proof as in Proposition 2.3.1.(a). □

We end this section by the following remark.

Remark 2.3.4 *If we compare the result of Theorem 2.2.8 and Theorem A.2.7 then the following question arise:*

Is the semigroup (P_t) analytic or at least differentiable on $BUC(H)$?

The answer is negative (see [27]) and will be given in the following section (see Corollary 2.4.2).

2.4 THE SPECTRUM OF THE INFINITE DIMENSIONAL LAPLACIAN

Let H be a separable, infinite dimensional, real Hilbert space and let (e_k) be an orthonormal basis. We shall regard $BUC(\mathbb{R}^n)$ as a subspace of $BUC(H)$ via the isometric embedding

$$J_n : BUC(\mathbb{R}^n) \to BUC(H), \quad (J_n\varphi)(x) := \varphi(x_1, \ldots, x_n),$$

for $\varphi \in BUC(\mathbb{R}^n)$, $x \in H$, and $x_k := \langle x, e_k \rangle$. Let $\lambda_k > 0$ with $\sum_{k=1}^{\infty} \lambda_k < \infty$ be given. We know from Theorem 2.1.1 that the infinite dimensional heat equation (HE) on $BUC(H)$ is solved by the C_0-semigroup of contractions

$$P_t\varphi = \lim_{n\to\infty} P_t^n \varphi, \quad \varphi \in BUC(H),$$

where the above limit exists in $BUC(H)$ uniformly in t on bounded subsets of $[0, \infty)$. We recall that for $\varphi \in BUC(H)$, $x \in H$ and $t > 0$,

$$P_t^n \varphi(x) := (2\pi t)^{-\frac{n}{2}} (\lambda_1 \cdots \lambda_n)^{-\frac{1}{2}} \int_{\mathbb{R}^n} e^{-\sum_{k=1}^{n} \frac{y_k^2}{2t\lambda_k}} \varphi\Big(x - \sum_{k=1}^{n} y_k e_k\Big) dy. \tag{2.4}$$

Let compute the spectrum of the generator $(G, D(G)$ of the semigroup (P_t) on $BUC(H)$.

Theorem 2.4.1 *The spectrum of G is the left half plane $\{\lambda \in \mathbb{C} : Re\ \lambda \leq 0\}$ and $\sigma(P_t) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$. Moreover, every $\lambda \in \sigma(G)$ is an approximate eigenvalue.*

Proof: Note that the restriction of P_t to $BUC(\mathbb{R}^n)$ coincides with the semigroup generated by $G_n := \sum_{k=1}^n \lambda_k D_k^2$. In particular, G_n is the part of G in $BUC(\mathbb{R}^n)$ and, hence, $R(\lambda, G_n) = R(\lambda, G)_{|BUC(\mathbb{R}^n)}$ for $\lambda \in \rho(G) \cap \rho(G_n)$. Therefore, for these values of λ, the sequence $\|R(\lambda, G_n)\|$ is bounded.
Let $V : BUC(\mathbb{R}^n) \to BUC(\mathbb{R}^n)$ be the isometry defined by

$$(V\varphi)(x) := \varphi(\sqrt{\frac{\lambda_1}{2}}x_1, \dots, \sqrt{\frac{\lambda_n}{2}}x_n), \qquad \varphi \in BUC(\mathbb{R}^n),\ x \in \mathbb{R}^n.$$

A simple change of variables in (2.4) shows that $e^{tG_n} = V^{-1}e^{t\Delta_n}V$ for $t \geq 0, n \in \mathbb{N}$, where Δ_n denotes the Laplacian on $\mathbb{R}^n$. This implies that

$$R(\lambda, G_n) = V^{-1}R(\lambda, \Delta_n)V \quad \text{ for } \lambda \in \Sigma_\pi := \{0 \neq \lambda \in \mathbb{C} : |\arg\ \lambda| < \pi\},$$

so that $\|R(\lambda, G_n)\| = \|R(\lambda, \Delta_n)\|$ for $\lambda \in \Sigma_\pi$ and $n \in \mathbb{N}$.

Fix $\lambda \in \Sigma_\pi$ with $Re\ \lambda < 0$. For $n \in \mathbb{N}$, the function $g_{\lambda,n}(x) := e^{\frac{\lambda}{2n}|x|^2}$, $x \in \mathbb{R}^n$, belongs to $BUC(\mathbb{R}^n)$ and $\|g_{\lambda,n}\|_\infty = 1$. Setting

$$f_{\lambda,n}(x) := (\lambda - \Delta_n)g_{\lambda,n}(x) - \frac{\lambda^2}{n^2}|x|^2 e^{\frac{\lambda}{2n}|x|^2}, \quad x \in \mathbb{R}^n,$$

we compute

$$\|f_{\lambda,n}\|_\infty = \frac{2|\lambda|^2}{ne|Re\ \lambda|}.$$

So we derive

$$\|R(\lambda, G_n)\| = \|R(\lambda, \Delta_n)\| \geq \frac{\|R(\lambda, \Delta_n)f_{\lambda,n}\|_\infty}{\|f_{\lambda,n}\|_\infty} = \frac{ne|Re\ \lambda|}{2|\lambda|^2}.$$

Since the sequence $\|R(\lambda, G_n)\|$ is unbounded, λ must belong to the spectrum of G. From standard spectral theory of C_0-semigroups, cf. [16, Chap. IV], now follows the first and second assertion.

To prove the last assertion, we observe that $i\mathbb{R}$ is contained in the approximate point spectrum of G. Let $\lambda = -a^2 + ib$ for $a > 0$ and $b \in \mathbb{R}$. The first part of the proof applies to the operator $\tilde{G}$ on $BUC(H)$ corresponding to the sequence $(\lambda_2, \lambda_3, \cdots)$. Thus there exist $g_n \in D_0(\tilde{G})$ such that $\|g_n\|_\infty = 1$ and $\|\tilde{G}g_n - ibg_n\|_\infty \to 0$ as $n \to \infty$. We now define

$$f_n(x) : \exp(ia\lambda_1^{-\frac{1}{2}}\, x_1)\, g_n(x_2, x_3, \cdots), \quad x \in H.$$

Clearly, $f_n \in D_0(G)$, $\|f_n\|_\infty = 1$, and

$$Gf_n(x) = \sum_{k=1}^{\infty} \lambda_k\, D_k^2\, f_n(x) = -a^2 f_n(x) + \exp(ia\lambda_1^{-\frac{1}{2}}\, x_1)\, (\tilde{G}g_n)(x_2, x_3, \cdots),$$

$x \in H$.
As a result, λ is an approximate eigenvalue of G. □

As a consequence of Theorem A.2.10 and (11) we immediately obtain the following result from [14], see also [18], [29] and [2].

Corollary 2.4.2 *The semigroup* (P_t) *is not eventually norm continuous an hence not eventually differentiable on* $BUC(H)$*.*