CHAPTER C

Durfee Rectangles and Classical Partition Identities

For a partition λ, its Durfee square is the maximum square contained in the Ferrers diagram of λ. It can be generalized similarly to the Durfee rectangles. They will be used, in this chapter, to classify partitions and establish classical partition identities.

C1. q-Series identities of Cauchy and Kummer: Unification

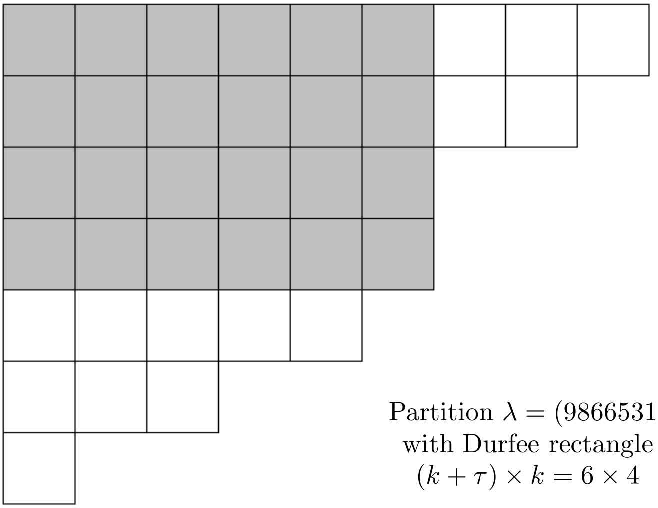
C1.1. Theorem. For the partitions into parts $\leq n$, classify them with respect to the Durfee rectangles of $(k+\tau)\times k$ for a fixed τ. We can derive the following

$$\frac{1}{(qx;\,q)_n} = \sum_{k=0}^{n-\tau} \begin{bmatrix} n-\tau \\ k \end{bmatrix} \frac{q^{k(k+\tau)}}{(qx;\,q)_{k+\tau}}\, x^k. \tag{C1.1}$$

Proof. The partitions into parts $\leq n$ with Durfee rectangles of $(k+\tau)\times k$ for a fixed τ are composed by three pieces. One of them is the Durfee rectangle $(k+\tau)\times k$ in common with enumerator $x^k q^{k(k+\tau)}$. Another is the piece right to Durfee rectangle which are partitions of length $\leq k$ with parts $\leq n-k-\tau$, whose univariate generating function is $\begin{bmatrix} n-\tau \\ k \end{bmatrix}$ in view of (B4.2b) (only the univariate function is considered because the length of partitions has been counted by the Durfee rectangle). The last piece corresponds to the partitions with parts $\leq k+\tau$ whose bivariate generating function is $1/(qx;q)_{k+\tau}$. Classifying the partitions into parts $\leq n$ with respect to Durfee rectangles of $(k+\tau)\times k$ with $0 \leq k \leq n-\tau$, we find

$$\frac{1}{(qx;q)_n} = \sum_{k=0}^{n-\tau} \begin{bmatrix} n-\tau \\ k \end{bmatrix} \frac{x^k q^{k(k+\tau)}}{(qx;q)_{k+\tau}}$$

which is exactly the identity required in the theorem. $\square$



Partition $\lambda = (9866531)$
with Durfee rectangle
$(k+\tau) \times k = 6 \times 4$

C1.2. Corollary. The formula just established contains the following known results as special cases:

- The finite version of Kummer's theorem ($\tau = 0$)

$$\frac{1}{(qx;\ q)_n} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \frac{x^k\, q^{k^2}}{(qx;\ q)_k}. \tag{C1.2}$$

- The identity due to Gordon and Houten [1968] ($n \to \infty$)

$$\frac{1}{(qx;\ q)_\infty} = \sum_{k=0}^{\infty} \frac{x^k\, q^{k(k+\tau)}}{(q;\ q)_k (qx;\ q)_{k+\tau}} \tag{C1.3}$$

which reduces further to the Cauchy formula with $\tau = 0$.

C2. q-Binomial convolutions and the Jacobi triple product

C2.1. Theorem. For the partitions into parts $\leq n$, with at most $\alpha+\gamma-n$ parts, classify them according to the Durfee rectangles of $(n-k)\times(\alpha-k)$. We obtain the first q-Vandermonde convolution formula

$$\begin{bmatrix} \alpha+\gamma \\ n \end{bmatrix} = \sum_{k=0}^{n} \begin{bmatrix} \alpha \\ k \end{bmatrix} \begin{bmatrix} \gamma \\ n-k \end{bmatrix} q^{(\alpha-k)(n-k)}. \tag{C2.1}$$

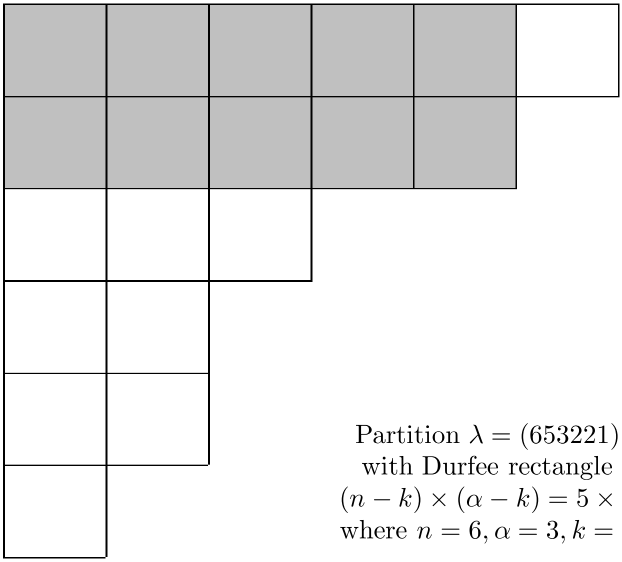
PROOF. The univariate generating function of the partitions into parts $\leq n$ with at most $\alpha+\gamma-n$ parts is equal to $\begin{bmatrix}\alpha+\gamma\\ n\end{bmatrix}$ by (B4.2b). Fixing the Durfee rectangle of $(n-k)\times(\alpha-k)$ we see that the corresponding partitions into parts $\leq n$ with at most $\alpha+\gamma-n$ parts consist of three pieces. The first piece is the rectangle of $(n-k)\times(\alpha-k)$ on the top-left with univariate enumerator $q^{(\alpha-k)(n-k)}$. The second piece right to the rectangle is a partition into parts $\leq k$ with at most $\alpha-k$ parts enumerated by $\begin{bmatrix}\alpha\\ k\end{bmatrix}$. The third and the last piece under the rectangle is a partition into parts $\leq n-k$ with at most $\gamma-n+k=(\alpha+\gamma-n)-(\alpha-k)$ parts enumerated by $\begin{bmatrix}\gamma\\ n-k\end{bmatrix}$. Classifying the partitions according to the Durfee rectangles of $(n-k)\times(\alpha-k)$ and summing the product of three generating functions over $0\leq k\leq n$, we find the following identity:

$$\begin{bmatrix}\alpha+\gamma\\ n\end{bmatrix} = \sum_{k=0}^{n}\begin{bmatrix}\alpha\\ k\end{bmatrix}\begin{bmatrix}\gamma\\ n-k\end{bmatrix}q^{(\alpha-k)(n-k)}.$$

Its limiting case $q\to 1$ reduces to

$$\binom{\alpha+\gamma}{n} = \sum_{k=0}^{n}\binom{\alpha}{k}\binom{\gamma}{n-k}$$

which is the well-known Chu-Vandermonde convolution formula. □



Partition $\lambda = (653221)$
with Durfee rectangle
$(n-k) \times (\alpha - k) = 5 \times 2$
where $n = 6, \alpha = 3, k = 1$

C2.2. Proposition. Instead, considering the Durfee rectangle of $k \times (\gamma - n)$ for the same partitions, we derive the second q-Vandermonde convolution formula

$$\begin{bmatrix} \alpha+\gamma \\ n \end{bmatrix} = \sum_{k=0}^{n} \begin{bmatrix} \alpha+k \\ k \end{bmatrix} \begin{bmatrix} \gamma-k-1 \\ n-k \end{bmatrix} q^{k(\gamma-n)}. \tag{C2.2}$$

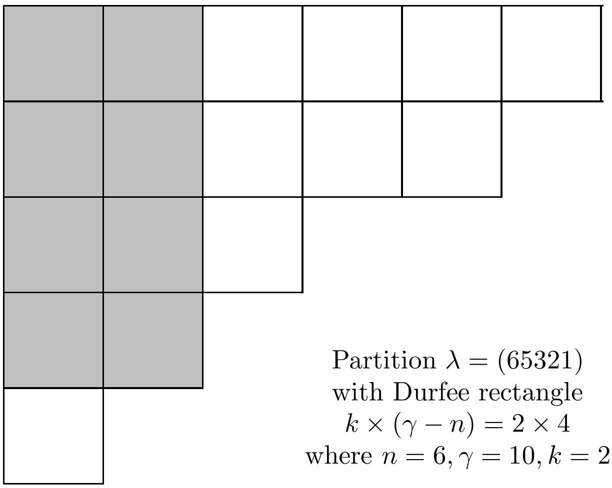
Proof. The univariate generating function of the partitions into parts $\leq n$ with at most $\alpha+\gamma-n$ parts is equal to $\begin{bmatrix} \alpha+\gamma \\ n \end{bmatrix}$ by (B4.2b). For a fixed Durfee rectangle of $k \times (\gamma - n)$ the corresponding partition into parts $\leq n$ with at most $\alpha+\gamma-n$ parts consists of three pieces: the first piece is the rectangle of $k \times (\gamma - n)$ on the top-left with univariate enumerator $q^{k(\gamma-n)}$, the second piece right to the rectangle is a partition into parts $\leq n-k$ with at most $\gamma - n - 1$ parts enumerated by $\begin{bmatrix} \gamma-k-1 \\ n-k \end{bmatrix}$, where we can easily justify that the partition length can not be $\gamma - n$, otherwise, we would have a larger Durfee rectangle $(k+1) \times (\gamma - n)$, and the third part under the rectangle is a partition into parts $\leq k$ with at most α parts enumerated by $\begin{bmatrix}\alpha+k\\ k\end{bmatrix}$. Classifying the partitions with respect to Durfee rectangles of $k \times (\gamma - n)$ and then summing the product of three generating functions over $0 \le k \le n$, we find the following identity:

$$\begin{bmatrix}\alpha+\gamma\\ n\end{bmatrix} = \sum_{k=0}^{n} \begin{bmatrix}\alpha+k\\ k\end{bmatrix} \begin{bmatrix}\gamma-k-1\\ n-k\end{bmatrix} q^{k(\gamma-n)}.$$

For $q \to 1$, the limiting case reads as

$$\binom{\alpha+\gamma}{n} = \sum_{k=0}^{n} \binom{\alpha+k}{k} \binom{\gamma-k-1}{n-k}$$

which is another binomial convolution formula. □



Partition $\lambda = (65321)$
with Durfee rectangle
$k \times (\gamma - n) = 2 \times 4$
where $n = 6, \gamma = 10, k = 2$

C2.3. Corollary. Given the diagram of $(m-\tau)\times(n+\tau)$, consider the partitions contained in it. The classification with respect to Durfee rectangles of $k \times (k + \tau)$ leads us to the following finite summation formula

$$\begin{bmatrix}m+n\\ n+\tau\end{bmatrix} = \sum_{k=0}^{n} \begin{bmatrix}m\\ k+\tau\end{bmatrix} \begin{bmatrix}n\\ k\end{bmatrix} q^{k(k+\tau)} \qquad \text{(C2.3)}$$

which is a special case of the first q-Chu-Vandermonde convolution formula.

Proof. For the partitions into parts $\le m-\tau$ with at most $n+\tau$ parts, the univariate generating function is equal to $\begin{bmatrix} m+n \\ n+\tau \end{bmatrix}$ by (B4.2b). Fixing a Durfee rectangle of $k\times(k+\tau)$, we observe that the partitions into parts $\le m-\tau$ with at most $n+\tau$ parts consist of three pieces. The first piece is the rectangle of $k\times(k+\tau)$ on the top-left with univariate enumerator $q^{k(k+\tau)}$. The second piece right to the rectangle is a partition into parts $\le m-\tau-k$ with at most $k+\tau$ parts enumerated by $\begin{bmatrix} m \\ k+\tau \end{bmatrix}$ and the third one under the rectangle is a partition into parts $\le k$ with at most $n-k$ parts enumerated by $\begin{bmatrix} n \\ k \end{bmatrix}$. Classifying the partitions according to the Durfee rectangles of $k\times(k+\tau)$ for $0\le k\le n$ and then summing the product of three generating functions over $0\le k\le n$, we find the following identity:

$$\begin{bmatrix} m+n \\ n+\tau \end{bmatrix} = \sum_{k=0}^{n} \begin{bmatrix} m \\ k+\tau \end{bmatrix} \begin{bmatrix} n \\ k \end{bmatrix} q^{k(k+\tau)}$$

which is exactly the identity stated in the theorem.

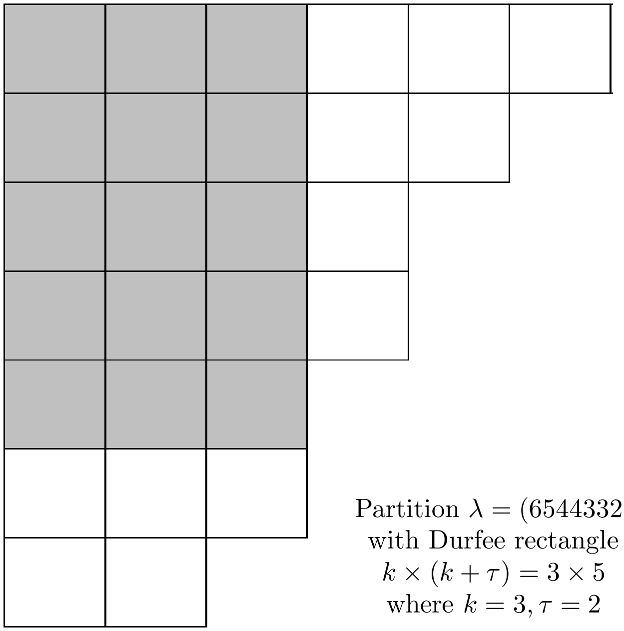
We remark that this identity is a special case of the first q-Vandermonde convolution formula stated in Theorem C2.1. In fact replacing n with ℓ, we can state the reversal of the q-Vandermonde convolution formula in Theorem C2.1 as follows:

$$\begin{bmatrix} \alpha+\gamma \\ \ell \end{bmatrix} = \sum_{k=0}^{\ell} \begin{bmatrix} \alpha \\ \ell-k \end{bmatrix} \begin{bmatrix} \gamma \\ k \end{bmatrix} q^{k(\alpha+k-\ell)}.$$

Performing parameter replacements

$$\alpha\to m,\quad \gamma\to n \quad\text{and}\quad \ell\to m-\tau$$

we obtain immediately the identity stated in Corollary C2.3. □



Partition $\lambda = (6544332)$
with Durfee rectangle
$k \times (k+\tau) = 3 \times 5$
where $k = 3, \tau = 2$

C2.4. The Jacobi-triple product identity. From the last q-binomial convolution identity, we can derive the following bilateral summation formula

$$(x;\, q)_m\, (q/x;\, q)_n \quad = \quad \sum_{k=-n}^{m} (-1)^k\, q^{\binom{k}{2}} \begin{bmatrix} m+n \\ n+k \end{bmatrix} x^k. \tag{C2.4}$$

It can be considered as a finite form of the well-known Jacobi triple product identity

$$(q;\, q)_\infty\, (x;\, q)_\infty\, (q/x;\, q)_\infty \quad = \quad \sum_{n=-\infty}^{+\infty} (-1)^n\, q^{\binom{n}{2}}\, x^n \tag{C2.5}$$

whose limiting case $x \to 1$ reads as the cubic form of the triple product (Jacobi):

$$(q;\, q)_\infty^3 \quad = \quad \sum_{n=0}^{\infty} (-1)^n \{1+2n\} q^{\binom{1+n}{2}}. \tag{C2.6}$$

PROOF. According to the Euler q-finite differences (B5.3), we have two finite expansions

$$\begin{aligned}(x,q)_m &= \sum_{i=0}^{m}(-1)^i\begin{bmatrix}m\\i\end{bmatrix}q^{\binom{i}{2}}x^i\\(q/x,q)_n &= \sum_{j=0}^{n}(-1)^j\begin{bmatrix}n\\j\end{bmatrix}q^{\binom{1+j}{2}}x^{-j}.\end{aligned}$$

Their product reads as the following double sum

$$\begin{aligned}(x,q)_m(q/x,q)_n &= \sum_{i=0}^{m}\sum_{j=0}^{n}(-1)^{i+j}\begin{bmatrix}m\\i\end{bmatrix}\begin{bmatrix}n\\j\end{bmatrix}q^{\binom{i}{2}+\binom{1+j}{2}}x^{i-j}\\&= \sum_{k=-n}^{m}(-1)^k x^k\sum_{j=0}^{n}\begin{bmatrix}m\\k+j\end{bmatrix}\begin{bmatrix}n\\j\end{bmatrix}q^{\binom{k+j}{2}+\binom{1+j}{2}}\end{aligned}$$

where the last line is justified by the replacement $k=i-j$. Observe that

$$\binom{k+j}{2}+\binom{1+j}{2} = \binom{k}{2}+\binom{j}{2}+kj+\binom{1+j}{2} = \binom{k}{2}+j(j+k).$$

Reformulating the double sum and then applying the convolution formula stated in Corollary C2.3, we derive the finite bilateral summation formula (C2.4)

$$\begin{aligned}(x,q)_m(q/x,q)_n &= \sum_{k=-n}^{m}(-1)^k q^{\binom{k}{2}}x^k\sum_{j=0}^{n}\begin{bmatrix}m\\k+j\end{bmatrix}\begin{bmatrix}n\\j\end{bmatrix}q^{j(j+k)}\\&= \sum_{k=-n}^{m}(-1)^k q^{\binom{k}{2}}\begin{bmatrix}m+n\\n+k\end{bmatrix}x^k.\end{aligned}$$

When m and n tend to infinity, the limit of q-binomial coefficient reads as

$$\begin{bmatrix}m+n\\n+k\end{bmatrix} = \frac{(q;q)_{m+n}}{(q;q)_{n+k}(q;q)_{m-k}} \to \frac{1}{(q;q)_\infty}.$$

Applying the Tannery Theorem, we therefore have

$$(x,q)_\infty(q/x,q)_\infty = \sum_{k=-\infty}^{+\infty}(-1)^k\frac{q^{\binom{k}{2}}x^k}{(q;q)_\infty}$$

which is equivalent to the Jacobi-triple product identity (C2.5).

In order to prove (C2.6), we rewrite the Jacobi triple product identity as

$$\begin{aligned}(q;\,q)_\infty(x;\,q)_\infty(q/x;\,q)_\infty &= \sum_{n=-\infty}^{+\infty}(-1)^n\,q^{\binom{1+n}{2}}\,x^{-n}\\ &= \sum_{n=0}^{+\infty}(-1)^n\,q^{\binom{1+n}{2}}\,x^{-n}\\ &+ \sum_{n=1}^{+\infty}(-1)^n\,q^{\binom{1-n}{2}}\,x^{n}.\end{aligned}$$

Replacing the summation index n by $1+m$ in the last sum:

$$\sum_{n=1}^{\infty}(-1)^n\,q^{\binom{n}{2}}\,x^n = -\sum_{m=0}^{\infty}(-1)^m\,q^{\binom{1+m}{2}}\,x^{m+1}$$

we can combine two sums into one unilateral sum

$$(q;\,q)_\infty(x;\,q)_\infty(q/x;\,q)_\infty = \sum_{n=0}^{\infty}(-1)^n\,q^{\binom{1+n}{2}}\big\{x^{-n}-x^{n+1}\big\}.$$

Dividing both sides by $1-x$, we get

$$(q;\,q)_\infty(qx;\,q)_\infty(q/x;\,q)_\infty = \sum_{n=0}^{\infty}(-1)^n\,q^{\binom{1+n}{2}}\frac{x^{-n}-x^{n+1}}{1-x}.$$

Applying L'Hôspital's rule for the limit, we have

$$\lim_{x\to 1}\frac{x^{-n}-x^{n+1}}{1-x}=2n+1.$$

Considering that the series is uniformly convergent and then evaluating the limit $x\to 1$ term by term, we establish

$$(q;\,q)_\infty^3 = \sum_{n=0}^{\infty}(-1)^n\big\{2n+1\big\}q^{\binom{1+n}{2}}$$

which is the cubic form of triple product. □

Remark The shortest proof of the Jacobi triple product identity is due to Cauchy (1843) and Gauss (1866). It can be reproduced in the sequel.

Recall the q-binomial theorem (finite q-differences) displayed in (B5.3)

$$(x;q)_\ell = \sum_{k=0}^{\ell}(-1)^k\begin{bmatrix}\ell\\ k\end{bmatrix}q^{\binom{k}{2}}x^k.$$

Replacing ℓ by $m+n$ and x by xq^{-n} respectively, and then noting the relation

$$(q^{-n}x;q)_{m+n} = (q^{-n}x;q)_n(x;q)_m = (-1)^n q^{-\binom{1+n}{2}} x^n (q/x;q)_n(x;q)_m$$

we can reformulate the q-binomial theorem as

$$(x;q)_m(q/x;q)_n = \sum_{k=0}^{m+n} (-1)^{k-n} \begin{bmatrix} m+n \\ k \end{bmatrix} q^{\binom{k-n}{2}} x^{k-n}$$

which becomes, under summation index substitution $k \to n+k$, the following finite form of the Jacobi triple product identity

$$(x;q)_m(q/x;q)_n = \sum_{k=-n}^{m} (-1)^k \begin{bmatrix} m+n \\ n+k \end{bmatrix} q^{\binom{k}{2}} x^k.$$

This is exactly the finite form (C2.4) of the Jacobi triple product identity.

C2.5. Corollary. From Jacobi's triple product identity, we may further derive the following infinite series identities:

- Triangle number theorem (Gauss)

$$\frac{(q^2;\, q^2)_\infty}{(q;\, q^2)_\infty} = \sum_{n=0}^{\infty} q^{\binom{1+n}{2}}.$$

- Pentagon number theorem (Euler)

$$(q;\, q)_\infty = \sum_{n=-\infty}^{+\infty} (-1)^n\, q^{\frac{n}{2}(3n+1)}.$$

Proof. Reformulate the factorial fraction in this way:

$$\begin{aligned}\frac{(q^2;\, q^2)_\infty}{(q;\, q^2)_\infty} &= \frac{(q;\, q)_\infty(-q;\, q)_\infty}{(q;\, q^2)_\infty} = (q^2;\, q^2)_\infty(-q;\, q)_\infty \\ &= (q;\, q)_\infty(-q;\, q)_\infty(-q;\, q)_\infty \\ &= \frac{1}{2}(q;\, q)_\infty(-1;\, q)_\infty(-q;\, q)_\infty.\end{aligned}$$

Applying the Jacobi triple product identity, we have

$$\begin{aligned}\frac{(q^2;\, q^2)_\infty}{(q;\, q^2)_\infty} = \frac{1}{2}\sum_{n=-\infty}^{+\infty} q^{\binom{n}{2}} &= \frac{1}{2}\Big\{\sum_{n=1}^{+\infty} q^{\binom{n}{2}} + \sum_{n=0}^{+\infty} q^{\binom{-n}{2}}\Big\} \\ &= \frac{1}{2}\Big\{\sum_{n=0}^{+\infty} q^{\binom{1+n}{2}} + \sum_{n=0}^{+\infty} q^{\binom{n+1}{2}}\Big\}\end{aligned}$$

where the substitution $n \to 1+n$ has been made for the first sum and $\binom{-n}{2} = \binom{1+n}{2}$ for the second sum. Canceling the factor 1/2 by two times of the same sum, we have the triangle number theorem.

Now, we prove pentagon number theorem. Classifying the factors of product $(q;q)_\infty$ according to the residues of the indices modulo 3, we have

$$(q;\, q)_\infty \;=\; (q^3;\, q^3)_\infty (q;\, q^3)_\infty (q^2;\, q^3)_\infty.$$

Then the Jacobi triple product identity (C2.5) yields

$$(q;\, q)_\infty \;=\; \sum_{n=-\infty}^{\infty} (-1)^n q^{3\binom{n}{2}+n} \;=\; \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n}{2}(3n+1)}$$

which is Euler's pentagon number theorem. □

C2.6. The quintuple product identity. Furthermore, we can derive the quintuple product identity

$$\begin{aligned}
[q,\, z,\, q/z; q]_\infty \, \left[qz^2,\, q/z^2; q^2\right]_\infty &= \sum_{n=-\infty}^{+\infty} \left\{1 - zq^n\right\} q^{3\binom{n}{2}} \left(qz^3\right)^n \\
&= \sum_{n=-\infty}^{+\infty} \left\{1 - z^{1+6n}\right\} q^{3\binom{n}{2}} \left(q^2/z^3\right)^n
\end{aligned}$$

and its limit form

$$(q;\, q)_\infty^3 \,(q;\, q^2)_\infty^2 \;=\; \sum_{n=-\infty}^{+\infty} \{1+6n\} q^{\frac{n}{2}(3n+1)}.$$

C2.7. Proof. Multiplying two copies of the Jacobi triple products

$$\begin{aligned}
[q,\, z,\, q/z; q]_\infty &= \sum_{i=-\infty}^{+\infty} (-1)^i q^{\binom{i}{2}} z^i \\
\left[q^2,\, qz^2,\, q/z^2; q^2\right]_\infty &= \sum_{j=-\infty}^{+\infty} (-1)^j q^{j^2} z^{2j}
\end{aligned}$$

we have the double sum expression

$$[q,\, z,\, q/z; q]_\infty \, \left[q^2,\, qz^2,\, q/z^2; q^2\right]_\infty = \sum_{i,\, j=-\infty}^{+\infty} (-1)^{i+j} q^{\binom{i}{2}+j^2} z^{i+2j}.$$

Defining a new summation index $k = i+2j$ and then rearranging the double sum, we can write

$$[q,\, z,\, q/z; q]_\infty\; \left[q^2,\, qz^2,\, q/z^2; q^2\right]_\infty \;=\; \sum_{k=-\infty}^{+\infty} (-1)^k z^k \sum_{j=-\infty}^{+\infty} (-1)^j q^{\binom{k-2j}{2}+j^2}.$$

Noting the binomial relation

$$\binom{k-2j}{2} \;=\; \binom{k}{2} + \binom{2j+1}{2} - 2kj \;=\; \binom{k}{2} + 2j^2 + j - 2kj$$

we find that

$$\begin{aligned}[q,\, z,\, q/z; q]_\infty\; \left[q^2,\, qz^2,\, q/z^2; q^2\right]_\infty \;&=\; \sum_{k=-\infty}^{+\infty} (-1)^k q^{\binom{k}{2}} z^k \\ &\times \sum_{j=-\infty}^{+\infty} (-1)^j q^{3j^2+j-2kj}.\end{aligned}$$

Applying the Jacobi product identity to the inner sum, we get

$$\begin{aligned}\sum_{j=-\infty}^{+\infty} (-1)^j q^{3j^2+j-2kj} \;&=\; \sum_{j=-\infty}^{+\infty} (-1)^j q^{6\binom{j}{2}+2(2-k)j} \\ &=\; [q^6, q^{2+2k}, q^{4-2k}, q^6]_\infty.\end{aligned}$$

This product can be simplified according to the residues of k modulo 3.

- $k = 3m$ with $m \in \mathbb{Z}$:

$$\begin{aligned}[q^6, q^{2+2k}, q^{4-2k}, q^6]_\infty \;&=\; [q^6, q^{2+6m}, q^{4-6m}, q^6]_\infty \\ &=\; \frac{(q^{4-6m}; q^6)_m}{(q^2; q^6)_m}[q^6, q^2, q^4, q^6]_\infty \\ &=\; (-1)^m\, (q^2; q^2)_\infty\, q^{m-3m^2}.\end{aligned}$$

- $k = 1+3m$ with $m \in \mathbb{Z}$:

$$\begin{aligned}[q^6, q^{2+2k}, q^{4-2k}, q^6]_\infty \;&=\; [q^6, q^{4+6m}, q^{2-6m}, q^6]_\infty \\ &=\; \frac{(q^{2-6m}; q^6)_m}{(q^4; q^6)_m}[q^6, q^2, q^4, q^6]_\infty \\ &=\; (-1)^m\, (q^2; q^2)_\infty\, q^{-m-3m^2}.\end{aligned}$$

- $k = 2+3m$ with $m \in \mathbb{Z}$:

$$[q^6, q^{2+2k}, q^{4-2k}, q^6]_\infty \;=\; [q^6, q^{6+6m}, q^{-6m}, q^6]_\infty \;=\; 0$$

because of the presence of zero-factor:

$$(q^{-6m};\ q)_\infty = 0, \quad m \geq 0$$
$$(q^{6+6m}; q)_\infty = 0, \quad m < 0.$$

Substituting these results into the infinity series expression, we obtain

$$\begin{aligned} & [q,\, z,\, q/z; q]_\infty \, \left[q^2,\, qz^2,\, q/z^2; q^2\right]_\infty \\ = & \sum_{k=-\infty}^{+\infty} (-1)^k q^{\binom{k}{2}} z^k [q^6, q^{2+2k}, q^{4-2k}, q^6]_\infty \\ = & (q^2;q^2)_\infty \sum_{m=-\infty}^{+\infty} q^{\binom{3m}{2}+m-3m^2}\ z^{3m} \\ - & (q^2;q^2)_\infty \sum_{m=-\infty}^{+\infty} q^{\binom{1+3m}{2}-m-3m^2} z^{1+3m} \\ = & (q^2;q^2)_\infty \sum_{m=-\infty}^{+\infty} q^{\frac{3m^2-m}{2}} \{1-zq^m\} z^{3m}. \end{aligned}$$

Dividing both sides by $(q^2;q^2)_\infty$, we get the quintuple product identity:

$$[q,\, z,\, q/z; q]_\infty \, \left[qz^2,\, q/z^2; q^2\right]_\infty = \sum_{m=-\infty}^{+\infty} q^{3\binom{m}{2}} \{1-zq^m\} (qz^3)^m.$$

Splitting the last sum into two and then reverse the first sum, we have

$$\begin{aligned} & [q,\, z,\, q/z; q]_\infty \times \left[qz^2,\, q/z^2; q^2\right]_\infty \\ = & \sum_{m=-\infty}^{+\infty} q^{3\binom{m}{2}} \{1-zq^m\} (qz^3)^m \\ = & \sum_{m=-\infty}^{+\infty} q^{3\binom{m}{2}+m} z^{3m} - \sum_{m=-\infty}^{+\infty} q^{3\binom{m}{2}+2m} z^{1+3m} \\ = & \sum_{n=-\infty}^{+\infty} q^{3\binom{n}{2}+2n} z^{-3n} - \sum_{m=-\infty}^{+\infty} q^{3\binom{m}{2}+2m} z^{1+3m} \\ = & \sum_{n=-\infty}^{+\infty} q^{3\binom{n}{2}} \{1-z^{1+6n}\} \left(q^2/z^3\right)^n \end{aligned}$$

which is exactly the second version of the quintuple product identity.

Finally, dividing both sides by $1-z$

$$[q,\, qz,\, q/z; q]_\infty\ \left[qz^2,\, q/z^2; q^2\right]_\infty \;=\; \sum_{n=-\infty}^{+\infty} q^{3\binom{n}{2}} \frac{1-z^{1+6n}}{1-z} \left(q^2/z^3\right)^n$$

and then letting $z \to 1$, we get the limiting case of the quintuple product identity

$$(q;\, q)_\infty^3\, (q;\, q^2)_\infty^2 \;=\; \sum_{n=-\infty}^{+\infty} \{1+6n\}\, q^{\frac{n}{2}(3n+1)}.$$

C3. The finite form of Euler's pentagon number theorem

C3.1. Theorem. The classification of partitions enumerated by $(-qx;q)_n$ with respect to the Durfee rectangles of $(k+\epsilon)\times k$ leads us to the following finite form of the Euler pentagon number theorem.

Denote by $[\theta]$ the integral part of real number θ. Then there holds

$$\begin{aligned}(-qx;\, q)_n \;&=\; \sum_{k=0}^{[\frac{n-\epsilon}{2}]} q^{k(k+\epsilon)+\binom{k}{2}} \begin{bmatrix} n-k-\epsilon \\ k \end{bmatrix} (-qx;\, q)_{k+\epsilon} \\ &\times\; \frac{1+xq^{2k+\epsilon}-q^{1+n-k-\epsilon}(1+xq^{k+\epsilon})}{(1+xq^{k+\epsilon})\,(1-q^{1+n-2k-\epsilon})}\, x^k.\end{aligned}$$

C3.2. Proof. For the partitions into distinct parts $\le n$ enumerated by $(-qx;q)_n$, they are divided by the Durfee rectangles of $(k+\epsilon)\times k$ into three pieces:

A: the Durfee rectangle $(k+\epsilon)\times k$ itself with enumerator $x^k q^{k(k+\epsilon)}$.
B: the piece of partitions right to the Durfee rectangle counted by

$$\begin{cases} \begin{bmatrix} n-k-\epsilon \\ k \end{bmatrix} q^{\binom{1+k}{2}}, & \text{with } k \text{ parts}, \\ \begin{bmatrix} n-k-\epsilon \\ k-1 \end{bmatrix} q^{\binom{k}{2}}, & \text{with } k-1 \text{ parts}. \end{cases}$$

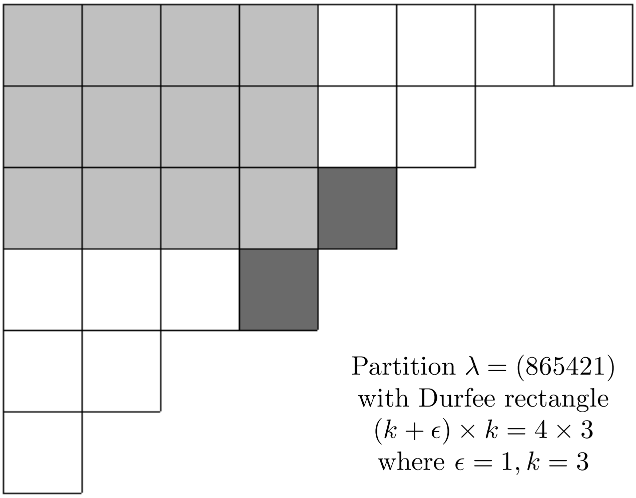
C: the piece of partitions below the Durfee rectangle enumerated by

$$\begin{cases} (-qx;\, q)_{k+\epsilon}, & \text{when } \mathbf{B} \text{ has } k \text{ parts}, \\ (-qx;\, q)_{k+\epsilon-1}, & \text{when } \mathbf{B} \text{ has } k-1 \text{ parts}. \end{cases}$$

Therefore for the fixed Durfee rectangle **A**, the enumerator for the rest of partitions is given by the combination of **B** and **C** as follows

$$\begin{aligned} & q^{\binom{1+k}{2}} \begin{bmatrix} n-k-\epsilon \\ k \end{bmatrix} (-qx;\, q)_{k+\epsilon} + q^{\binom{k}{2}} \begin{bmatrix} n-k-\epsilon \\ k-1 \end{bmatrix} (-qx;\, q)_{k+\epsilon-1} \\ = \quad & q^{\binom{k}{2}} \begin{bmatrix} n-k-\epsilon \\ k \end{bmatrix} \frac{1+xq^{\epsilon+2k} - q^{1+n-k-\epsilon}(1+xq^{k+\epsilon})}{(1+xq^{k+\epsilon})\,(1-q^{1+n-2k-\epsilon})} (-qx;\, q)_{k+\epsilon}. \end{aligned}$$

Summing the last expression over $0 \le k \le [(n-\epsilon)/2]$, we get the identity stated in Theorem C3.1.



Partition $\lambda = (865421)$
with Durfee rectangle
$(k+\epsilon) \times k = 4 \times 3$
where $\epsilon = 1, k = 3$

C3.3. Corollary. This formula contains the following well-known results as special cases:

- The limiting version with two parameters ($n \to \infty$)

$$(-qx;\, q)_\infty \;=\; \sum_{n=0}^{\infty} q^{n(n+\epsilon)+\binom{n}{2}} \frac{1+xq^{2n+\epsilon}}{1+xq^{n+\epsilon}} \frac{(-qx;\, q)_{n+\epsilon}}{(q;\, q)_n}\, x^n.$$

- The Sylvester formula ($\epsilon = 1$, $x = -y/q$ and $n \to \infty$)

$$(y;\, q)_\infty \;=\; \sum_{n=0}^{\infty} (-y)^n \left\{1 - yq^{2n}\right\} \frac{(y;\, q)_n}{(q;\, q)_n} q^{\frac{3n^2-n}{2}}.$$

- The Euler pentagon number theorem ($\epsilon = 0$, $x = -1$ and $n \to \infty$)

$$(q;\, q)_\infty \quad = \quad 1 + \sum_{n=1}^{\infty} (-1)^n \left\{1 + q^n\right\} q^{\frac{3n^2-n}{2}}.$$

Remark The Euler pentagon number theorem is also a particular case of the Sylvester formula. In fact, for $y \to 1$, the limit can be computed term by term as follows:

$$\begin{aligned}(q;\, q)_\infty &= \sum_{n=0}^{\infty} \frac{q^{\frac{3n^2-n}{2}}}{(q;\, q)_n} \lim_{y \to 1} (-y)^n \frac{(1 - yq^{2n})(y;\, q)_n}{1 - y} \\ &= 1 + \sum_{n=1}^{\infty} (-1)^n \big\{1 - q^{2n}\big\} \frac{(q;\, q)_{n-1}}{(q;\, q)_n} q^{\frac{3n^2-n}{2}} \\ &= 1 + \sum_{n=1}^{\infty} (-1)^n \big\{1 + q^n\big\} q^{\frac{3n^2-n}{2}}.\end{aligned}$$