CHAPTER 1: ALMOST MINIMAL BOUNDARIES

We begin with an informal presentation of the material to be discussed in the sequel. While doing this, free use will be made of concepts and elementary results that will be discussed more deeply later on (especially in Chapter 3).

1.1. We fix an open set Ω in $\mathbb{R}^n$, $n \geq 2$, and consider sets $E, F, \ldots$ whose boundaries $\partial E, \partial F, \ldots$ have locally finite "surface area" in Ω:

$$|D\phi_E|(A) < + \infty \qquad \forall A \subset\subset \Omega .$$

The quantity $|D\phi_E|(A)$ may be thought of as the area (in some generalized sense) of $\partial E \cap A$, where A is an arbitrary open an bounded set, strictly contained in Ω; indeed, it coincides with $H_{n-1}(\partial E \cap A)$ whenever ∂E is, locally within Ω, a smooth hypersurface (section 3.1).

A basic definition and a corresponding fundamental result are now in order.

1.2. Definition of minimal boundaries.

We say that the boundary of E is *minimal* in Ω iff

(1.1) $\qquad |D\phi_E|(A) \leq |D\phi_F|(A) \qquad \forall A \subset\subset \Omega, \forall F : F \triangle E \subset\subset A$

.e., iff any local variation of E in Ω increases surface area (Fig.1).
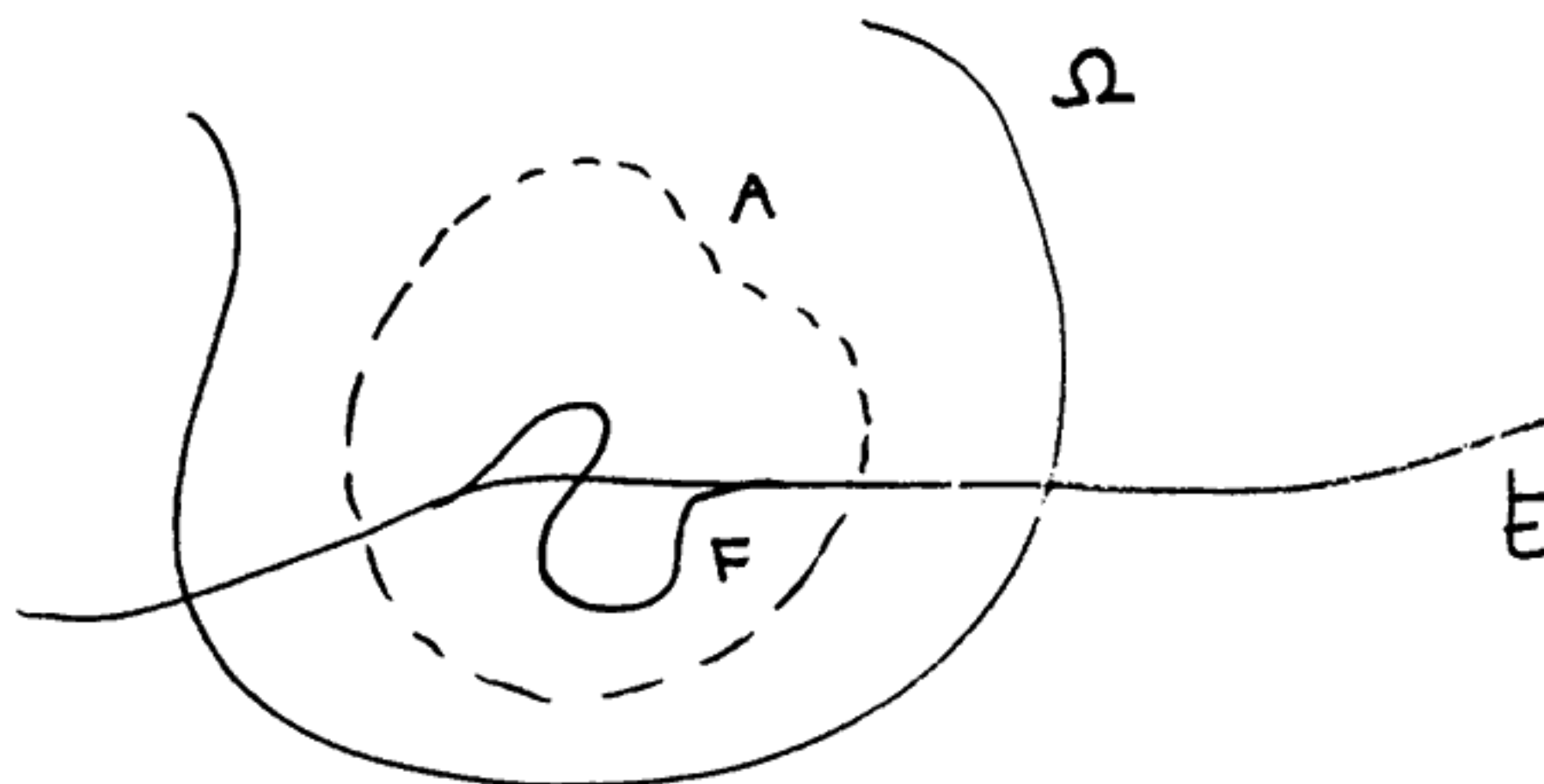
FIGURE 1

1.3. Regularity of minimal boundaries.

Let E have minimal boundary in $\Omega \subset \mathbb{R}^n$. Then

(r_1) $\qquad \partial^*E \cap \Omega$ is an analytic hypersurface

(r_2) $\qquad H_{n-1}\big[(\partial E \smallsetminus \partial^*E) \cap \Omega\big]= 0$.

Furthermore, assuming that $\{E_h\}$ be a sequence of sets with minimal boundary in Ω, and that $\{x_h\}$ be a corresponding sequence of points, satisfying:

$$E_h \to E_\infty \ , \quad x_h \in \partial E_h; \quad x_h \to x_\infty \in \Omega$$

then

(r_3) $\qquad x_\infty \in \partial E_\infty$.

If in addition $x_\infty \in \partial^*E_\infty$, then

(r_4) $\qquad x_h \in \partial^*E_h$ for any large h, and $\nu_{E_h}(x_h) \to \nu_{E_\infty}(x_\infty)$.

We recall that ∂^*E denotes the "reduced boundary" of E, i.e. the

collection of those points x e ∂E where an approximate inner unit normal vector $\nu_E(x)$ exists, and that the convergence $E_h \to E_\infty$ is to be intended in the L^1_{loc} - sense on Ω. See section 3.1 again.

1.4. Conclusion (r_1) above, which undoubtedly contains the essence of the Regularity Theorem, was proved by E. de Giorgi in 1960-61 (see [8] and [9]), and then rederived together with (r_2) in 1965 by M.Miranda (see [28])[1]. Two years after, Miranda proved (r_3) and (r_4) as well, see [29]. Thus, in 1967 the Regularity Theorem for minimal boundaries (in the form appearing above) was completely demonstrated. In the setting of Caccioppoli sets, i.e. sets with finite surface area, Theorem 1.3 may well be called the basic regularity result.

In the meantime, various different settings were proposed, in which the classical questions related to Plateau's problem (the problem of finding a surface of least area among those surfaces which span a given curve) could receive a satisfactory answer. We mention the work of Reinfenberg [32,33], Féderer-Fleming [14], Fleming [15], Almgren [3], Allard [1], etc. A considerable effort was directed toward a complete understanding of·the structure of the singularities of minimal boundaries: the work Fleming [15], Triscari [38], Almgren [2] and Simons [35] culminated in the celebrated proof -first given by Bombieri, De Giorgi and Giusti in 1969 (see [6] and also [26])- of the minimality of the cone

$$C = \{x \in \mathbb{R}^8 : x_1^2 + \ldots + x_4^2 < x_5^2 + \ldots + x_8^2\}$$

(*Simons' cone*), which is singular at the origin. As a consequence, the best possible estimate of the Hausdorff dimension of the singular set $\partial E \setminus \partial^*E$ could be obtained by H.Federer [13], thus improving (r_2) above.

More general variational problems of "least area" type have since been considered, especially those concerning surfaces of prescribed mean curvature, possibly with obstacles or subject to given constraints. In this respect, the work of Almgren [4] is really impressive, for both the deepness and the generality of the results obtained. Working with different methods, E.Bombieri [5] and R. Schoen - L. Simon [34] developed quite recently a simplified version of (part of) Almgren's Regularity Theory.[2]

Restricting our attention to the theory of Caccioppoli sets in $\mathbb{R}^n$, we should mention the important contribution of Miranda [30] and Massari [23, 24], on the obstacle problem for minimal boundaries and, respectively, on the regularity of boundaries of prescribed mean curvature. These two problems will be properly discussed later on in this chapter.

Urged by the consideration of these and other particular cases, one is naturally led to the search of a class of "almost minimal boundaries", for which a Regularity Theorem like Theorem 1.3 could be proved. In this respect, the following definition seems quite natural:

1.5. Definition of almost minimal boundaries[3].

The boundary of E is said to be *almost minimal* in $\Omega \subset \mathbb{R}^n$ iff for every A cc Ω there exist T $\in$ (0,dist(A,$\partial\Omega$)) and $\alpha:(0,T)\to[0,+\infty)$, with α non-decreasing and[4] $\alpha(t) = o(1)$, such that

$$(1.2) \qquad |D\phi_E|(B_{x,t}) \leq |D\phi_F|(B_{x,t}) + \alpha(t)\cdot t^{n-1}$$

for every x $\in$ A, every t $\in$ (0,T), and every F : F $\triangle$ E cc $B_{x,t}$

As usual, $B_{x,t}$ denotes the open n-ball with centre x and radius t. See Fig. 2.
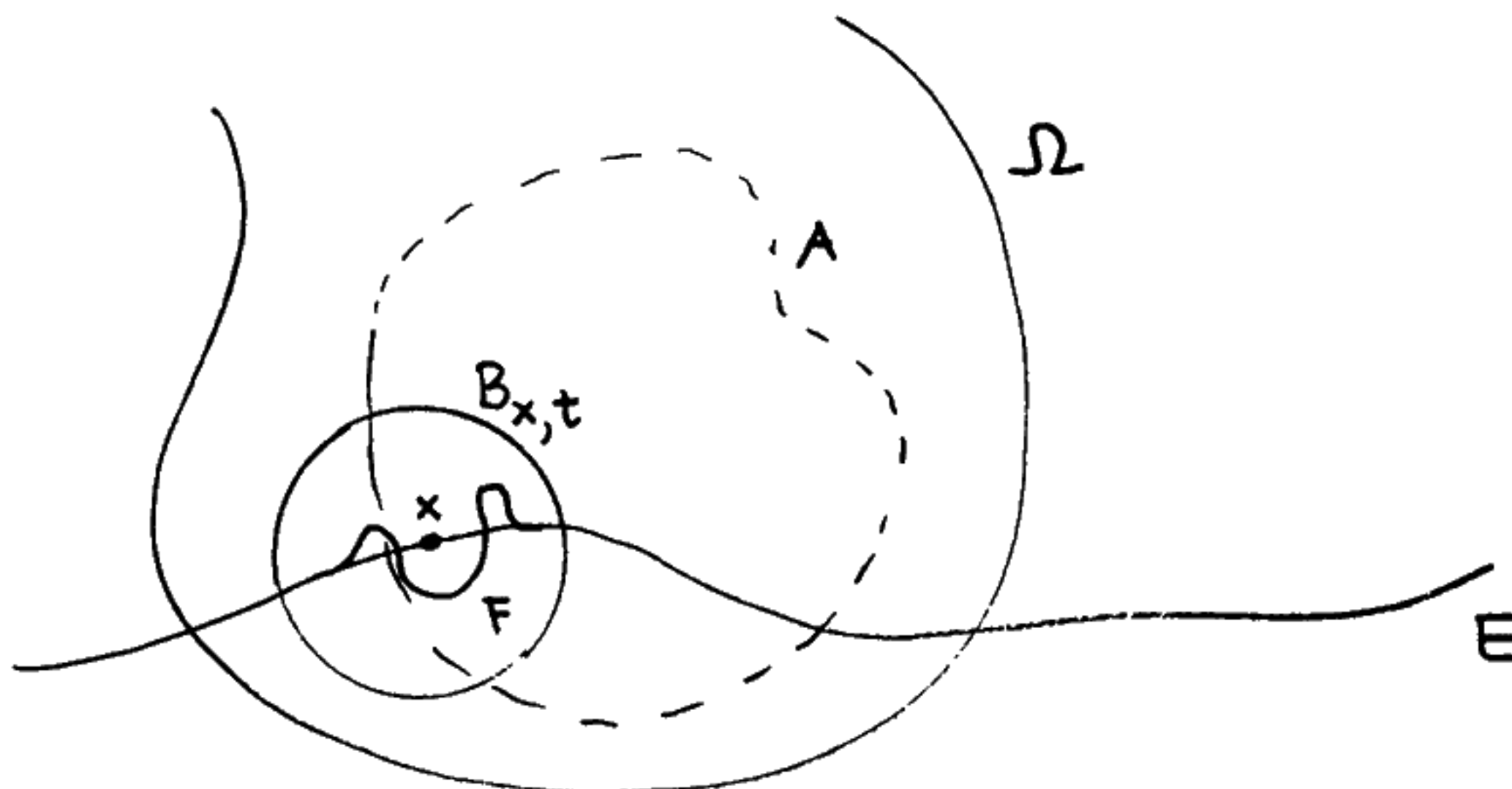
FIGURE 2.

1.6. The assumptions on $\alpha(t)$ are in some sense the minimal ones we can make if we want to prove regularity. Indeed, let us consider a smooth hypersurface S in $\mathbb{R}^n$, and let us choose R > 0 and the reference system so that

$$S \cap Q_R = \text{graph of } u \text{ over } B_R'$$

with $u \in C^1(B_R')$, $u(0)=0$, $Du(0)=0$. Here, Q_R denotes the "vertical" cylinder

$$Q_R = \{x = (x',x_n) \in \mathbb{R}^n : |x'| < R, |x_n| < R\}$$

and B_R' its projection on the space of the first (n-1) variables:

$$B_R' = \{x' = (x_1,\ldots,x_{n-1}) \in \mathbb{R}^{n-1}, |x'| < R\} .$$

See Fig. 3.
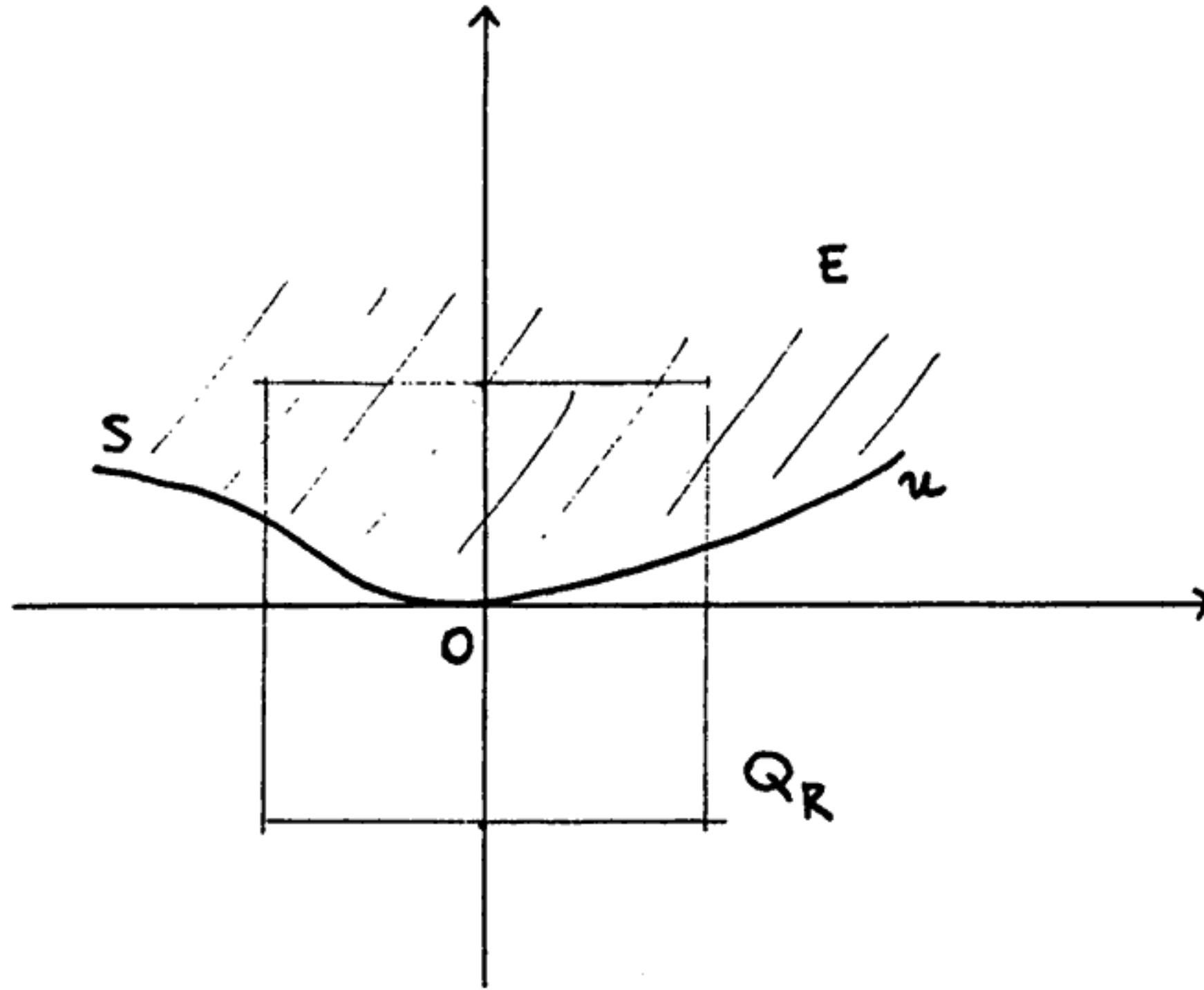
FIGURE 3.

Now, given an (n-1)-ball $B'_t \subset B'_R$ (with centre O), and a function $v \in C^1(B'_R)$ such that the support of u-v lies in B'_t, we have:

$$\int_{B'_t} (1+|Du|^2)^{\frac{1}{2}}dx' - \int_{B'_t} (1+|Dv|^2)^{\frac{1}{2}} dx' \leq$$

$$\leq \int_{B'_t} [(1+|Du|^2)^{\frac{1}{2}} - 1]dx' \leq \frac{1}{2} \int_{B'_t} |Du|^2 dx' \leq \alpha(t) \cdot t^{n-1}$$

provided we take:

$$\alpha(t) \equiv \frac{1}{2} \omega_{n-1} \cdot \sup_{B'_t} |Du(x')|^2 .$$

From this one easily concludes that (1.2) holds for E = epi(u), with

a function $\alpha(t)$ of the type described in Definition 1.5. Consequently, any set with smooth (C^1) boundary has almost minimal boundary, in the sense of Def. 1.5.

1.7. On the other hand, we see that the cone

$$E = \{x = (x_1, x_2) : |x_1| < |x_2|\} \subset \mathbb{R}^2$$

whose boundary has a singularity at 0, satisfies

$$|D\phi_E|(B_t) \leq |D\phi_F|(B_t) + \alpha(t) \cdot t \qquad \forall F : F \triangle E \subset\subset B_t$$

the *best* choice fot $\alpha(t)$ being the *constant* $2(2-\sqrt{2})$ (see Fig. 4). This also shows the "necessity" of the assumptions on $\alpha(t)$.
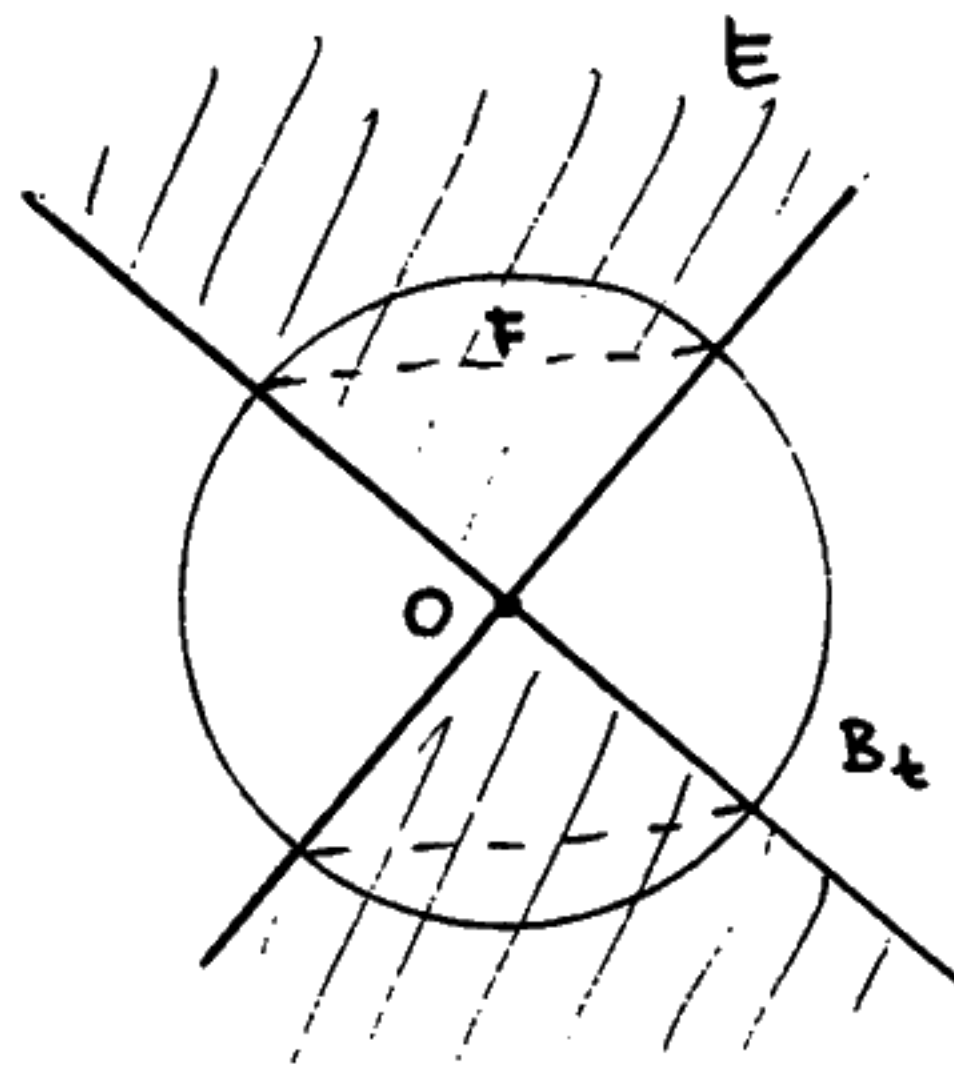

FIGURE 4.

.8. A second important class of almost minimal boundaries is .istituted by surfaces of prescribed mean curvature which satisfies suitable integrability condition. To be specific, let us consider .ven function $H \in L^1_{loc}(\Omega)$, together with a *local minimizer* E of the following functional[5]

$$(1.3) \qquad \mathscr{F}_H(F,A) = |D\phi_F|(A) + \int_{A \cap F} H(x)dx$$

which is defined for any Caccioppoli set F and any A cc Ω . That is, suppose that

$$(1.4) \qquad \mathscr{F}_H(E,A) \leq \mathscr{F}_H(F,A) \qquad \forall A \text{ cc } \Omega \text{ , } \forall F : F \vartriangle E \text{ cc } A.$$

Whenever $\partial E \cap A$ is locally of class C^2, and H is a continuous function on Ω , we see immediately that the mean curvature of ∂E coincides, at any point $x \in \partial E \cap \Omega$, with H(x) (just compute the first variation of $\mathscr{F}_H$!). Motivated by the preceding observation, we call the local minimizers of $\mathscr{F}_H$: "sets of generalized mean curvature H in Ω".

Now, for any ball B_t cc Ω, we have from (1.3),(1.4):

$$(1.5) \qquad |D\phi_E|(B_t) \leq |D\phi_F|(B_t) + \int_{B_t} |H(x)|dx$$

Assuming $H \in L^n_{loc}(\Omega)$, we find (according to Hölder's inequality) that (1.2) holds true, with (essentially)

$$\alpha(t) = \omega_n^{1-1/n} \left(\int_{B_t} |H|^n dx \right)^{1/n}$$

More precisely, we observe that if E is a local minimizer of $\mathscr{F}_H$ in Ω, A cc Ω , and $T \in (0,\text{dist}(A,\partial\Omega))$, then $\int_{B_{x,t}} |H|^n dy$ is, for any $t \in (0,T)$, a continuous function of $x \in \bar{A}$, which will then achieve its maximum value at some point $x_t \in \bar{A}$. Setting

$$(1.6) \qquad \alpha(t) = \omega_n^{1-1/n} \cdot \left(\int_{B_{x_t,t}} |H|^n dx \right)^{1/n}$$

we get (1.2), with a function $\alpha(t)$ depending on A, non-decreasing on (0,T) and infinitesimal at 0. We shall return to this problem later on, in section 1.14.

We are now in a position to state our main result (compare with Theorem 1.3).

1.9. Regularity Theorem for almost minimal boundaries.

Let E have almost minimal boundary in $\Omega \subset \mathbb{R}^n$, in the sense of Def. 1.5. Assume in addition that $\alpha(t)$ (the function appearing in (1.2)) be such that

$$t^{-1} \cdot \alpha(t) \quad \text{is non-increasing on} \quad (0,T)$$

and

$$\int_0^T t^{-1} \cdot \alpha^{1/2}(t) \, dt < + \infty.$$

Then

$(R_1) \quad \partial^* E \cap \Omega \quad$ is a C^1 hypersurface

$(R_2) \quad H_s \left[(\partial E \smallsetminus \partial^* E) \cap \Omega \right] = 0 \qquad \forall s > n - 8.$

Furthermore, assuming that $\{E_h\}$ be a sequence of sets with *uniformly* almost minimal boundaries in Ω (i.e., such that (1.2) holds for every E_h, with T and $\alpha(t)$ independent of h), and that $\{x_h\}$ be a corresponding sequence of points, satisfying

$$E_h \to E_\infty \ , \ x_h \in \partial E_h, \ x_h \to x_\infty \in \Omega$$

then

$$(R_3) \qquad\qquad x_\infty \in \partial E_\infty \ .$$

If in addition $x_\infty \in \partial^* E_\infty$, then

$$(R_4) \qquad x_h \in \partial^* E_h \quad \text{for any large h, and} \quad \nu_{E_h}(x_h) \to \nu_{E_\infty}(x_\infty).$$

1.10. Regarding (R_1), we have specifically the following estimate of the oscillation of the unit normals to ∂E :

$$(1.7) \qquad |\nu_E(x) - \nu_E(y)| \le c_1 \int_0^{|x-y|} t^{-1} \alpha^{1/2}(t) dt + c_2 |x-y|^{1/2}$$

which holds for every $x, y \in \partial^* E$ such that $|x-y|$ is sufficiently small. Thus, we see that the integrability of $t^{-1} \alpha^{1/2}(t)$ is an essential ingredient of the Regularity Theorem.

The other hypotesis of the Theorem, namely, the monotonicity of $t^{-1}\alpha(t)$, is more a convenience than a necessity, and it is assumed only with the aim of simplifying calculations. At any rate, we see that when α is a (non decreasing, infinitesimal at 0, and) concave function of $t \in (0,T)$, then it certainly satisfies that condition. See Fig. 5.
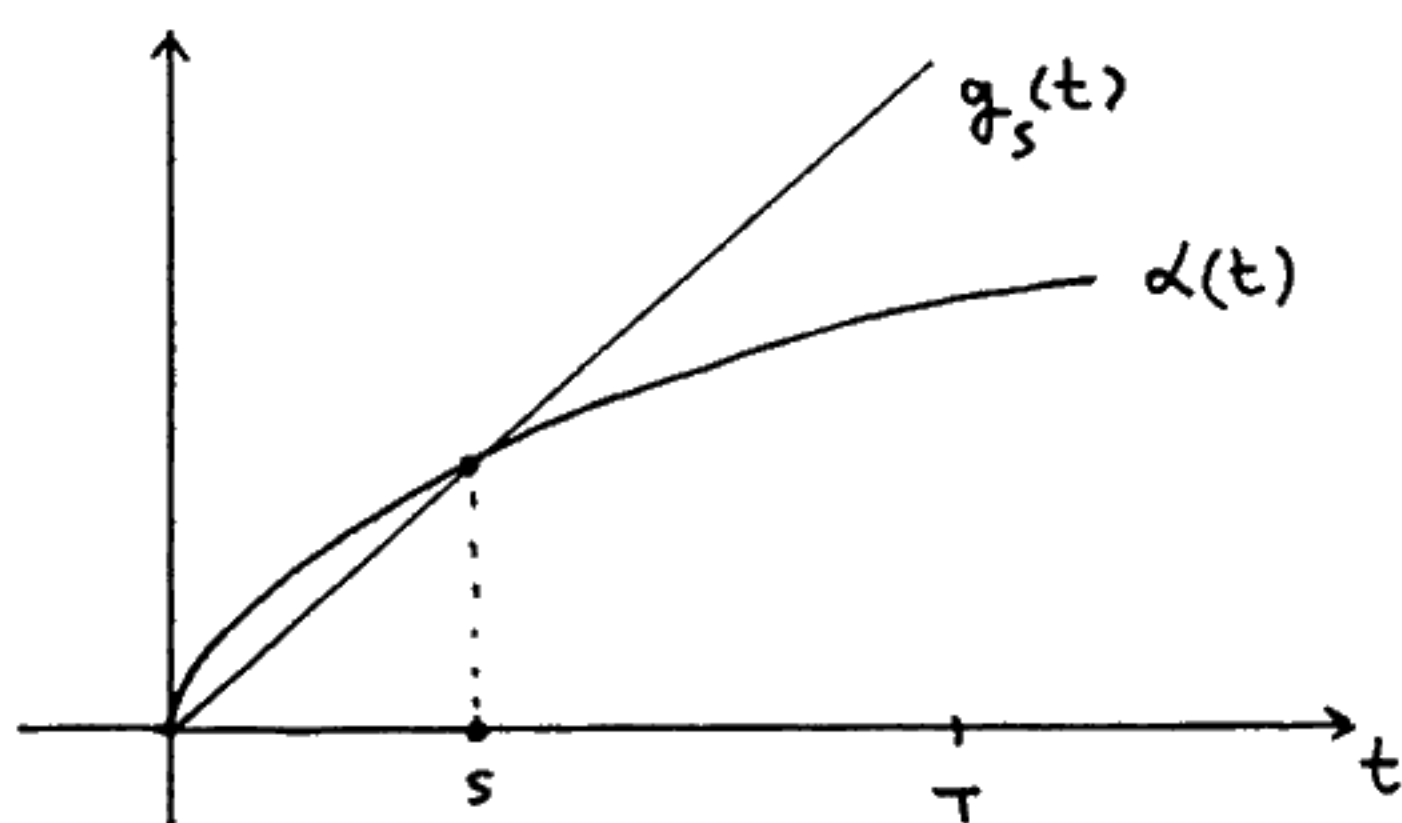
$$\alpha(t) \geq g_s(t)$$

$$= (t/s)\cdot\alpha(s)$$

$$\forall t \in (0,s).$$

FIGURE 5.

The proof of Theorem 1.9 occupies the second part of the present work Chapters 3 and 4). Before starting with the formal demonstration, it seems appropriate to illustrate with examples the applicability of the Theorem itself, and to discuss in some details the method of the proof.

For convenience ot the reader, we list the assumptions on $\alpha(t)$ under which Theorem 1.9 will be proved.

1.11. <u>Complete set of hypotheses on $\alpha(t)$</u>.

(α_1) $\alpha : (0,1) \to \mathbb{R}$ is non-decreasing and bounded

(α_2) $\alpha(t) = o(1)$

(α_3) $t^{-1}\alpha(t)$ is non-increasing on $(0,1)$

(α_4) $\int_0^1 t^{-1} \alpha^{1/2}(t)dt < + \infty.$

1.12. Now we consider some explicit examples.

The simplest choice is perhaps $\alpha(t) = c.t^{2\alpha}$; conditions above are then all satisfied, for any $c \geq o$ and any $\alpha \in (0,\frac{1}{2}]$.

Moreover, in this case[6] (1.7) becomes

(1.8) $$|v_E(x) - v_E(y)| \leq \text{const} \cdot |x-y|^{\alpha}$$

which amounts to saying that $\partial^*E \in C^{1,\alpha}$. In a sense, this is an optimal result, since the converse is also true (and easy to prove, see Example 1.14 (v) below); that is, if $E = \text{epi}(u)$ with $u \in C^{1,\alpha}$, then ∂E is almost minimal, with $\alpha(t) \leq \text{const} \cdot t^{2\alpha}$.

For $\varepsilon > 0$, the function

$$\alpha(t) = c \left[\lg(e/t)\right]^{-2(1+\varepsilon)}$$

(truncated at a suitable level, in order to save concavity), also satisfies $(\alpha_1 - \alpha_4)$. On the contrary, it does not satisfy (α_4) when $\varepsilon = 0$.

A similar behaviour is exhibited by the function

$$\alpha(t) = c \left[\lg(e/t)\right]^{-2} \cdot \left[\lg(e \ \lg(e/t))\right]^{-2(1+\varepsilon)} .$$

1.13. Before giving examples of almost minimal boundaries, we introduce the following functional

(1.9) $\quad \psi(E,A) = |D\phi_E|(A) - \inf \{ |D\phi_F|(A) : F \triangle E \ \subset\subset A \}$

which is defined for every Caccioppoli set E and every $A \subset\subset \mathbb{R}^n$. With the aid of ψ , the definition of almost minimality can be restated in a more compact form, just by replacing (1.2) of Def. 1.5 by

(1.10) $\quad \psi(E,B_{x,t}) \leq \alpha(t) \cdot t^{n-1} \qquad \forall x \in A , \ \forall t \in (0,T).$

The quantity $t^{1-n} \psi(E,B_{x,t})$ may be called "Deviation from minimality of E in $B_{x,t}$", abbreviated: Dev(E,x,t). Thus, almost minimal boundaries are those boundaries whose deviation from minimality is controlled from above, in any ball $B_{x,t}$, by a suitable function of the radius t, which is non-decreasing in t and infinitesimal at 0.

Simple properties of the functional ψ are stated and proved in Chapter 3, see (3.10)-(3.13). For the present usage, we anticipate that ψ is non-decreasing with respect to the second variable, i.e. $\psi(E,A_1) \le \psi(E,A_2)$ whenever $A_1 \subset A_2$.

1.14. Examples of almost minimal boundaries.

(i) *Minimal boundaries* (Def. 1.2) are evidently almost minimal, with $\alpha(t) \equiv 0$ (and $\psi \equiv 0$).

(ii) *Boundaries with prescribed generalized mean curvature.*

We saw in the preceding pages that any local minimizer of the functional $\mathscr{F}_H$ (defined by (1.3)) in Ω, corresponding to a mean curvature function $H \in L^n_{loc}(\Omega)$, has almost minimal boundary in Ω, since it verifies (1.2) (or equivalently (1.10)), with a function $\alpha(t)$ - given by (1.6) - satisfying (α_1) and (α_2) of Hypotheses 1.11. It may happen however, that the remaining assumptions $(\alpha_3),(\alpha_4)$ are not satisfied. We see this with the aid of the following example.

Consider, for $n = 2$, the function

(1.11) $$H(x) = H(x_1,x_2) = -Mr^{-1}(\lg(e/r))^{-5/2}$$

where $r = |x| \in (0,1]$ and $M > 1$. For every $t \in (0,1]$ we have

$$\int_{B_t} |H|^2 \, dx = (\pi/2)M^2(\lg(e/t))^{-4}$$

so that $H \in L^2(B_1)$. However, with $\alpha(t) = (\int_{B_t} |H|^2 dx)^{1/2}$, see (1.6),

we find

$$\int_0^1 t^{-1} \alpha^{1/2}(t) \, dt = +\infty$$

thus violating condition (α_4).

Nevertheless, putting

(1.12) $H(x) = 0$ for $r = |x| \in (1,2]$

and taking into account the symmetry of the problem, we realize immediately that $E = B_1$ is local minimizer of $\mathscr{F}_H$ in $\Omega = B_2$, at least when M is large enough. Indeed, if

$$M > [\int_0^1 (\lg(e/s))^{-5/2} \, ds]^{-1}$$

then B_1 is the *unique* solution to the problem [5]

$$|D\phi_E|(\bar{B}_2) + \int_E H(x) dx \to \min, \quad \text{with } E \subset B_2.$$

Therefore, the choice (1.11),(1.12) for H provides no counterexample to regularity! Actually, the question whether a regularity theorem holds for boundaries of prescribed mean curvature $H \in L^n_{loc}(\Omega)$ has not been settled in full generality. We shall return to this question in a moment, after discussing the general case $H \in L^p_{loc}(\Omega)$,with either $1 \le p < n$ or $n < p \le +\infty$.

In the former instance, simple examples show that singularities may appear, even in low dimension; while, on the contrary, the conclusions ot Theorem 1.9 hold in the latter case.

We see this as follows: first, notice that the lipschitz function

$w(x) = |x|$, $x \in \mathbb{R}^n$, $n \geq 2$, is a *weak solution* of the non-homogeneous minimal surface equation

$$(1.13) \qquad \text{Div} \ \frac{Dw(x)}{(1+|Dw(x)|^2)^{\frac{1}{2}}} = h(x)$$

corresponding to $h(x) = (n-1)/\sqrt{2}|x|$ - a function which belongs to the Lebesgue class $L_{loc}^{\tau n}(\mathbb{R}^n)$ for every $\tau < 1$. By this we mean, as usual, that

$$\int < \frac{Dw}{(1+|Dw|^2)^{\frac{1}{2}}} \ , \ D\phi> \ dx = - \int h\phi \ dx \qquad \forall \phi \in C_0^1(\mathbb{R}^n).$$

(w is of course a classical C^2 solution of (1.13) in $\mathbb{R}^n \smallsetminus \{0\}$). Conse_quently, the mean curvature of the graph of w (an n-dimensional cartesian surface in $\mathbb{R}^{n+1}$) is summable to any power less than n (notice that when n=1, the corresponding mean curvature is given by $\sqrt{2}$ times the Dirac mass at 0).

Next, put $E = \text{epi}(u) \subset \mathbb{R}^n$, with $u(x') = |x'|$, $x' \in \mathbb{R}^{n-1}$, $n \geq 2$, and consider a solution G of the following problem [5]

$$|D\phi_G|(\bar{B}_1) + \int_{B_1 \cap G} H(x)dx \rightarrow \min, \quad \text{with} \ G\smallsetminus\bar{B}_1 = E\smallsetminus\bar{B}_1$$

where $H(x) = -c/|x|$ if $x \in E$, = 0 otherwise (c being a large positive constant). Plainly, $G \cap B_1 \subset E \cap B_1$, since E is convex and H vanishes outside E. Furthermore, $0 \in \partial G$, suice from

$$|D\phi_{E_r}| + \int_{E_r} Hdx < 0 \qquad\qquad (E_r \equiv E \cap B_{o,r})$$

which holds $\forall r \in (0,1)$ if c is large enough, we derive

$$|\Gamma_{G \cup E_r}|\,(\bar{B}_1) + \int\limits_{(G \cup E_r) \cap B_1} Hdx < |D\phi_G|\,(\bar{B}_1) + \int\limits_{G \cap B_1} Hdx$$

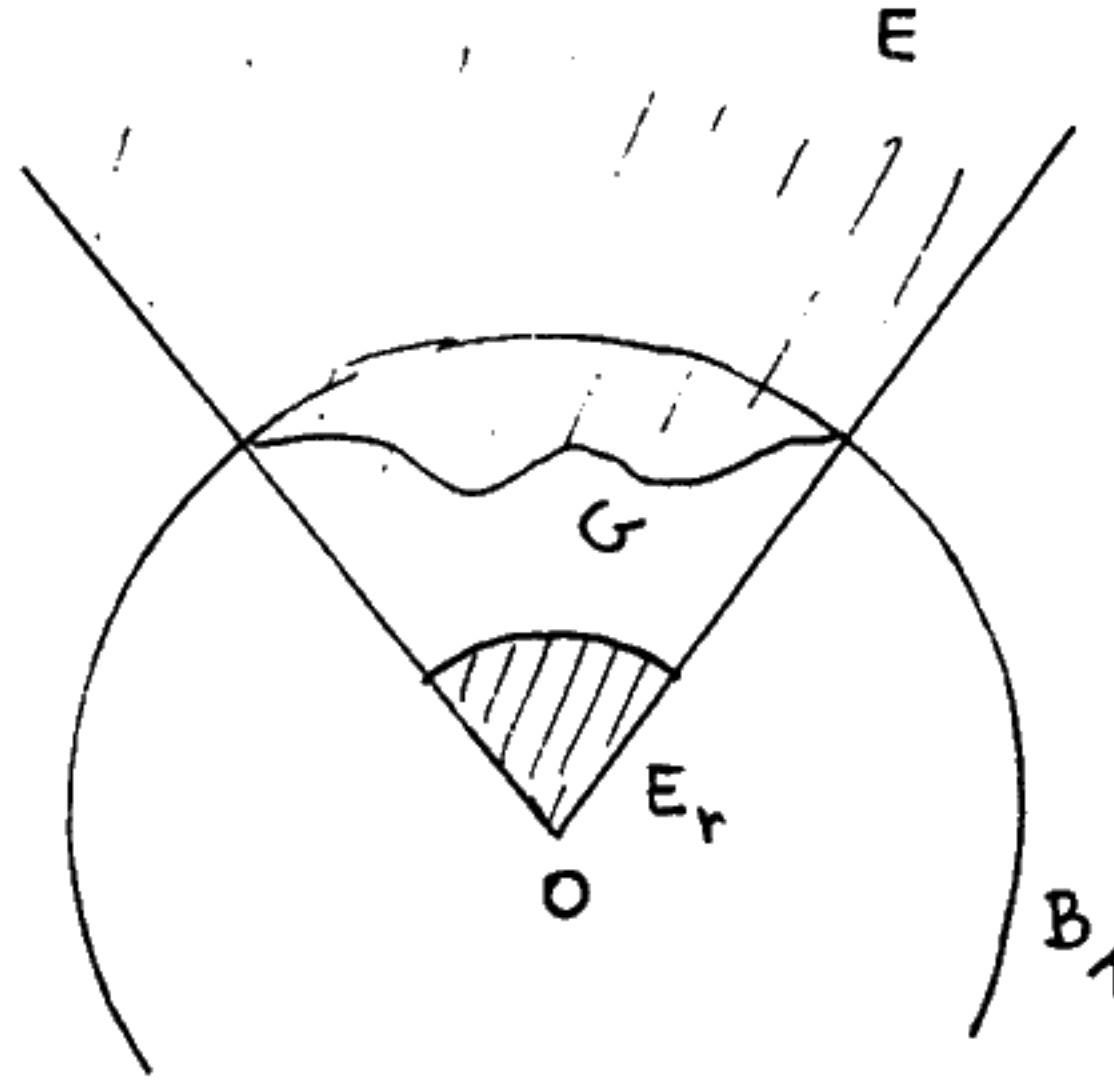whenever $G \cap E_{2r} = \emptyset$ (see Fig. 6).

FIGURE 6.

In conclusion, G is a local minimizer of $\mathscr{F}_H$ in B_1, with $H \in L^{\tau n}$ (B

$\forall \tau < 1$, and 0 is a singular point of ∂G, thus showing that Theorem 1.9 (particularly, conclusion (R_2)) does not hold for boundaries of prescribed mean curvature $H \in L^p_{loc}(\Omega)$, with $p < n$.

On the other hand, if E is a local minimizer of $\mathscr{F}_H$ in Ω, with $H \in L^p_{loc}(\Omega)$ and $p > n$, and if $A \subset\subset \Omega$, $T < dist(A, \partial\Omega)$, $x \in A$, $t \in (0, T)$ then (1.5), (1.10) and Hölder inequality yield:

$$\psi(E,B_{x,t}) \leq \int_{B_{x,t}} |H|\,dy \leq \omega_n^{1-1/p} \|H\|_{L^p(B_{x,t})} t^{n-n/p}$$

$$\leq \text{const}(n, \|H\|_{L^p(A_T)}) \cdot t^{1-n/p} \cdot t^{n-1}$$

$$\equiv \alpha(t) \cdot t^{n-1}$$

where $A_T = \{x : \text{dist}(x,A) < T\} \subset\subset \Omega$. Since $1-n/p \in (0,1)$ in this case, we know (recall 1.12) that $\alpha(t) = c \cdot t^{1-n/p}$ satisfies $(\alpha_1 \geq \alpha_4)$, so that the Regularity Theorem 1.9 applies to this case.

We can also consider mean curvature functions H belonging to more general function spaces. Let us introduce e.g. the Morrey space $L_{loc}^{p,\lambda}(\Omega)$ $(p \geq 1, \lambda \geq 0)$:

$$u \in L_{loc}^{p,\lambda}(\Omega) \text{ iff } u \in L_{loc}^p(\Omega) \text{ and } \forall A \subset\subset \Omega :$$

$$(1.14) \qquad \sup_{\substack{x \in A \\ o<t<\text{diam}A}} \left(t^{-\lambda} \int_{A \cap B_{x,t}} |u|^p\,dy\right) < +\infty$$

Some elementary properties of Morrey spaces can be found in [20], Chapter 4.

It should be clear by the foregoing considererations that any local minimizer of $\mathscr{F}_H$ in Ω , with $H \in L_{loc}^{1,n-1+\alpha}(\Omega)$ and $\alpha > 0$, satisfies (1.10) with $\alpha(t) = \text{const} \cdot t^\alpha$, so that Theorem 1.9 applies equally well to the present situation. Notice that $L^p \subset L_{loc}^{1,n-1+(p-n)/p}$,

by Hölder inequality, so that the case $H \in L^p_{loc}(\Omega)$ with $p > n$ appears as a particular instance in this general picture.

As previously seen, things are not so clear in the borderline case $H \in L^n_{loc}(\Omega)$: in particular, we do not know yet whether conclusions (R_1) and (R_2) of Theorem 1.9 extend to local minimizers of $\mathscr{F}_H$, when $H \in L^n_{loc}(\Omega)$. The following example may shed some light on the question.

Let $E \subset \mathbb{R}^n$, $n \geq 3$, be the epigraph of a radial function $u = u(r)$, where $r = |x'|$ and $x' \in \mathbb{R}^{n-1}$. Assume that $u \in C^1(0,+\infty)$, with derivative $u' > 0$, and that

$$(1.15) \qquad u(r) \to 0 \; , \quad u'(r) \to M \in [0,+\infty]$$

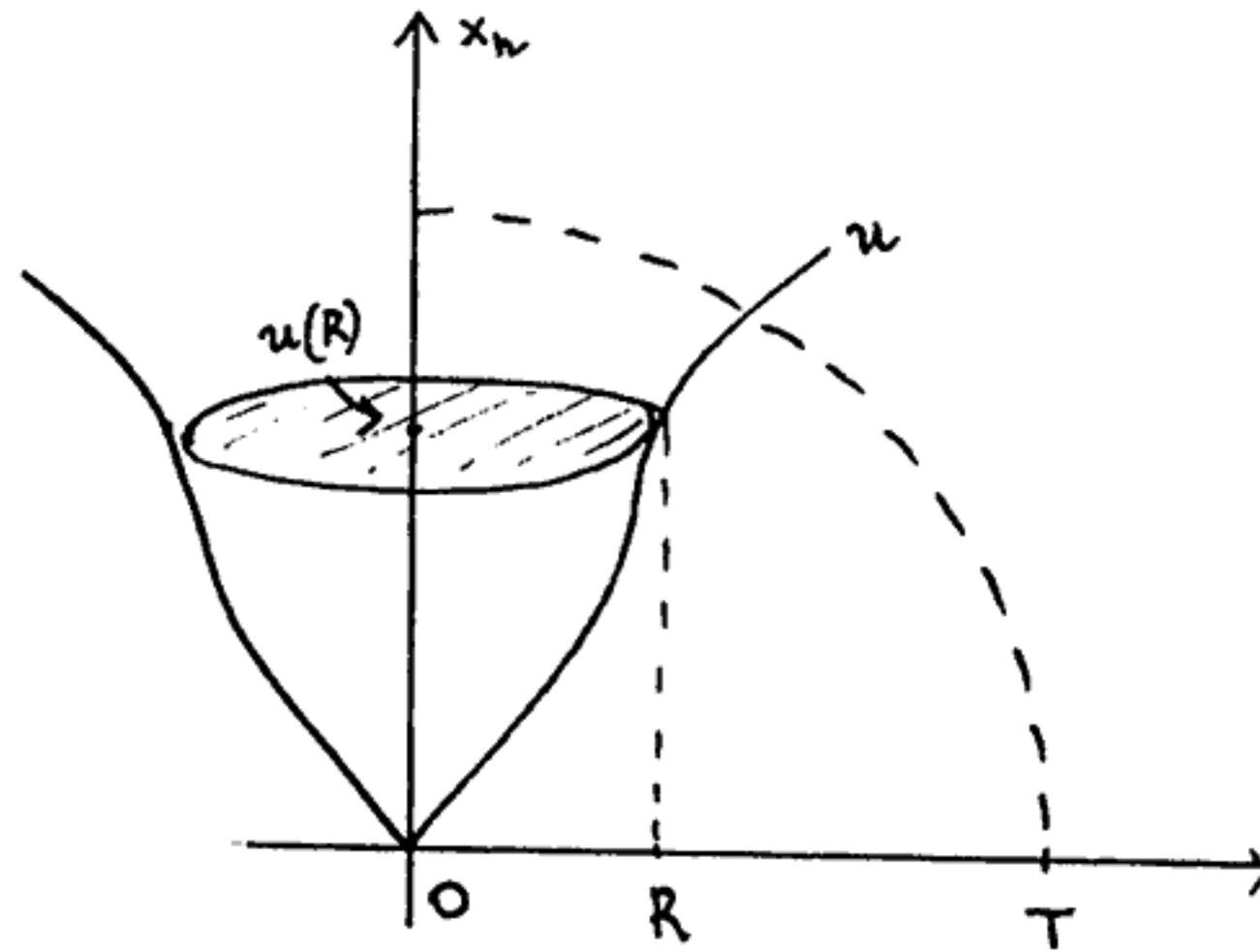as $r \to o^+$. See Fig. 7.

FIGURE 7.

If E is a local minimizer of $\mathscr{F}_H$ in $\mathbb{R}^n$, with $H \in L^n_{loc}(\mathbb{R}^n)$, and if E_R denotes the set

$$E_R = E \cap \{x : x_n > u(R)\}$$

then $\forall R > 0$ and $\forall T > (R^2 + u^2(R))^{\frac{1}{2}}$ we find (see (1.3) and Fig. 7):

$$0 \leq \mathscr{F}_H(E_R, B_T) - \mathscr{F}_H(E, B_T)$$

(1.16)

$$= \omega_{n-1} R^{n-1} - (n-1)\omega_{n-1} \int_0^R r^{n-2}(1+u'^2(r))^{\frac{1}{2}}dr - \int_{E \smallsetminus E_R} H(x)dx$$

On using succesively Hölder inequality and the isoperimetric inequality we get

(1.17)
$$\int_{E \smallsetminus E_R} |H|dx \leq n^{-1}\omega_n^{-1/n} \|H\|_{L^n(E \smallsetminus E_R)} |D\phi_{E \smallsetminus E_R}| \equiv g_n(R)|D\phi_{E \smallsetminus E_R}|$$

which, combined with (1.16), yields

(1.18) $(n-1)[1-g_n(R)] \displaystyle\int_0^R r^{n-2}(1+u'^2)^{\frac{1}{2}}dr \leq [1+g_n(R)] \cdot R^{n-1}.$

Were M in (1.15) positive, we would deduce from (1.18)

$$[1-g_n(R)] \cdot (1+\varepsilon^2)^{\frac{1}{2}} \leq 1+g_n(R) \qquad \forall \varepsilon \in (0,M), \; \forall R < R_\varepsilon,$$

a contradiction, since $g_n(R) \to 0$ when $R \to 0$ (see (1.17)).

Therefore, $M = 0$, thus showing that ∂E is everywhere smooth.

(iii) <u>Minimal boundaries with a volume constraint.</u>

It is a well-known fact that among the sets having a given measure v in $\mathbb{R}^n$, the n-ball B_R of radius $R = (v/\omega_n)^{1/n}$ is the one which minimizes surface area. A variety of (less trivial) examples of the same type are usually encountered in Capillarity Theory. For instance, one can think of a liquid drop of given mass and resting on a given surface (as shown in Fig. 8), as a local minimizer of an "energy functional" (whose analytic expression is, roughly speaking, the sum of "surface terms" plus "curvature terms", corresponding respectively to the surface forces - like surface tension - and body forces - like gravity - acting on the drop) in a certain class of admissible configurations, all with the same fixed mass (see e.g. [11]).

FIGURE 8

Let us now introduce the following abstract definition:

E *has minimal boundary in* Ω *with a volume constraint iff*

(1.19) $|D\phi_E|(A) \leq |D\phi_F|(A)$ $\forall A \subset\subset \Omega$, $\forall F : F \vartriangle E \subset\subset A$ *and* $|F \cap A| = |E \cap A|$

where $|G|$ denotes the Lebesgue measure of $G \subset \mathbb{R}^n$. The preceding definition extends to "curvature functionals" like (1.3) in the obvious way.

We proved in [21] that such an E satisfies (1.10) with $\alpha(t) = ct$: Theor. 1.9 then yields the regularity of ∂^*E, together with the usual estimate of the Hausdorff dimension of $\partial E \smallsetminus \partial^*E$.

Actually, the main body of [21] was devoted to the proof of the fact, that whenever (1.19) holds (with, of course, $|E \cap A| > 0$ and $|A \smallsetminus E| > 0$), then two balls B_1, B_2 of arbitrarily small radius r can be found, such that $B_1 \subset E \cap A$ and $B_2 \subset A \smallsetminus E$.
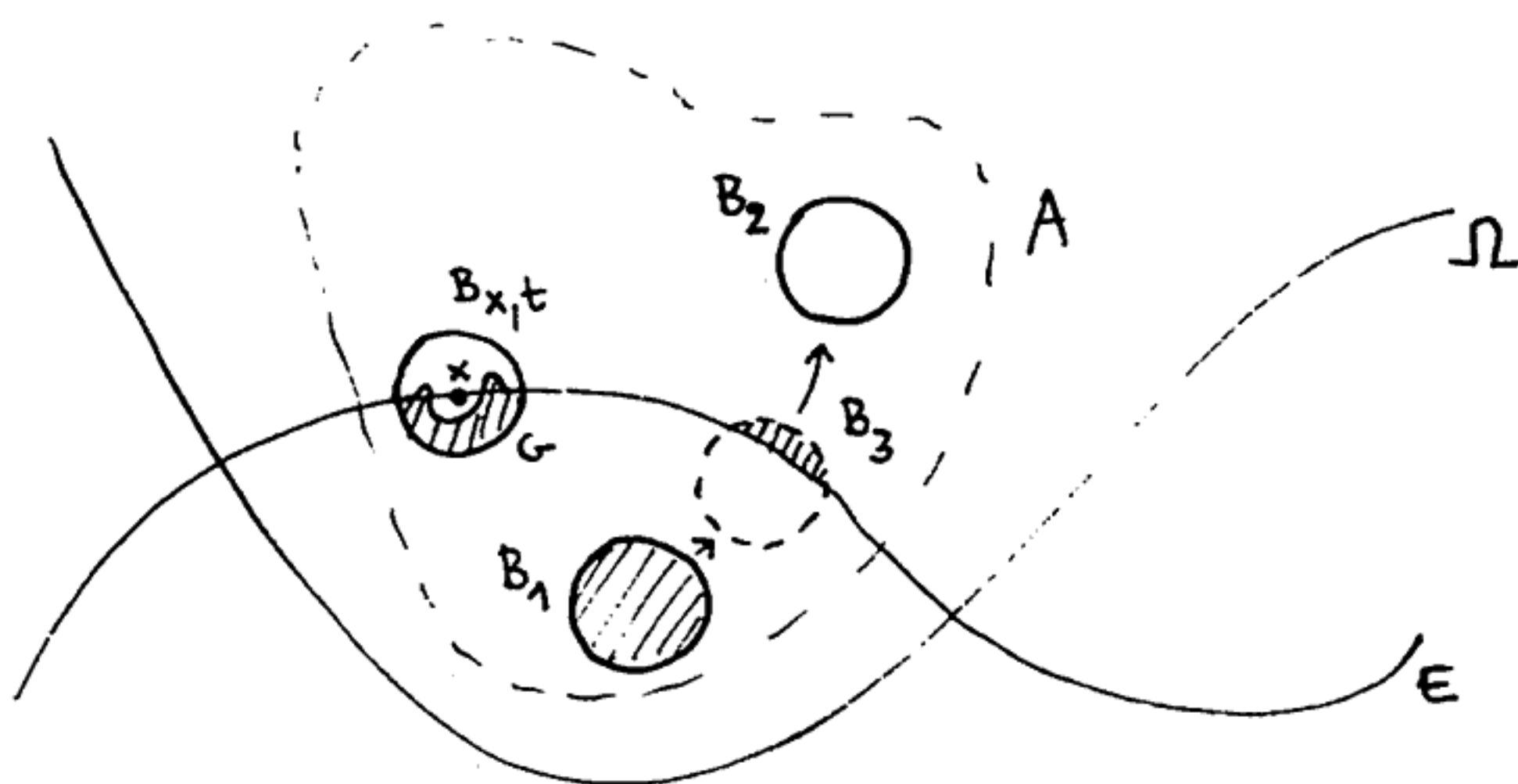

FIGURE 9.

Assuming this, and having fixed $x \in \partial E \cap A$, $t \in (0,r)$, and G such that

$$(1.20) \qquad B_{x,t} \subset\subset A \smallsetminus (\overline{B_1 \cup B_2}), \qquad G \vartriangle E \subset\subset B_{x,t}, |G \cap B_{x,t}| < |E \cap B_{x,t}$$

we see that we can move B_1 toward B_2 (in a continuous fashion, and taking care of remaining strictly within $A \smallsetminus \bar{B}_{x,t}$; see Fig. 9), until a new position, denoted by B_3, is reached, such that

$$|F \cap A| = |E \cap A|, \text{ where } F \equiv (G \cap B_{x,t}) \cup (E \smallsetminus B_{x,t}) \cup B_3.$$

From (1.19) we then derive essentially (see [21], prop. 1, for the precise calculations):

$$|D\phi_E|(B_{x,t}) \leq |D\phi_G|(B_{x,t}) + \int_{\partial B_3} \phi_{E \cup B_3} \, dH_{n-1} \cdot \; - \; \cdot |D\phi_E|(B_3)$$

(1.21) $$\leq |D\phi_G|(B_{x,t}) + (n/r)|B_3 \setminus E|$$

(see section 1.15) to follow for the proof of this last inequality). As the case when $|G \cap B_{x,t}| \geq |E \cap B_{x,t}|$ can be treated similarly

(just by interchanging the role of B_1, B_2), we see that (1.21) holds $\forall G : G \triangle E \; cc \; B_{x,t}$. In addition, the way B_3 was chosen shows that $\cdot |B_3 \setminus E| \leq \omega_n t^n$, which, combined with (1.21), yields

$$\psi(E, B_{x,t}) \leq (n/r)\omega_n \, t^n$$

as claimed.

(iv) Minimal boundaries with obstacles.

A second constraint we can impose on our solution is that it has to avoid some "obstacle". Stated more precisely:

E ! has minimal boundary in Ω with respect to the obstacle L iff

(1.22) $$E \supset L \cap \Omega \quad and \quad |D\phi_E|(A) \leq |D\phi_F|(A)$$

$\forall A \; cc \; \Omega, \forall F : F \triangle E \; cc \; A \quad and \quad F \supset L \cap A.$

Assuming this, and having fixed $A \; cc \; \Omega$ and $F : F \triangle E \; cc \; A$, we get

$$|D\phi_E|(A) \leq |D\phi_{F \cup L}|(A)$$

since $F \cup L$ is an admissible variation of E (i.e. $F \cup L \supset L \cap A$; see Fig. 10).
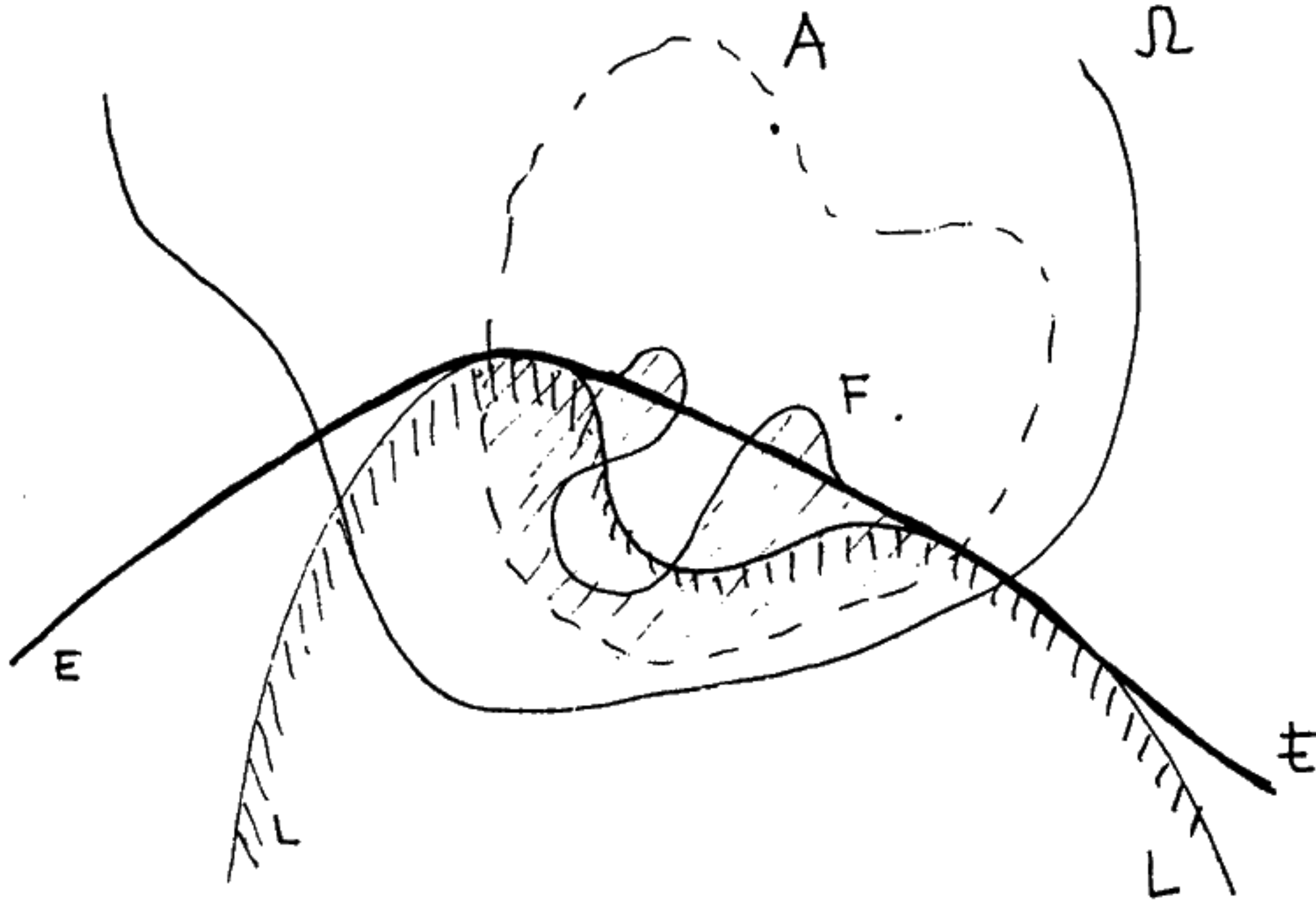

FIGURE 10.

By virtue of the inequality

$$|D\phi_{E_1 \cup E_2}|(A) + |D\phi_{E_1 \cap E_2}|(A) \le |D\phi_{E_1}|(A) + |D\phi_{E_2}|(A)$$

(see [27], 2.1.2 (10)), we then find

$$|D\phi_E|(A) - |D\phi_F|(A) \le |D\phi_L|(A) - |D\phi_{F \cap L}|(A)$$

(1.23)
$$\le \psi_o(L,A)$$

where (see (1.9)):

(1.24) $\psi_o(L,A) = |D\phi_L|(A) - \inf\{|D\phi_G|(A) : G \triangleleft L \subset\subset A, G \subset L\} \le \psi(L,A).$

On taking the supremum of the left-hand side of (1.23), as F varies *freely* among the local variations of E in A, we get in conclusion:

(1.25) $$\psi(E,A) \le \psi_o(L,A) \le \psi(L,A).$$

Thus, we realize that in any ball B $\subset\subset \Omega$ the deviation from minimality of E (a solution of the least area problem with obstacle L) is controlled by the deviation from minimality of the obstacle itself! (see 1.13).

Therefore, whenever ∂L is almost minimal in Ω, the same is true for ∂E. In fact, in view of example (v) immediately following, by using (1.25), the Regularity Theorem 1.9, and (essentially) the fact that a set with minimal boundary in $\mathbb{R}^n$, which in addition contains a half space, is itself a half space (see [30], Theor. 1), we can prove that if ∂L is of class $C^{1,\alpha}$ in Ω ($0 < \alpha < 1$), then ∂E is likewise of class $C^{1,\alpha}$ in a neighbourhood of ∂L. We refer to section 3 of [37] for a deeper analysis of the regularity of minimal boundaries with obstacles.

We remark that the result just quoted holds for $\alpha = 0$ as well (see [30], Theor.2). The proof in this case requires special attention, and in fact that result *cannot* be deduced directly from (1.25) and Theor. 1.9 alone (the reason being that the deviation from minimality of a set with C^1 boundary cannot be controlled, in general, by a "good" function $\alpha(t)$ - in particular, one satisfying (α_4) of Hypotheses 1.11; see example (v) below). The case $\alpha = 1$ also requires a special analysis, see [7]. Finally, we remark that the regularity result just quoted does *not* generally hold, for obstacles with lipschitz boundary (in constrast to what happens in the "cartesian case", see e.g. [10,16,18]).

To see this, merely consider the lipschitz cone

$$L = \{x \in \mathbb{R}^8 : x_8 > (x_1^2 + \ldots + x_7^2)^{\frac{1}{2}}\}$$

which is contained in Simons'cone C (see 1.4): although C has minimal boundary in $\mathbb{R}^n$ (and thus also with respect to L), ∂C is not lipschitz (not even a local graph near $0 \in \partial L \cap \partial C$.

(v) Smooth hypersurfaces

We saw in section 1.6 that whenever ∂E is of class C^1, then ∂E is in particular an almost minimal boundary. Generally however, the function $\alpha(t)$ which controls the deviation from minimality of E does not satisfy the integrability hypothesis (α_4) of 1.11. Here we have a simple example of such a situation.

Consider the function $u : (-1,1) \rightarrow (0,1)$ defined by

$$\begin{cases} u(t) = \int_0^t [\lg(e/s)]^{-1} \, ds & \text{when} \quad t \in (0,1) \\ \\ u(0) = 0, \quad u(t) = u(-t) \end{cases}$$

and put $E = \mathrm{epi}(u) \subset \mathbb{R}^2$, so that ∂E is of class C^1 in the open square $\Omega = (-1,1)^2$.
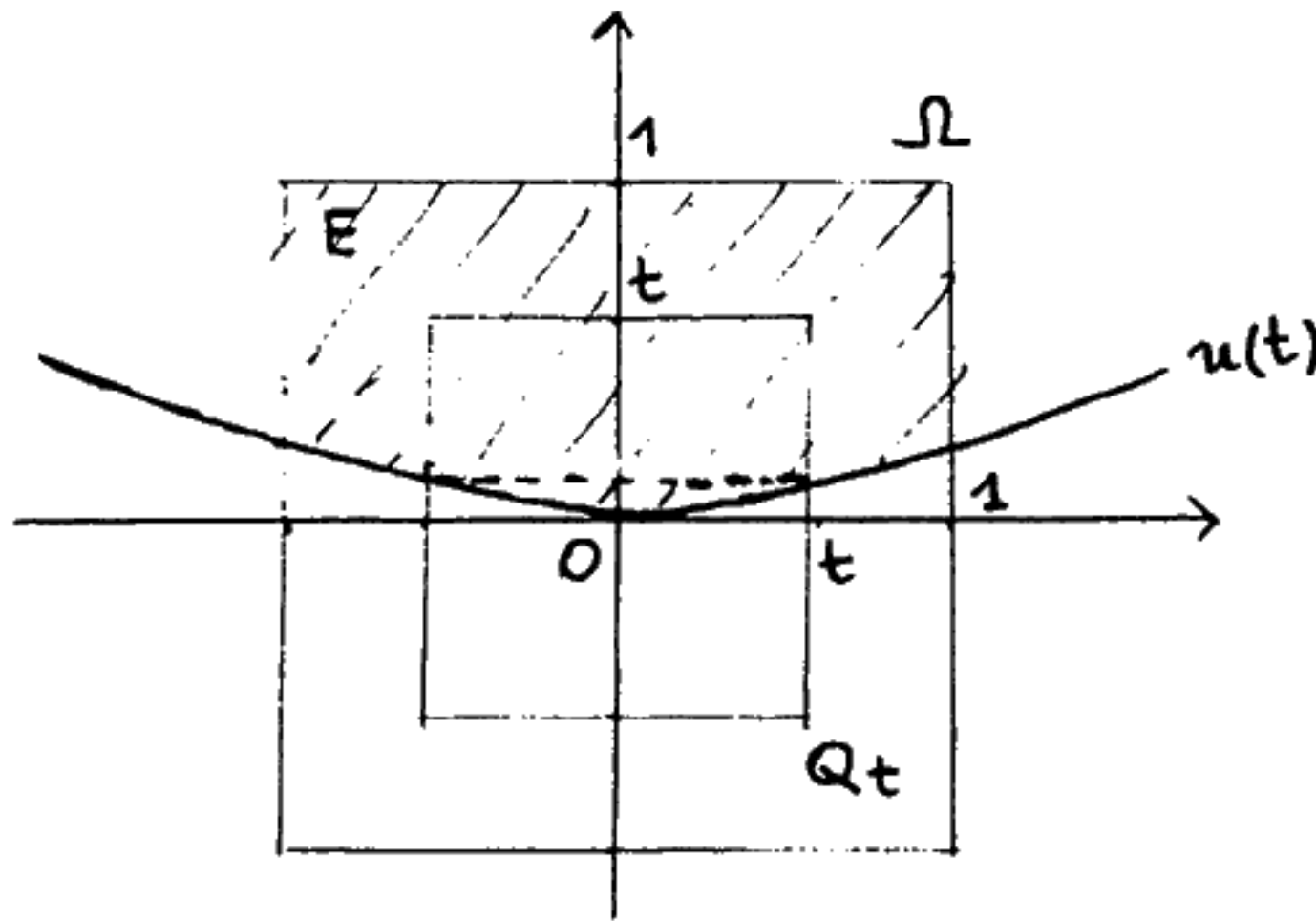

FIGURE 11.

Setting $Q_t = (-t,t)^2$, we find (see Fig. 11):

$$\psi(E,Q_t) = 2 \int_0^t \{[1+(lg(e/s))^{-2}]^{\frac{1}{2}} - 1\} \, ds$$

that is [4], $\psi(E,Q_t) \sim t[lg(e/t)]^{-2}$. Consequently, the function controlling the deviation from minimality of E is essentially

$$\alpha(t) = [lg(e/t)]^{-2}$$

which does not satisfy (α_4) (recall 1.12).

Things are better in the case when ∂E is of class $C^{1,\alpha}$, $0<\alpha<1$, i.e. when (locally) $\partial E = epi(u)$, with $u \in C^1(\mathbb{R}^{n-1})$ and

(1.26) $$|Du(y') - Du(z')| \leq L |y'-z'|^\alpha .$$

In this case, arguing as in 1.6, we find

$$\int_{B_t'} (1+|Du|^2)^{\frac{1}{2}} \, dx' - \int_{B_t'} (1+|Dv|^2)^{\frac{1}{2}} \, dx' \leq \frac{1}{2} \int_{B_t'} |Du|^2 dx'$$

$$\leq \frac{1}{2} \omega_{n-1} L^2 t^{n-1} \cdot t^{2\alpha}$$

(recall we are assuming $Du(o) = 0$), so that [7]

$$\psi(E,B_{x,t}) \leq c \cdot t^{n-1+2\alpha} \qquad \text{whenever} \qquad \partial E \in C^{1,\alpha} .$$

As the preceding example (iv) indicates, this fact is of considerable importance in connection with the obstacle problem for minimal boundaries. Also recall the remark following (1.8).

We conclude the present section with the short proof of an inequality, which find application in several cases (see [36], section 1),

and which was used in Example 1.14 (iii) above.

1.15. An isoperimetric inequality.[8]

Given a ball B_R of radius R in $\mathbb{R}^n$ and a subset L of B_R, there holds:

$$(1.27) \qquad \int_{\partial B_R} \phi_L \, dH_{n-1} \leq |D\phi_L|(B_R) + (n/R)|L|$$

Equality holds in (1.27) iff either $L = \emptyset$ or $L = B_R$.

Proof. Clearly, (1.27) is homogeneous in R, hence it suffices to prove it when R=1. Assuming this, we apply the Gauss-Green Theorem to the vector field $\phi(x) = \phi_L(x) \cdot x$, $x \in \mathbb{R}^n$, thus obtaining

$$\int_{\partial B_1} \phi_L(x) dH_{n-1}(x) = n \int_{B_1} \phi_L(x) dx + \int_{B_1} \langle x, D\phi_L(x) \rangle$$

$$(1.28)$$

$$\leq n|L| + |D\phi_L|(B_1)$$

which is (1.27) for R = 1. Recalling that (see [27], 2.3):

$$(1.29) \qquad D\phi_L = \nu_L H_{n-1}\big|_{\partial^* L} \quad , |D\phi_L| = H_{n-1}\big|_{\partial^* L}$$

we conclude that equality holds in (1.28) only if

$$\langle x, \nu_L(x) \rangle = 1 \qquad H_{n-1} - \text{a.e. on} \quad \partial^* L \cap B_1$$

which is possible only when

$$|x| = 1 \qquad H_{n-1} - \text{a.e. on} \quad \partial^* L \cap B_1$$

i.e., only when $H_{n-1}(\partial^* L \cap B_1) = 0$. This last assertion implies tha

L is equivalent either to the empty set or to B_1 itself.

The converse being obvious, we are done . From (1.27), observing that $L \subset B_R$ implies $|L| \le \omega_n R^n$, we deduce

$$(1.30) \qquad \int_{\partial B_R} \phi_L \, dH_{n-1} \le |D\phi_L|(B_R) + n\omega_n^{1/n} |L|^{(n-1)/n} \qquad \forall L \subset B_R$$

still with equality iff either $L = \emptyset$ or $L = B_R$. We see in addition that the only bounded sets $\Omega \subset \mathbb{R}^n$ for which the inequality

$$(1.31) \qquad \int_{\partial\Omega} \phi_L \, dH_{n-1} \le |D\phi_L|(\Omega) + n\omega_n^{1/n} |L|^{(n-1)/n}$$

holds, for every choice of $L \subset \Omega$, are exactly the n-balls. To be convinced, put $L = \Omega$ in (1.31) and recall the isoperimetric inequality:

$$(1.32) \qquad |D\phi_E| \ge n\omega_n^{1/n} |E|^{(n-1)/n}$$

which is valid for every bounded $E \subset \mathbb{R}^n$, with equality iff E is an n-ball. See [27], 2.2.2(2).