to occur when larger dimensions of the market, L, push researchers to increase their efforts in searching for process innovations; firstly, to avoid the decrease in the value of new patents which otherwise would be generated by smaller profits due to the higher competition and, secondly, to increase their purchasing power on a larger number of new more productive varieties.

We know that the no arbitrage condition between patents and a safe asset implies that the following Fisher equation must be satisfied for every value of m

$$\frac{\pi_m}{v_m} + \frac{\dot{v}_m}{v_m} = r \tag{22}$$

We recall that while for $m \neq i$ innovation does not introduce any new varieties, these are developed for the $i - th$ group of firms.

4 Moving equilibrium

In this section we describe the properties of the equilibrium of the model, which will be characterized as a *moving equilibrium*, given that we assume that the number of firms is the slow variable of the economy, while all other variables are the fast variables.[5]

In particular, we know from expression (11) that in equilibrium the labor market is clearing. From (10), (6) and (17) we obtain that employment in the final sector is

$$L_C = \frac{\alpha}{w} = \frac{\alpha a}{v_i n_i} \tag{23}$$

Thus, in any periods between the two subsequent reduction in γ_i, the market clearing condition (11) can be rewritten as

$$L = \frac{\dot{n}_i}{n_i} a + \frac{\alpha a}{v_i n_i}$$

Let us denote with V_i the inverse of the value of the aggregate existing stock of patents of firms of type i, $V_i = \frac{1}{v_i n_i}$. Then, from the previous equation, we derive the growth rate of firms of type

[5] See Schlicht (1985, 1997).

i in any periods between the two subsequent reduction in γ_i, that is

$$g_i = \frac{L}{a} - \alpha V_i \tag{24}$$

where $g_i = \dot{n}_i / n_i$.

For any time before a subsequent reduction in γ_i, we know from expressions (22) and (23) that the rate of change of V_i for firms of type i is

$$\frac{\dot{V}_i}{V_i} = V_i \frac{n_i(1-\alpha)}{\sum_{j=1}^{i} n_j \left(\frac{\gamma_i}{\gamma_j}\right)^{\sigma-1}} - g_i - r \tag{25}$$

From the previous expression, we note that we have Grossman and Helpman's (1991) results only if $\gamma_i = \gamma_j \; \forall j = 1, ..., i-1$, because this would also imply that $n_i = n_j \; \forall j = 1, ..., i-1$. Moreover, in the same particular case, we know that the interception between the two curves (24) and (25) would be unique when $\dot{V}_i = 0$, as is required in equilibrium.

However, when, as it happens in our case, $\gamma_i \neq \gamma_j$, the intersection between the two curves (24) and (25) is not unique and it moves over time as n_i increases in any period between two following values of γ_i are made available. Therefore, more than a fixed steady state equilibrium, as in Grossman and Helpman (1991), our assumptions lead to identify a *moving equilibrium*, which is characterized by continuous changes in the number of firms with different productivities.

Let us define the index b_i as

$$b_i \equiv \frac{n_i \gamma_i^{1-\sigma}}{\sum_{j=1}^{i} n_j \gamma_j^{1-\sigma}} \tag{26}$$

It is readily verifiable that $0 \leq b_i \leq 1$ and that b_i approaches 1 when n_i goes to infinity. b_i gives us some information on the relative weight of firms of type i on the total number of firms, where the weights are given by the productivity measure $\gamma_i^{1-\sigma}$. Hence, given that the value of b_i continuously changes, we have a *moving equilibrium* characterized by continuous changes in the fast variables due to movements in the slow variable b_i. Particularly, we have a moving equilibrium when all variables assume their equilibrium values conditioned to the number of patents already introduced in the R&D sector or, in an equivalent fashion, conditioned to the value of b_i, which depends on the number of patents. Then, changes in the number of available varieties, change b_i and, consequently, other variables, as we show in the rest of this section. Expressions (24), (25) and (5) tell us that for any of those moving equilibria the following condition must be satisfied

$$\dot{V}_i = V_i \left[V_i \left((1-\alpha) \frac{n_i \gamma_i^{1-\sigma}}{\sum_{j=1}^{i} n_j \gamma_j^{1-\sigma}} + \alpha \right) - \frac{L}{a} - \rho \right] \tag{27}$$

Moreover, we use (26) to rewrite profits (8) in the following way

$$\pi_m = \frac{(1-\alpha)}{n_m} b_m < 1 \tag{28}$$

Substituting (26), (6) and (17) into (3), we obtain that the demand for any firm of type m is

$$x_m = \frac{n_m p_m^{1-\sigma}}{n_m p_m \sum_{j=1}^{i} n_j p_j^{1-\sigma}} = \frac{a\alpha b_m}{n_m \gamma_m} V_i \tag{29}$$

Using this expression we derive the total demand $x_m n_m$ for all firms of type m, that is

$$n_m x_m = \frac{a\alpha b_m}{\gamma_m} V_i \tag{30}$$

Expression (30) tells us that when new more productive varieties are made available by the innovative sector, as the innovative process goes on, the total demand $x_m n_m$ for the oldest firms of type m, characterized by the highest values of γ, tends to decrease to zero, because b_m becomes smaller and smaller. On the contrary, the total demand $x_i n_i$ for firms of type i on the technological frontier, tends to increase as b_i increases.

Substituting (26), expression (27) becomes

$$\dot{V}_i = V_i \left\{ V_i \left[(1-\alpha) b_i + \alpha \right] - \frac{L}{a} - \rho \right\} \tag{31}$$

Expression (31) is an upward opening parabola, with $\dot{V}_i = 0$ either when $V_i = 0$ or when $V_i^* = \frac{L/a+\rho}{(1-\alpha)b_i+\alpha} > 0$. The graph is plotted in Figure 1 only for positive values of V_i, because negative values of V_i would have no meaning.

Insert Figure 1 about here

Moreover, in Figure 1 we also plot the actual value of V_i derived from (23), that is

$$V_i = \frac{L_C}{\alpha a} \tag{32}$$

We know from (32) that (11) and (25) are, respectively,

$$L_R = L - L_C \tag{33}$$

and

$$\dot{V}_i = \left(\frac{L_C}{\alpha a}\right)\left[\frac{b_i L_C}{\alpha a}(1-\alpha) - \left(\frac{L_R}{a} + \rho\right)\right] \tag{34}$$

Hence, as both pairs of equations above (31)-(32) and (33)-(34) show, the equilibrium outcomes which we describe within this framework are not stationary, given that b_i changes as the innovating process goes on determining the introduction of new varieties which increase n_i. Thus, between any pair of subsequent process innovations which lead to changes in γ_i, the equilibria we consider are moving equilibria which we need, indeed, to identify.[6]

We notice that we need to know L_C (or L_R) in order to define the exact position of the vertical line (32) in Figure 1, otherwise, we could either have that $L_C/(\alpha a) < V_i^*$ or that $L_C/(\alpha a) > V_i^*$. These two options would imply opposite changes in V_i. In fact, while V_i is increasing when $V_i > V_i^*$, it is decreasing when $V_i < V_i^*$. However, as we show in two steps, V_i must be equal to

[6] In other words, we can consider the economy as described by the following equation system in two vectors of variables x and y:
$\dot{x} = f(x, y)$ and $\dot{y} = g(x, y)$
where the vector of fast variable is $x' = (p_m, x_m, v_m, V_m, w, \pi_R, L_R, L_C, g_i)$ and the vector of slow variable is $y' = (b_i)$. Note that the slow variable b_i is obtained as a transformation of the number of all variables, n_m with $m = 1, 2, ..., i$, which, thus, are considered as slow variables too.
In the paper we assume that the fast vector has already reached its equilibrium for any given and fixed value of the slow variable, b_i, and we prove that the equilibrium is univocally identified for any given value of b_i in the following paragraphs in the text of the paper when we show that $V_i = V_i^* = \frac{L/a+\rho}{(1-\alpha)b_i+\alpha}$.
Particularly, the equilibrium value of the fast vector x is
$x = X(b_i)$, with $f(X(b_i), b_i) = 0$.
Then, given that the slow variable b_i changes over time, then $x = X(b_i)$ gives the corresponding moving equilibrium of x. (See Schlicht (1985, 1997))

V_i^*. In particular, first we recall that this is true in Grossman and Helpman's (1991, ch. 3) case. Then we prove that this is true in our general case.

First of all, we recall that if we were in Grossman and Helpman's (1991, ch. 3) case, γ would assume only one value, that is $\gamma_m = \gamma_i = \gamma_1 \ \forall m$. Moreover, in this case the steady state equilibrium is characterized by $\dot{V}_1 = 0$ and $L_R = L(1-\alpha) - a\alpha\rho$. In fact, we know that in this case the expectations of agents are fulfilled only if the economy jumps immediately to the point in which $\dot{V}_1 = 0$, because if $\dot{V}_1$ were positive we would have V_1 growing to infinity, while if $\dot{V}_1$ were negative, we would end up with $V_1 = 0$. However, Grossman and Helpman (1991) show that both cases are impossible, given that: in the first case we cannot have V_1 growing to infinity because L_R would be drawn to zero, n_1 would stop growing, and v_1 would be different from zero (given that with a finite number of variety, profits are strictly positive); in the second case, we cannot have $V_1 = 0$, because L_R would assume its maximum potential value, L, with $L_C = 0$, and expectations would be contradicted.[7] Finally, we notice that in this case, $b_1 = 1$. If we consider the pair of equations (31)-(32) which describes the equilibrium condition, they would intersect in $V_1 = \frac{L-L_R}{\alpha a} = V_1^* = L/a + \rho$ with $L_R = L(1-\alpha) - a\alpha\rho$ derived from the second pair of equations (33)-(34).

Let us now consider the case in which, in the framework so far described, an innovation process takes place producing new patents characterized by $\gamma_2 < \gamma_1$, which perturbs previous stationary equilibrium.[8] These new patents allow n_2 firms to employ the technology of type 2. We know from (21) that $n_2 = \frac{1}{a} n_1 L_R$ and that $b_2 = \frac{n_2 \gamma_2^{1-\sigma}}{n_1 \gamma_1^{1-\sigma} + n_2 \gamma_2^{1-\sigma}} < 1$.

[7] In fact, Grossman and Helpman (1991, p. 61) recall that if $L_R = L$, the number of varieties would grow continuously and, at the same time we would have $v_1(t) = \int_t^\infty e^{-r(\tau - t)} \frac{1-\alpha}{n_1(t)} d\tau < \frac{1-\alpha}{r n_1(t)}$.

In fact, $v_1(t) = \int_t^\infty e^{-r(\tau-t)} \frac{1-\alpha}{n_1(t)} d\tau = \left[-\frac{e^{-r(\tau-t)}}{r}\right]_t^\infty \frac{1-\alpha}{n_1(t)} = \left[-\frac{e^{-r\infty}}{r} + \frac{1}{r}\right] \frac{1-\alpha}{n_1(t)} < \frac{1-\alpha}{r n_1(t)}$.

Therefore, we would have that $v_1(t) n_1(t) < \frac{1-\alpha}{r}$ which is equivalent to saying $V_1(t) > \frac{r}{1-\alpha} > 0$ which contradicts the fact that $V_1 = 0$.

[8] In the particular example described by (20) when $n_1 L = \chi_1$. In any case, we recall that we do not need to use this particular specification of the more general expression (19).

After the change in γ, the innovative sector continues to produce new patents of type 2 according to (12). The inverse of the aggregate value of patents of type 1 is equal to $V_1 = \frac{1}{v_1 n_1}$, where n_1 is now a constant. Moreover, from (28), we know that profits of firms of type 1 are from now on $\pi_1 = \frac{(1-\alpha)}{n_1} b_1$. At the same time, there will be continuous increases in n_2, or in other more productive types of firms whenever there are further innovations leading to further reductions in γ. These processes will reduce b_1, reducing profits of firms of type 1 and, therefore, the value of patents of firms of type 1, v_1, thereby, increasing V_1. Therefore, we know that V_1 is increasing in b_1. Moreover, for any given value of b_1, n_1 and v_1 are given and, thus, V_1 is univocally determined. In other words, we are able to rule out bubble paths for the aggregate value of firms which are no longer on the technological frontier, such as firms of type 1, once technology with γ_2 can be used. Furthermore, we may say that this is generally true for any firms of type m different from i, that is, firms at the technological frontier from the production process point of view, because their number n_m does not increase anymore, and because their value v_m must decrease due to the ongoing growth in variety. At the limit, when the weight b_m of firms adopting older technology than the firms at the frontier, γ_i tends to decrease toward zero, and V_m tends to infinity.

Lemma 1 *For any variety which is not at the technological frontier, that is, for any variety produced with $\gamma_m > \gamma_i$, profits decrease and V_m increases as the weight b_m of the group decreases as a consequence of subsequent innovations in the R&D sector which increase the number of patents.*

Moreover, returning to our example, we further observe that once the new patents of type 2 become available at the technological frontier, with $i = 2$, and $b_2 < 1$, then $V_2^* = \frac{L/a+\rho}{(1-\alpha)b_2+\alpha} > V_1^*$.

Then we notice that while firms of type 2 remain at the frontier, for a given value of b_2 , if $V_2 = \frac{1}{v_2 n_2}$ does not immediately jump to V_2^*, there could be two other possible cases which we should consider: either we have $V_2 < V_2^*$ (with $\dot{V}_2 < 0$ which would draw V_2 to zero), or $V_2 > V_2^*$ (with $\dot{V}_2 > 0$ and V_2 growing to infinity). We note, in passing, that the following arguments can be generalized to the case in which firms of type i are at the frontier, for given b_i values.

We rule out the first case, that is $V_2 < V_2^*$, because we want to exclude asset bubble paths both in the subcase in which all firms of type 2 will always continue to be on the technological frontier in the future and in the subcase in which, sometime in the future, these firms will no longer be on the technological frontier due to further process innovations, which further reduce γ for future varieties. In the first subcase, V_2 cannot be drawn to zero because this would be possible, for a finite number of firms n_2, only with v_2 increasing to infinity; but with ongoing patent innovations this is impossible. Following the same reasoning, we can exclude asset bubble paths in the present (while firms of type 2 are at the frontier) also in the subcase in which the same firms were no longer supposed to be on the technological frontier at sometime in the future. In fact, if in the future the number of varieties is growing, we can exclude a continuous growth in v_2 and consequent decreases in V_2 in the present, for given value of n_2 and b_2, because profits of firms of type 2 are superiorly limited by variety growth.

We can also rule out the second case, that is $V_2 > V_2^*$, once more both in the subcase in which all firms of type 2 will always continue to be on the technological frontier in the future and in the subcase in which, at sometime in the future, these firms will no longer be on the technological frontier due to further process innovations. In the first subcase, when firms of type 2 remain on the frontier with $i = 2$, this will rapidly lead to employ all workers in the manufacturing sector with no more growth in variety (given that we know from (32) that $V_2 = \frac{L_C}{\alpha a}$) and V_2 increasing to infinity. However, this would be possible only if v_2 were equal to zero and we exclude this case because if the number of varieties stops growing, profits must always be strictly positive. Finally, in the second subcase if n_2 stops growing at sometime in the future because firms of type 2 are displaced from the frontier, in the present (while these firms are at the frontier), we know that V_2 will increase to infinity in the future, because it is in the future that v_2 tends to zero as b_2 (and profits) tends to zero. However, these future increases in V_2 cannot be anticipated in the present, because, otherwise, V_2 would be lead to infinity in the present which would be inconsistent with $L_C < L$. In other words, from (32) we could have it only with $L_C = L$, but at the expense of no innovation at all in the present (because $L_R = 0$) which would exclude the potential future process innovations.

So we have that when firms of type two are at the technological frontier, V_2 immediately jumps to V_2^* and that, for any given value of b_2, $\dot{V}_2 = 0$. We know that b_2 changes in the present with ongoing patent innovations. In particular, it continuously increases, because n_2 increases. As long as firms of type two are at the technological frontier, the continuous increase in b_2 is associated with continuous reductions in $V_2 = V_2^*$, with V_2 having a lower limit. In fact, $V_2 = V_2^*$ tends to V_1^* as n_2 increases.

We already noticed that previous arguments can be generalized to the case in which, instead of firms of type 2 we considers firms of type i at the frontier for given b_i values. For we know now that we have

$$V_i = V_i^* = \frac{L/a + \rho}{(1-\alpha)b_i + \alpha} \tag{35}$$

where $V_i = V_i^*$ is increasing in L, ρ (with $\rho = r$) and decreasing in a, b_i and α.

We may write the following lemma:

Lemma 2 *In general, the inverse of the aggregate value of firms at the frontier, V_i, is $V_i = V_i^* > V_1^*$ with $\dot{V}_i = 0$ when b_i is given. However, given that b_i increases when patent innovations take place for firms at the technological frontier increasing n_i, V_i changes approaching V_1^* as the weight b_i increases.*

The previous lemma is extremely important given that it allows us to identify not a unique steady state equilibrium, but a series of moving equilibria, which can be considered as perturbations of the original steady state equilibrium in Grossman and Helpman (1991, ch. 3), and which continue to change as long as b_i changes and as long as we can have different process innovations which continue to change the technology at the frontier. The implications of our results will be discussed in the following section.

Moreover, from expressions (30) and (35) we obtain the total demand for varieties of type i

$$x_i n_i = \frac{a\alpha}{\gamma_i}\left[\frac{L/a + \rho}{(1-\alpha) + \alpha/b_i}\right]$$

which clearly shows that as b_i increases, total demand for varieties of type i increases. Thus, gradually in our model, while the market share and the demand of previously developed varieties decreases, the market share and the demand of new and more productive varieties made available increases as long as they are on the technological frontier.

5 Structural changes and the scale effect

One of the most striking characteristics of the moving equilibrium we have so far described is that it allows us to represent the effects of ongoing patent innovations which take place together with process innovations. Considering both kinds of innovations gives a more complete picture of the effects of R&D activities and it produces a setup in which the rate of growth of patent innovations varies across time according to workers' distribution between the final and the innovative sectors considered in the model.

In the period in which technology of type i is available, we know from expression (12) that the rate of innovation is proportional to the number of workers employed in the R&D sector, and this number L_R, derived from (33)-(34) when $\dot{V}_i = 0$, depends on the value of b_i, that is

$$L_R = \frac{Lb_i(1-\alpha) - a\rho\alpha}{(1-\alpha)b_i + \alpha} \tag{36}$$

As in Grossman and Helpman (1991), we assume that L is sufficiently large to allow patent innovations to take place: this requires that $L > a\rho\alpha / b_i (1-\alpha)$. Once more, it is readily verifiable that when $b_i = 1$ we obtain the same results as in Grossman and Helpman (1991).

Expression (36) shows that the number of workers employed in the innovative sector is an increasing function of b_i because

$$\frac{\partial L_R}{\partial b_i} = \frac{(1-\alpha)\alpha(L + a\rho)}{((1-\alpha)b_i + \alpha)^2} > 0$$

Therefore, when there are at least two different types of firms producing using different technologies, and the innovative sector intensifies its research in finding new patents for the production of new goods employing the more productive technologies, then any time a new patent is produced and implemented the value of b_i increases. As b_i increases, the final sector in aggregate

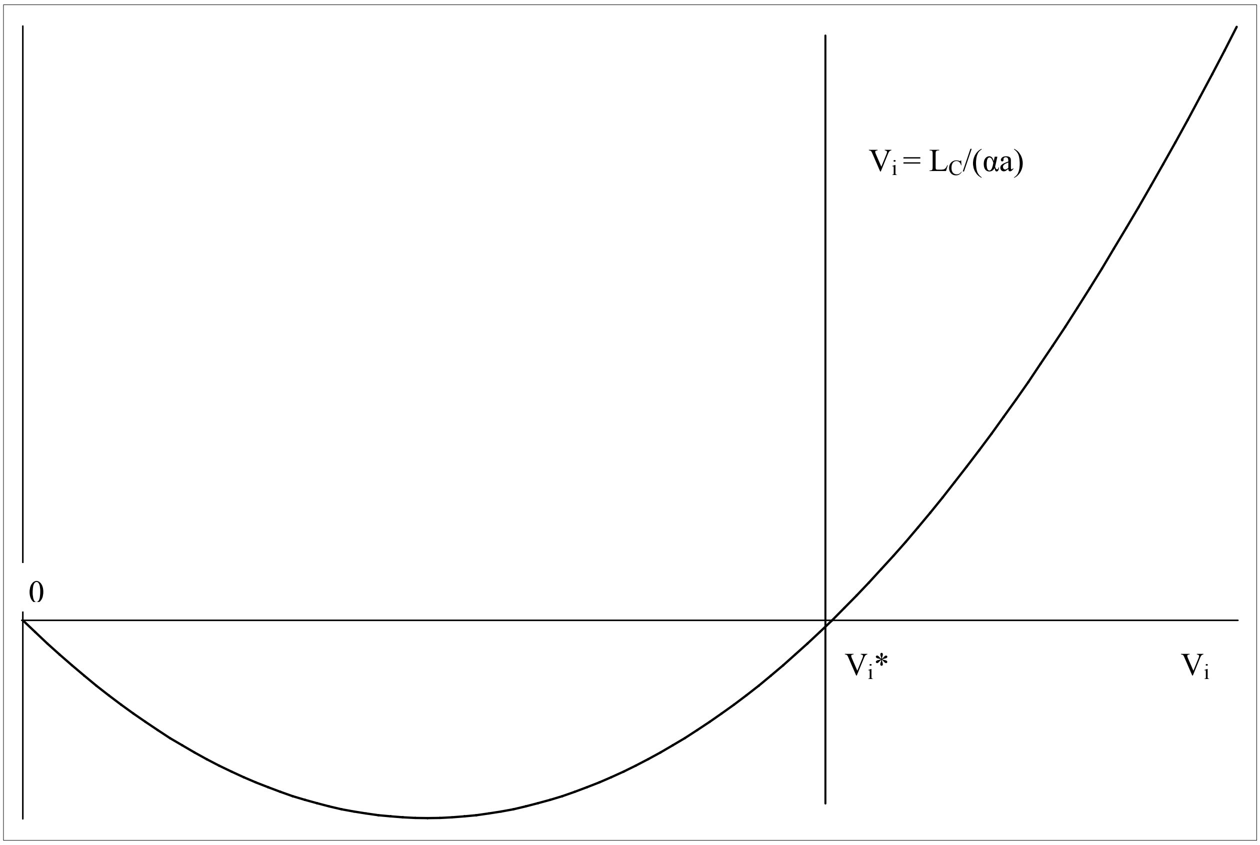

Figure 1