1 Introduction

Among the many dimensions along which the agents' heterogeneity can affect market demand, income heterogeneity is surely a crucial one.The relationship between the personal income distribution and market demand is usually studied, by pointing out how distributional changes yield their effects *via* the shape of the consumers' Engel curves – which of course implies emphasis on the size of the market for a given price level (e.g., Lambert and Pfähler, 1997).

In this paper we take a different approach, by emphasizing how income distribution affects also the sensitivity of demand to price changes. This focus on the price elasticity of demand allows to draw some general relationship between the behaviour of firms in noncompetitive markets, and some key features of the distribution of income. The degree of market power (as measured by the standard Lerner index) turns out to depend on such factors as the weight of the 'middle' class and, more generally, the degree of income dispersion. In particular, under fairly general conditions we show that, as incomes become more concentrated around the given mean, market demand and its elasticity both increase for a relevant intermediate range of prices. In this sense, the existence of a large middle class may support a more competitive market environment, conducive to lower monopoly profits. A related implication is that distributive changes can provide an example of a demand shift, able to deliver a negative comovement of demand and mark-up levels – an issue widely debated both in the micro and the macro literature.

Beyond the theoretical interest of studying the link between income dispersion and market competitiveness, our results may contribute to the debate on the implications of the well known phenomena of income polarization and 'shrinking middle class', which characterized several countries over the last two decades [1] – in some markets, income polarization may account for wider allocative inefficiencies.

The paper is organized as follows. In Section 2 we formalize distributive changes in terms of mean preserving spreads of the income distribution, and build a simple discrete-choice demand model. The co-movements of demand and price elasticity are studied in Section 3, where we rely upon some properties of the so-called income share elasticity function. Section 4 presents a fully developed example, based on the lognormal distribution. Some final comments are gathered in Section 5.

[1] For recent assessments of increasing income inequality and polarization, see, e.g., Atkinson (1998), Gottschalk and Smeeding (2000), and Pearson and Förster (2000).

2 Personal income distribution and market demand

We model income distribution as a continuous differentiable unimodal density function $f(y, \theta)$, defined over some positive interval (y_m, y_M), $0 \leq y_m < y_M \leq \infty$. The parameter θ is a mean preserving spread. As is well known, in probability theory this is a measure of the degree of riskiness of a distribution. The reason why θ can be fruitfully applied to model income distributions, is that *via* changes in θ one can study the effects of changes income dispersion, as distinct from changes in aggregate (average) income – loosely speaking, an increase in θ shifts income frequencies towards the tails of the density function, while a decrease in θ raises the frequency of central income values.Using a mean preserving spread as defined here, amounts to ranking equal-mean distributions by second-order stochastic dominance. In the literature on income distribution, it is well known that such ranking is equivalent to Lorenz dominance: θ is thus a proper inequality index satisfying the Pigou-Dalton's "principle of transfers" (Atkinson, 1970).[2]

Formally, letting $h \in (y_m, y_M)$ denote the modal income, and letting subscripts denote partial derivatives, the following holds:

$$\left.\begin{array}{l} f_y(h, \theta) = 0 \\ f_y(y, \theta) > 0 \text{ for } y < h \\ f_y(y, \theta) < 0 \text{ for } y > h \end{array}\right\} \tag{1}$$

$$\left.\begin{array}{ll} \displaystyle\int_{y_m}^{y} F_\theta(x, \theta)dx \geq 0, & y < y_M \\ \displaystyle\int_{y_m}^{y_M} F_\theta(x, \theta)dx = 0 & \end{array}\right\} \tag{2}$$

where $F(y, \theta) = \int_{y_m}^{y} f(x, \theta)dx$ is the cumulative distribution function.

We specialize our model by imposing some regularity conditions on $F(\cdot, \theta)$. First, we assume that the mean preserving spread is of the simple type (Rothschild and Stiglitz, 1971), i.e. the crossing of distributions implied by (2) takes place only once. Secondly, we assume that the shift of the frequencies towards the tails associated to an increase in θ is such that the old and new density functions intersect only twice.

[2] Accordingly, an increase in θ shifts unambiguously down the Lorenz curve. The link between inequality orderings and stochastic dominance has been recently sudied, e.g., by Formby *et al.* (1999).

To see the implications of these restrictions, consider the function F_θ. Both assumptions are captured by this taking a shape like that exhibited in Fig. 1c: single crossing of the distributions (Fig. 1a) implies that F_θ crosses zero in the interior of (y_m, y_M) only once; double crossing of the densities (Fig. 1b) implies that this function has only one maximum and one minimum over (y_m, y_M). It should be stressed that this behaviour is shared by many commonly used distributions subject to mean preserving shocks.[3]

Figure 1 about here

This simple figure brings out a very general property of the effects of changes in θ under our assumptions. Four intervals can be identified according as F_θ and f_θ have the same or the opposite sign. Indeed, for any $y \in (y_m, y_A]$, $F_\theta > 0$ and $f_\theta \geq 0$ (with equality only for $y = y_A$); for any $y \in (y_A, y_B]$, $F_\theta \geq 0$ (with equality only for $y = y_B$) and $f_\theta < 0$; for any $y \in (y_B, y_C]$, $F_\theta < 0$ and $f_\theta \leq 0$ (with equality only for $y = y_C$); for any $y > y_C$, $F_\theta \leq 0$ (with equality only for $y = y_M$) and $f_\theta > 0$ (clearly, this holds in the limit if $y_M = \infty$). For ease of notation, we label intervals as $A = (y_m, y_A]$, $B = (y_A, y_B]$, $C = (y_B, y_C]$, $D = (y_C, y_M)$. Of course, the boundary values of these intervals in the interior of (y_m, y_M) are functions of θ, i.e. $y_i = y_i(\theta)$, $i = A, B, C$.

Income distribution is immediately connected to market demand, whenever each consumer chooses discretely between buying or not buying one unit of the commodity, according as the quoted price is lower or higher than his reservation price – the distribution of reservation prices across consumers can reasonably be thought of as mirroring somehow that of the consumers' incomes.

We assume throughout that the reservation price coincides with income, so that market demand is

$$Q(p, \theta) = 1 - F(p, \theta) \tag{3}$$

where population has been normalized to unity. This is clearly the sharpest way to model the relationship between income, reservation prices and demand. The weaker assumption of strict proportionality of reservation prices to income comes out, e.g., in models for durabes, such as that suggested by Deaton and Muellbauer (1980, pp.366-69). In this case the distribution

[3] To quote some examples, Beta, Gamma, Chi square, F, Lognormal, all follow this general pattern (one maximum, one zero crossing and one minimum) if subject to appropriately defined mean preserving spreads. Clearly, asymmetric distributions, while preserving the pattern, trace out a more irregular function than that plotted in the figure.

of reservation prices is isomorphic to the income distribution, when the latter is lognormal (Cowell 1995, pp.71-78). One should however notice that, independently of the specific form of the income distribution, our general argument rests only on the idea that a mean preserving spread on incomes generates single and double crossings of cumulative and density distributions of the reservation prices – which is always the case if the reservation price is monotonically increasing in income.[4]

Our demand function is clearly continuous and continuously differentiable. Moreover,

$$\begin{array}{l} Q_p = -f(p,\theta) \\ Q_{pp} < 0 \quad for \quad p < h(\theta); \quad Q_{pp} > 0 \quad for \quad p > h(\theta); \\ Q_{pp} = 0 \quad for \quad p = h(\theta) \end{array}$$

This definition of market demand makes it clear that both its position and its features depend on the parameter θ.

Our focus is on studying how changes in income dispersion affect the optimal behaviour of noncompetitive firms. As is well known, changes of this kind – which have apparently taken place in many countries – may be attributed to long-run structural factors (such as the skill distribution of workers, the insitutional framework of wage negotiations, the relative weight of capitale vs labour income, etc.), as well as to changes in the shorter-run redistributive effects of fiscal policy. As to the latter, one might think of y as disposable income, and accordingly interpret changes in θ as (equal yield) changes, e.g., in the degree of progression of the income tax schedule as measured by the residual progression index.[5]

The crucial issue we are interested in is then how changes in θ translate into comovements of demand and its elasticity. This may have some relevance in its own, as pointing out a mechanism through which income distribution affects the degree of competition; however, it should also be recalled that in a non-competitive general equilibrium setting the pro- or counter-cyclical

[4] In this sense the assumption $p = y$ drastically simplifies the exposition, with no substantial loss of generality. On the other hand, the binary-choice model of demand is widely applied in the literature (e.g., Anderson *et al.*, 1992); in the analysis of the relationship between income distribution and demand it is very convenient, as continuous individual demand curves are standardly derived from omothetic preferences, which prevent any discussion of distributional issues.

[5] As is well known, residual progression is (inversely) measured by the elasticity of post-tax income to pre-tax income; an increase in this index shifts unambiguously down the concentration curve (see, e.g., Lambert, 1990, chs 7 and 9). This Lorenz dominance is equivalent, under a equal-yield constraint, to a mean preserving spread of the distribution of disposable income.

behaviour of demand elasticity is the key element to assess the role of demand in determining the equilibrium output.[6]

Given market demand (3), the (positive) price elasticity of demand is given by

$$\eta(p,\theta) = \frac{pf(p,\theta)}{1-F(p,\theta)} \tag{4}$$

By differentiating with respect to θ one easily obtains

$$\eta_\theta(p,\theta) = \left(\frac{f_\theta(p,\theta)}{f(p,\theta)} + \frac{F_\theta(p,\theta)}{1-F(p,\theta)}\right)\eta(p,\theta) \tag{5}$$

Simple inspection of (5) reveals that changes in θ affect η differently, depending on where p lies in the four intervals A, B, C and D identified above.

A priori, the sign of η_θ is clearly unambiguous whenever f_θ and F_θ have the same sign, ambiguous otherwise: hence we can say that $\eta_\theta > 0$ for $p \in A$ and $\eta_\theta < 0$ for $p \in C$; in intervals B and D the sign of η_θ is potentially ambiguous. However, since η_θ changes sign going from A to C, it follows trivially by continuity that at least one point $\widehat{p}$ exists in the interior of B such that $\eta_\theta = 0$, and hence an interval $\widehat{B} \subset B$ exists where $\eta_\theta < 0$ – the left boundary of $\widehat{B}$ being $\widehat{p}$. This allows us to establish the following

Proposition 1 For all distributions obeying (1) and (2), and such that a change in θ generates single crossing of distributions and double crossing of densities, there exists a non-empty interval $\widehat{B}$ where the normalized demand function (3) and its elasticity (4) move in the same direction following a change in θ.

Proof Follows trivially from the fact that, by the definition of B, $Q_\theta(p,\theta) = -F_\theta(p,\theta) < 0$ for $p \in B$, while by the definition of $\widehat{B} \subset B$, $\eta_\theta(p,\theta) < 0$ for $p \in \widehat{B}$. □

An immediate implication of this proposition is that, if the income distribution is subject to a change in dispersion as measured by θ, firms whose equilibrium price lies in the specified range face a positive comovement in demand and price elasticity. In particular, as incomes become less dispersed, for all initial prices $p \in \widehat{B}$ firms experience an increase in both the level and the elasticty of demand. Of course, Proposition 1 is a simple existence proof for $\widehat{p}$, which says nothing as to uniqueness in the set B and, *a fortiori*, over

[6]For a general discussion of different perspectives on the comovements of market demand and its price elasticity, see Benassi *et al.* (1994, ch 5) and the references therein.

the whole range of p – the sign of η_θ in area D is clearly still ambigous. Sorting this out would enable us to determine the behaviour of η over the whole range of p. The properties of the income distribution which deliver uniqueness of $\widehat{p}$ are discussed in the next section.

3 Income share elasticity and the price elasticity of demand

Ideally, one would expect to pin down a unique value $\widehat{p}$, such that $\eta_\theta > 0$ for all $p < \widehat{p}$ and $\eta_\theta < 0$ for all $p > \widehat{p}$. Given that $\eta_\theta > 0$ for $p \in A$ and $\eta_\theta < 0$ for $p \in \widehat{B}$, η_θ crosses zero from above at the left boundary of $\widehat{B}$, i.e.at $\widehat{p}$. In order to define the conditions for $\widehat{p}$ to be unique, we first notice that the derivative of η_θ with respect to p is

$$\begin{aligned}\eta_{\theta p} &= \frac{\eta(p,\theta)}{f(p,\theta)}\left(f_{p\theta}(p,\theta) - \frac{f_p(p,\theta)}{f(p,\theta)}f_\theta(p,\theta)\right) \\ &\quad + \left(\eta_p + \frac{\eta^2}{p}\right)\left[\frac{f_\theta(p,\theta)}{f(p,\theta)} + \frac{F_\theta(p,\theta)}{1-F(p,\theta)}\right]\end{aligned} \tag{6}$$

which, for $\eta_\theta = 0$ collapses to

$$\eta_{\theta p|\eta_\theta=0} = \frac{\eta(p,\theta)}{p}\Pi_\theta\left(p,\theta\right) \tag{7}$$

where $\Pi_\theta\left(y,\theta\right)$ is the derivative with respect to θ of the income share elasticity (Esteban, 1986). The latter is defined as

$$\Pi\left(y,\theta\right) = 1 + \frac{yf_y\left(y,\theta\right)}{f\left(y,\theta\right)}$$

and measures the percentage change of the income share accruing to individuals of income y, given a marginal change in y.[7] Esteban shows that there is a one-to-one correspondence between $f\left(y,\cdot\right)$ and $\Pi\left(y,\cdot\right)$, so that any given distribution can be characterized in terms of Π.

Therefore, given (7),

$$\text{at } \eta_\theta = 0, \quad sign\left[\eta_{\theta p}(p,\theta)\right] = sign\left[\Pi_\theta(p,\theta)\right] \tag{8}$$

This is particularly convenient, as the Π function typically exhibits some useful regularity properties.

[7]Formally, $\Pi = \lim_{h\to 0}\frac{1}{\mu}\int_y^{y+h} xf\left(x,\theta\right)dx$, where μ is the mean income (Esteban 1986, p.441).

We are now in the position to establish the following general proposition.

Proposition 2 If the distribution $f(y,\theta)$ and the corresponding income share elasticity $\Pi(y,\theta)$ are such that (a) $\Pi_\theta(y,\theta)$ is monotonically increasing in y and crosses zero, and (b) $\lim_{p\to y_M}\eta_\theta<0$, then there exists one value $\widehat{p}$ such that $\eta_\theta(p,\theta)>0$ for $p<\widehat{p}$ and $\eta_\theta(p,\theta)<0$ for $p>\widehat{p}$.
Proof By Proposition 1 there exists a $\widehat{p}$ which is the lowest p such that $\eta_\theta(p,\theta)$ crosses zero, obviously from above. Condition (a) together with (8) imply that $\Pi_\theta(\widetilde{p},\theta)=0$ at some unique $\widetilde{p}>\widehat{p}$. This implies that $\widehat{p}$ is the unique value of p at which η_θ is zero. To see this, notice that by condition (b), if additional such points existed, they should be even in number. Suppose they are two (the proof applies trivially for any even number), and call them $\widehat{p}_1$ and $\widehat{p}_2$, $\widehat{p}_1<\widehat{p}_2$. Obviously, $\eta_{\theta p}$ will be positive at $\widehat{p}_1$ and negative at $\widehat{p}_2$. Two possibilities arise: (*i*) $\widehat{p}_1$ and $\widehat{p}_2$ are both lower or higher than $\widetilde{p}$; (*ii*) $\widehat{p}_1<\widetilde{p}<\widehat{p}_2$. Case (*i*) is ruled out by (8); case (*ii*) is ruled out by (8) together with condition (a). □

It should be noticed that conditions (a) and (b) of Proposition 2 are verified for many widely used distributions, such as those quoted in f.note 2.

One implication of Proposition 2 is that the interval $\widehat{B}$ identified by Proposition 1 is unambiguously defined as $(\widehat{p},y_B)$. Figure 2 brings this out by showing a possible behaviour of the sensitivity of the elasticity of market demand to θ, for different values of p.

Figure 2 about here

Why is it that, for prices lying between $\widehat{p}$ and y_B, an increase in income concentration generates both an increase in demand and an increase in its elasticity (which for constant marginal costs would imply non competitive firms setting a lower price)? Prices in that interval are prices at which the higher income individuals getting poorer are still able to buy, while lower income individuals getting richer are eventually allowed to enter the market. The additional demand accruing at these prices is therefore due to the latter - those who descend into the middle class from the upper tail of the distribution were already buyers, and keep buying after the distributional change. However, this overall movement from the tails towards the central area of the distribution is such that for prices belonging to $\widehat{B}$, there are more consumers actually buying, whose reservation price is close to the set price. This implies, for example, that non competitive firms perceive a weaker incentive to exploit an intensive margin on higher income consumers, and a stronger incentive to acquire new consumers at the margin by keeping lower prices.[8] Demand increases and becomes more elastic simply because there are indeed new consumers entering the market, but also more consumers whose decision to enter or exit the market is now very sensible to small variations in prices. Notice that these observations are consistent with the fact that a positive comovement of demand and demand elasticity is observed only in $\widehat{B}$, i.e., it is peculiar of an intermediate portion of the demand curve, as defined by $\widehat{B}$. Moreover, they apply to whatever unimodal distribution, once concentration towards central income values is considered, and this explains the generality of our result. As a notable example, in the next section we apply the results of Propositions 1 and 2 to the lognormal distribution.

4 An example: income dispersion with lognormal distribution

Assume that income is distributed lognormally. This is a particularly remarkable case, since – as is well known – the lognormal distribution is perhaps the model most frequently used to describe actual income frequencies.[9]

We standardize mean income equal to unity, so that the density and distribution functions take the form[10]

$$f(y,\theta) = \frac{1}{y\sqrt{2\pi\ln\theta}} \exp\left(-\frac{\left(\ln y + \frac{1}{2}\ln\theta\right)^2}{2\ln\theta}\right)$$

$$F(y,\theta) = \int_0^y f(x,\theta)dx = \tfrac{1}{2}\left[1 + \Phi\left(\tfrac{1}{4}\sqrt{2}\frac{2\ln y + \ln\theta}{\ln\left(\frac{1}{2}\right)\theta}\right)\right]$$

[8] This may offer a general explanation for the empirical evidence discussed by Frankel and Gould (2001), who find a causal link running from income distribution in urban areas to retail prices: according to their estimates, greater inequality is indeed associated with an increase in retail prices paid by lower middle-class consumers.

[9] It is well known that the lognormal distribution fits satisfactorily the actual income distribution for central income values, while it is unsatisfactory in the tails, i.e. for extreme income values (for an evaluation of the empirical performance of various distributions, see e.g. Majumder and Chakravarty, 1990). Since the phenomenon we are interested in is peculiar of intermediate intervals, the lognormality assumption seems worth investigating. We recall that, if reservation prices are proportional to incomes, they also are lognormally distributed.

[10] Given a generic lognormal distribution $f(y,\theta) = (y\sqrt{2\pi\ln\theta})^{-1}\exp\left(-\frac{(\ln y-\zeta)^2}{2\ln\theta}\right)$, the mean is $\mu = e^{\zeta}\sqrt{\theta}$. Clearly, by imposing $\mu = 1$ one constrains the parameters θ and ζ according to the restriction $\varsigma = -\frac{1}{2}\ln\theta$. Note, in particular, that income variance is $\sigma^2 = \theta - 1 > 0$.

where $\Phi(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt$ is the so-called *error function* (Johnson *et al.* 1994, p.81). It can be checked that $\theta > 1$ is indeed a mean preserving spread, in that the conditions set out in (2) are satisfied. Also, single crossing of the cumulative distributions following a change in θ, as well as double crossing of the densities, are satisfied. In particular, for this lognormal distribution we have $y_A(\theta) = e^{-\frac{1}{2}\sqrt{\left(4\ln\theta+\ln^2\theta\right)}}$, $y_B(\theta) = \sqrt{\theta}$ and $y_C(\theta) = e^{\frac{1}{2}\sqrt{\left(4\ln\theta+\ln^2\theta\right)}}$. It is immediate to define the demand curve as

$$Q(p,\theta) = \tfrac{1}{2}\left[1 - \Phi\left(\tfrac{1}{4}\sqrt{2}\frac{2\ln p+\ln\theta}{\ln\left(\frac{1}{2}\right)\theta}\right)\right]$$

the elasticity of which is

$$\eta(p,\theta) = \frac{\dfrac{1}{\sqrt{2\pi\ln\theta}}\exp\left(-\dfrac{\left(\ln p+\frac{1}{2}\ln\theta\right)^2}{2\ln\theta}\right)}{1-\frac{1}{2}\left[1+\Phi\left(\frac{1}{4}\sqrt{2}\frac{2\ln p+\ln\theta}{\ln\left(\frac{1}{2}\right)\theta}\right)\right]}$$

The corresponding function $\eta_\theta(p,\theta)$ is derived in the Appendix, where it is shown to tend to minus infinity as $p \to \infty$. This function is in general analytically difficult to treat – and indeed one advantage of Proposition 2 is that it offers a simple, general characterization of its qualitative behaviour in terms of the income share elasticity. Actually, the Π function takes the simple form given by

$$\Pi(y,\theta) = -\frac{1}{2}\frac{2\ln y+\ln\theta}{\ln\theta}$$

It is easy to check that $\Pi_\theta = \ln y/(\theta\ln^2\theta)$, which is monotonically increasing in y and crosses zero at $y = \mu = 1$. Therefore we can establish that a mean preserving shock generates a unique area of positive comovement of demand and demand elasticity, the left boundary of which lies between $y_A(\theta) = e^{-\frac{1}{2}\sqrt{\left(4\ln\theta+\ln^2\theta\right)}}$ and 1, and the right boundary of which is $y_B(\theta) = \sqrt{\theta}$. In order to assess the relevance of this phenomenon, one can notice that a numerical approximation performed under the arbitrary value of $\theta = 2.5$ gives to this area a size such that about the 40% of the population has income (reservation prices) falling in it.[11]

[11] The value is not wholly arbitrary, since $\theta = 2.5$ yields a coefficient of variation $\kappa = 1.225$ very similar to the value of 1.237 recorded by Champernowne and Cowell (1998, p.78) for the distribution of labour income in the UK.

5 Concluding remarks

In this paper, through a simple discrete-choice model of consumers' behaviour, we have derived some general results on how changes in demand elasticity may be associated with empirically relevant changes in income distribution. We have shown that the concentration of the households' incomes in the range which one would roughly identify as middle class, results in a relevant segment of demand expanding and becoming more elastic.

This may also contribute to explaining why markets previously patronized only by richer groups of consumers, typically benefit from the middle class entering them, in terms of both market size and lower prices. According to our interpretation, the latter effect (typical, e.g., of some durables) may be seen as a consequence (and not the cause) of the new consumers being indeed middle class. Moreover, the link from income distribution to the degree of competitiveness may add a new perspective in evaluating the effects of redistibutive policies.

Clearly, the relationship between income distribution and market structure can be extended in several directions – to quote some of them, the change in profit margins may trigger entry and exit of firms; different distributions may alter the incentive to horizontal or vertical product differentiation; and, similarly, if income distribution affects price elasticity, it may affect the incentive towards price discrimination. We believe that the framework we have developed could fruitfully be enriched and applied to these research areas.

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Appendix

In this appendix we prove that, in the case of the lognormal distribution, $\lim_{y\to\infty} \eta_\theta = -\infty$.[12] Writing out the whole expression, we get

$$\eta_\theta(y,\theta) = \frac{1}{8}\sqrt{2}\frac{\exp\left(-\frac{1}{8}\frac{(2\ln y+\ln\theta)^2}{\ln\theta}\right)}{\ln^3\theta}\left\{\frac{4\left(\ln^{\frac{1}{2}}\theta\right)\pi\ln^2 y-4\left(\ln^{\frac{1}{2}}\theta\right)\pi\left(\ln^2 y\right)\Phi\left(\frac{1}{4}\sqrt{2}\frac{2\ln y+\ln\theta}{\ln^{\left(\frac{1}{2}\right)}\theta}\right)}{\theta\pi^{3/2}\left(\Phi\left(\frac{1}{4}\sqrt{2}\frac{2\ln y+\ln\theta}{\ln^{\left(\frac{1}{2}\right)}\theta}\right)-1\right)^2}+\right.$$

$$-\frac{\left(\ln^{\frac{5}{2}}\theta\right)\pi-\left(\ln^{\frac{5}{2}}\theta\right)\pi\Phi\left(\frac{1}{4}\sqrt{2}\frac{2\ln y+\ln\theta}{\ln^{\left(\frac{1}{2}\right)}\theta}\right)}{\theta\pi^{3/2}\left(\Phi\left(\frac{1}{4}\sqrt{2}\frac{2\ln y+\ln\theta}{\ln^{\left(\frac{1}{2}\right)}\theta}\right)-1\right)^2}-\frac{4\left(\ln^{\frac{3}{2}}\theta\right)\pi-4\left(\ln^{\frac{3}{2}}\theta\right)\pi\Phi\left(\frac{1}{4}\sqrt{2}\frac{2\ln y+\ln\theta}{\ln^{\left(\frac{1}{2}\right)}\theta}\right)}{\theta\pi^{3/2}\left(\Phi\left(\frac{1}{4}\sqrt{2}\frac{2\ln y+\ln\theta}{\ln^{\left(\frac{1}{2}\right)}\theta}\right)-1\right)^2}$$

$$\left.+\frac{2\sqrt{2}\exp\left(-\frac{1}{8}\frac{(2\ln y+\ln\theta)^2}{\ln\theta}\right)\left(\ln^2\theta\right)\sqrt{\pi}-4\sqrt{2}\exp\left(-\frac{1}{8}\frac{(2\ln y+\ln\theta)^2}{\ln\theta}\right)(\ln\theta)\sqrt{\pi}\ln y}{\theta\pi^{3/2}\left(\Phi\left(\frac{1}{4}\sqrt{2}\frac{2\ln y+\ln\theta}{\ln^{\left(\frac{1}{2}\right)}\theta}\right)-1\right)^2}\right\}$$

This function can be written as

$$\frac{\sqrt{2}}{8}\frac{e^{-z^2}}{\alpha^6}\left[\frac{4\alpha\pi(1-\Phi(z))\ln^2 y-\alpha^5\pi(1-\Phi(z))-4\alpha^3\pi(1-\Phi(z))+2\alpha^2\sqrt{2\pi}e^{-z^2}\left(\alpha^2-2\ln y\right)}{\theta\pi^{\frac{3}{2}}(\Phi(z)-1)^2}\right]$$

where $z=\frac{\sqrt{2}}{4}\frac{2\ln y+\ln\theta}{\alpha}$ and $\alpha=\sqrt{\ln\theta}$, so that $\ln y=\sqrt{2}\alpha z-\frac{\alpha^2}{2}$. Substituting for the latter, we get

$$\frac{\sqrt{2}}{8}\frac{e^{-z^2}}{\alpha^6}\left[\frac{\alpha\pi(1-\Phi(z))\left(8\alpha^2z^2-4\sqrt{2}\alpha^3z-4\alpha^2\right)+4\sqrt{2}\alpha^3e^{-z^2}\left(\alpha-\sqrt{2}z\right)}{\theta\pi^{\frac{3}{2}}(\Phi(z)-1)^2}\right]$$

which is in terms of z, and collapses to

$$\frac{\sqrt{2}}{2}\frac{e^{-z^2}}{\alpha^3}\left[\frac{\pi\left(1-\Phi\left(z\right)\right)\left(2z^2-\sqrt{2}\alpha z-1\right)-\sqrt{2\pi}e^{-z^2}\left(\sqrt{2}z-\alpha\right)}{\theta\pi^{\frac{3}{2}}\left(\Phi\left(z\right)-1\right)^2}\right]$$

$$=\frac{e^{-z^2}\sqrt{2\pi}}{2\theta\alpha^3\pi^{\frac{3}{2}}}\left[\frac{\sqrt{\pi}\left(1-\Phi\right)\left(2z^2-\sqrt{2}\alpha z-1\right)-e^{-z^2}\left(2z-\sqrt{2}\alpha\right)}{\left(1-\Phi(z)\right)^2}\right] \quad \text{(a.1)}$$

[12] We owe very useful suggestions about this proof to profs A.Leaci and D.Scolozzi.

To ease notation, let $k = \dfrac{\sqrt{2\pi}}{2\theta\alpha^3\pi^{\frac{3}{2}}}$, a constant. Then (a.1) can be written as

$$ke^{-z^2}\left[\frac{\left(2z-\sqrt{2}\alpha\right)\left(\sqrt{\pi}\,(1-\Phi(z))\,z-e^{-z^2}\right)-\sqrt{\pi}\,(1-\Phi(z))}{[1-\Phi(z)]^2}\right]$$

$$=\frac{ke^{-z^2}}{1-\Phi(z)}\left\{\frac{2z\left[\sqrt{\pi}(1-\Phi(z))z-e^{-z^2}\right]}{(1-\Phi(z))}-\frac{\sqrt{2}\alpha\left[\sqrt{\pi}(1-\Phi(z))z-e^{-z^2}\right]}{(1-\Phi(z))}-\sqrt{\pi}\right\} \quad (a.2)$$

Thus finding the $\lim_{y\to\infty}\eta_\theta = -\infty$ amounts to studying the behaviour of (a.2) as z tends to infinity. We now consider the curly brackets of (a.2), to calculate

$$\lim_{z\to\infty}\frac{2z\left[\sqrt{\pi}(1-\Phi(z))z-e^{-z^2}\right]}{(1-\Phi(z))} \quad \text{and} \quad \lim_{z\to\infty}\frac{\sqrt{2}\alpha\left[\sqrt{\pi}(1-\Phi(z))z-e^{-z^2}\right]}{(1-\Phi(z))}$$

As to the former, we can use Hôpital's rule to obtain

$$\lim_{z\to\infty}\frac{2z\left[\sqrt{\pi}(1-\Phi(z))z-e^{-z^2}\right]}{(1-\Phi(z))}=2\lim_{z\to\infty}\frac{\dfrac{2\sqrt{\pi}\,(1-\Phi(z))\,z}{e^{-z^2}}-1}{-\dfrac{2}{\sqrt{\pi}}}=-\sqrt{\pi} \quad (a.3)$$

where we use the fact that, by Hôpital's rule,

$$\lim_{z\to+\infty}\frac{(1-\Phi(z))\,z}{e^{-z^2}}=\frac{1}{\sqrt{\pi}} \quad (a.4)$$

As to the latter, notice that

$$\lim_{z\to\infty}\frac{\sqrt{2}\alpha\left[\sqrt{\pi}(1-\Phi(z))z-e^{-z^2}\right]}{(1-\Phi(z))}=\lim_{z\to\infty}\left\{\frac{\sqrt{2}\alpha}{z}\frac{z\left[\sqrt{\pi}(1-\Phi(z))z-e^{-z^2}\right]}{(1-\Phi(z))}\right\}=0 \quad (a.5)$$

where we use (a.3).

Finally, the term outside the curly brackets tends to infinity. To see this, notice that

$$\lim_{z\to\infty}\frac{ke^{-z^2}}{1-\Phi(z)}=k\lim_{z\to\infty}\frac{e^{-z^2}}{z(1-\Phi(z))}\cdot z=\infty$$

using (a.4). Putting togethere these limits, $\lim_{y\to\infty}\eta_\theta = -\infty$.

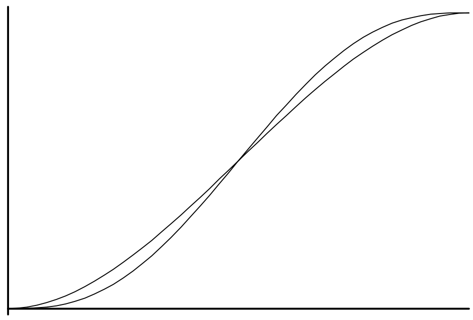

Fig. 1a. Single crossing of distributions

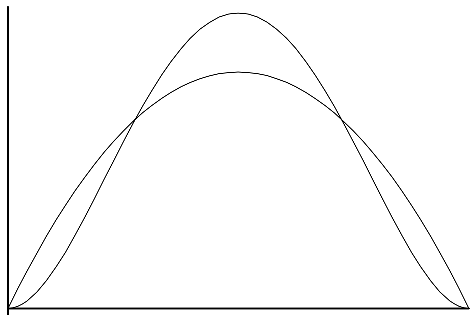

Fig. 1b. Double-crossing of densities

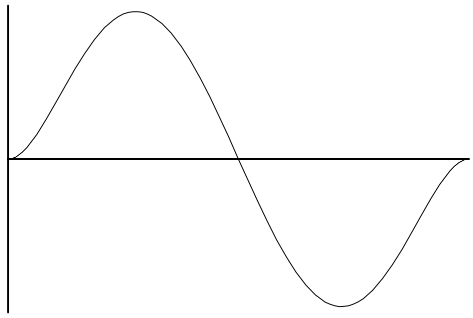

Fig. 1c. The function $F_\theta\left(y,\theta\right)$

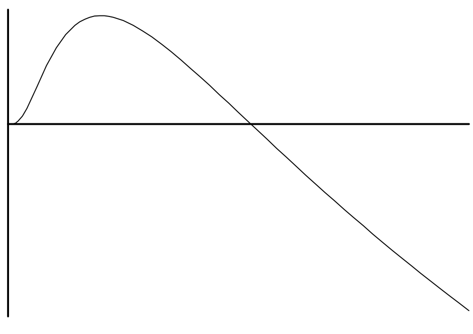

Fig. 2. The function $\eta_\theta(p,\theta)$