Chapter 4

Stability analysis of the Rayleigh-Bénard convection for a fluid with temperature and pressure dependent viscosity

4.1 Introduction

Consider a horizontal layer of fluid in which an adverse temperature gradient is mantained by heating the underside. The temperature gradient thus mantained is qualified as adverse since, on account of thermal expansion, the fluid at the bottom will be lighter than the fluid at the top. The basic state is then one of rest with light fluid below heavy fluid. When the adverse temperature gradient is great enough, the stabilizing effects of viscosity and thermal conductivity are overcome by the destabilizing buoyancy, and an overturning instability ensues as thermal convection. Convective instability was first described by Thomson in 1882 but the first experiments were made by Bénard in 1900. The experiments of Bénard established that the motions which ensue on surpassing the critical temperature gradient have a cellular stationary character. At the onset of instability the fluid layer resolves itself into a number of cells; and if the experiment is performed with sufficient care, the cells become equal and they align themselves to form a regular hexagonal pattern. This is called *Bénard convection* although Pearson [61] proved that most of the motions observed by Bénard were driven by the variation of surface tension with temperature and not by thermal instability of light fluid below heavy fluid.

The theoretical foundations for a correct interpretation of the convective instability were laid by Lord Rayleigh [69] who chose equations of motions and boundary conditions to model the experiments of Bénard and derived the linear equations for normal modes. He then showed that instability would occur only when the adverse temperature gradient was so large that the dimensionless parameter

$$\mathscr{R} = \frac{g\alpha d^4}{\kappa\nu}\left|\frac{\mathrm{d}T}{\mathrm{d}z}\right|,$$

now called Rayleigh number, exceeded a certain critical value. Here g is the acceleration due to gravity, α the coefficient of volumetric thermal expansion of the fluid, d the depth of the fluid layer, κ its thermal diffusivity, ν its kinematic viscosity and $|\mathrm{d}T/\mathrm{d}z|$ the magnitude of the vertical temperature gradient. Further theoretical and numerical studies of thermal convection for fluids with constant viscosity can be found in [11, 17] and references therein.

Fundamental early paper on convection in temperature-dependent viscosity fluids is that of Palm and coworkers [60] in which the following linear relationship

$$\nu(T) = \nu_0[1 - \gamma(T - T_0)],$$

ν_0, γ and T_0 being positive constants, is adopted. Richardson and Straughan [72] developed a conditional nonlinear stability analysis for such fluids and the result they obtained is very sharp in that it derives coincidence of the nonlinear stability and linear instability Rayleigh number thresholds. Capone and Gentile [8, 10] also develop a nonlinear stability analysis for fluids whose temperature-dependent kinematic viscosity is of the form

$$\nu(T) = \nu_0 \exp[-\gamma(T - T_0)],$$

whereas in [9] they treat a very general viscosity of the type

$$\nu(T) = \nu_0 f(T),$$

in which f is a convex non-increasing function. See also [15, 86, 92] and references therein for other important studies on the thermal convection for fluids with temperature-dependent viscosity.

On the contrary the stability analysis of the Bénard problem for fluids with pressure-dependent viscosity, to our knowledge, has not been received the same attention although it could be of practical interest in geophysics and in polymer melt processing. When the dependence of viscosity on pressure is taken into account, the Oberbeck-Boussinesq equations, i.e. the approximate equations of motion of a heat-conducting viscous fluid under the action of gravity, must be slightly modified as one needs to distinguish between the pressure due to gravity and the pressure due to the thermal expansion of the fluid, only the former contributes to variations in viscosity at a first approximation as we have shown in section 2.4. Then, by using the Oberbeck-Boussinesq-type equations we have derived in section 2.4 under the assumption that the coefficient of volumetric thermal expansion α, the heat conductivity k and the specific heat at constant pressure c_p are constants (such an assumption is reasonable as we have seen in section 3.1), we study the stability of the conduction solution in fluids whose viscosity is an analytic function of both temperature and pressure. In particular we first introduce the dimensionless perturbation equations of the Bénard problem for such a class of fluids. Thus we prove that the principle of exchange of stabilities holds and hence instability sets in as stationary convection. Furthermore, by following a standard procedure, we show how to find the critical Rayleigh number, the linear stability-instability threshold, by appealing to a variational analysis. Finally we study the nonlinear stability of the basic conduction solution by employing the energy method, and prove that the thresholds for linear theory and energy analysis coincide, provided the initial disturbance to the temperature field meets a specific restriction. We end this chapter with numerical results when the viscosity depends on temperature and pressure as in (3.12).

4.2 The problem

Let $Oxyz$ be a Cartesian frame of reference with fundamental unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, respectively, $\mathbf{k}$ pointed vertically upward in a direction opposed to that in which gravity acts. Let $\Omega_d = \mathbb{R}^2 \times (0, d)$ $(d > 0)$ be a horizontal layer of fluid whose viscosity is an analytic function of pressure and temperature and assume that the top and bottom surfaces of the fluid are held at constant temperature T_2 and T_1 $(T_1 > T_2)$, respectively. The equations governing the fluid motion in Ω_d are:

$$\begin{cases} \nabla p + \rho_0 g \mathbf{k} = 0 \\ \rho_0 \mathbf{v}_t + \rho_0 \mathbf{v} \cdot \nabla \mathbf{v} = -\alpha(T_1 - T_2)\nabla P + \mu(p, T)\Delta \mathbf{v} \\ \qquad\qquad +2\mathbf{D} \cdot \nabla \mu(p, T) + \rho_0 g \alpha (T - T_2)\mathbf{k} \\ \operatorname{div} \mathbf{v} = 0 \\ T_t + \mathbf{v} \cdot \nabla T = \kappa \Delta T \end{cases} \tag{4.1}$$

where ρ_0 is the density at the reference temperature T_2, $\kappa = k/(\rho_0 c_p)$ is the thermal diffusivity, g and p are, respectively, the acceleration and the pressure field due to gravity, P is the pressure due to the thermal expansion of the fluid and T is the temperature. Equations (4.1) have been established in section 2.4. The appropriate boundary conditions to append to system (4.1) are

$$\begin{cases} T(x_1,x_2,0,t)=T_1, \quad T(x_1,x_2,d,t)=T_2, \\ p(x_1,x_2,d,t)=p_0 \end{cases} \tag{4.2}$$

where p_0 is the reference pressure. Our aim is the study of stability of the steady static conduction solution m_0 to (4.1)-(4.2):

$$\begin{cases} \bar{p}=-\rho_0 g(z-d)+p_0 \\ \bar{\mathbf{v}}=\mathbf{0} \\ \bar{T}=-\dfrac{T_1-T_2}{d}z+T_1 \\ \bar{P}=-\rho_0 g z\left(\dfrac{z}{2d}-1\right)+P_0. \end{cases} \tag{4.3}$$

In order to study the stability of the conduction solution m_0 we introduce the perturbations $\mathbf{u}=u\mathbf{i}+v\mathbf{j}+w\mathbf{k}$, θ, p_1 and P_1 to $\bar{\mathbf{v}}$, $\bar{T}$, $\bar{p}$ and $\bar{P}$, respectively, i.e.,

$$\mathbf{v}=\bar{\mathbf{v}}+\mathbf{u}, \quad T=\bar{T}+\theta, \quad p=\bar{p}+p_1, \quad P=\bar{P}+P_1.$$

Setting $\mathbf{d}=[\nabla\mathbf{u}+(\nabla\mathbf{u})^T]/2$, from (3.1) the perturbations are found to satisfy

$$\begin{cases} \nabla p_1=0 \\ \rho_0\mathbf{u}_t+\rho_0\mathbf{u}\cdot\nabla\mathbf{u}=-\alpha(T_1-T_2)\nabla P_1+\mu(\bar{p}+p_1,\bar{T}+\theta)\Delta\mathbf{u} \\ \qquad\qquad +2\mathbf{d}\cdot\nabla\mu(\bar{p}+p_1,\bar{T}+\theta)+\rho_0 g\alpha\theta\mathbf{k} \\ \operatorname{div}\mathbf{u}=0 \\ \theta_t+\mathbf{u}\cdot\nabla\theta-\dfrac{T_1-T_2}{d}w=\kappa\Delta\theta \end{cases} \tag{4.4}$$

in $\mathbb{R}^2\times(0,d)\times(0,+\infty)$. To the previous system we append the initial conditions

$$\mathbf{u}(\mathbf{x},0)=\mathbf{u}_0, \quad \theta(\mathbf{x},0)=\theta_0(\mathbf{x}), \tag{4.5}$$

and the boundary conditions

$$\theta(x,y,0,t)=\theta(x,y,d,t)=0, \quad p_1(x,y,0,t)=0 \tag{4.6}$$

as the surface $z=0$ is mantained at constant temperature whereas the surface $z=d$ is mantained at constant pressure as well as at constant temperature. In (4.5) $\mathbf{u}_0$ and θ_0 are regular fields, $\mathbf{u}_0$ being divergence-free. From $(4.4)_1$ and $(4.6)_2$ it readily follows that $p_1\equiv 0$. As concerns the boundary conditions for the perturbation to velocity $\mathbf{u}$ we shall distinguish two kinds of bounding surfaces: *rigid surfaces* on which no slip occurs and *free surfaces* on which no tangential stresses act (see [11] for details).

For rigid bounding surfaces

$$\mathbf{u} = \mathbf{0} \quad \text{at } z = 0, d. \tag{4.7}$$

Since this condition must be satisfied for all x and y on the rigid surfaces $z = 0, d$, from the equation of continuity $(4.4)_3$ it follows that

$$\frac{\partial w}{\partial z} = 0 \quad \text{at } z = 0, d.$$

For free bounding surfaces

$$\frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = w = 0 \quad \text{at } z = 0, d, \tag{4.8}$$

by which, differentiating the equation of continuity $(4.4)_3$ with respect to z, we deduce that

$$\frac{\partial^2 w}{\partial z^2} = 0 \quad \text{at } z = 0, d.$$

Returning to equations (4.4), we non-dimensionalize them by introducing the following dimensionless quantities:

$$\left.\begin{aligned} &\mathbf{x}^* = \frac{\mathbf{x}}{d}, \quad t^* = \frac{\mu_0}{\rho_0 d^2} t, \quad \mathbf{u}^* = \frac{\rho_0 d}{\mu_0}\mathbf{u}, \quad \mu^* = \frac{\mu}{\mu_0}, \\ &\bar{p}^* = \frac{\bar{p} - p_0}{\rho_0 g d} = -(z^* - 1), \quad \bar{T}^* = \frac{\bar{T} - T_2}{T_1 - T_2} = -(z^* - 1), \\ &P_1^* = \frac{\alpha(T_1 - T_2)\rho_0 d^2}{\mu_0^2} P_1, \quad \theta^* = \frac{\rho_0 d}{\mu_0}\sqrt{\frac{\alpha \rho_0 g d \kappa}{\mu_0 (T_1 - T_2)}}\,\theta, \\ &\mathscr{R} = R^2 = \frac{\alpha(T_1 - T_2)\rho_0 g d^3}{\mu_0 \kappa}, \quad Pr = \frac{\mu_0}{\rho_0 \kappa}, \end{aligned}\right\} \tag{4.9}$$

where $\mu_0 = \mu(p_0, T_2)$ is the viscosity at the reference state (p_0, T_2). With this scaling the non-dimensional form of (4.4) becomes (omitting all asterisks)

$$\left\{\begin{aligned} &\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} = -\nabla P_1 + \mu(\bar{p}, \bar{T} + \theta)\Delta\mathbf{u} \\ &\qquad\qquad +2\mathbf{d}\cdot\nabla\mu(\bar{p}, \bar{T} + \theta) + R\theta\mathbf{k} \\ &\operatorname{div}\mathbf{u} = 0 \\ &Pr(\theta_t + \mathbf{u}\cdot\nabla\theta) - Rw = \Delta\theta \end{aligned}\right. \tag{4.10}$$

in $\mathbb{R}^2 \times (0,1) \times (0,+\infty)$ with boundary conditions

$$\theta = 0 \quad \text{at } z = 0, 1,$$

and

$$\mathbf{u} = \mathbf{0} \quad \text{at } z = 0, 1,$$

for rigid boundaries, or

$$\frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = w = 0 \quad \text{at } z = 0, 1,$$

for free bounding surfaces. In (4.10)

$$\mathscr{R} = R^2 = \frac{\alpha(T_1 - T_2)\rho_0 g d^3}{\mu_0 \kappa}$$

is the Rayleigh number and

$$Pr = \frac{\mu_0}{\rho_0 \kappa}$$

is the Prandtl number. Note that the Rayleigh number is positive since the lower boundary is hotter than the upper one and is seen to be the characteristic ratio of the buoyancy to the viscous forces. Also note that the Prandtl number is an intrinsic property of the fluid; it measures the ratio of the molecular diffusion of momentum and heat.

From now on, as usual, we shall assume that the perturbations $\mathbf{u}$, θ and P_1 have periods $2\pi/a_x$ and $2\pi/a_y$ in the x and y directions ($a_x > 0$, $a_y > 0$), denote by Ω the period cell

$$\Omega = \left[0, \frac{2\pi}{a_x}\right] \times \left[0, \frac{2\pi}{a_y}\right] \times [0, 1]$$

and by $a = (a_x^2 + a_y^2)^{1/2}$ the two-dimensional wave number. Moreover, since the stability of m_0 makes sense only in a class of solutions of (4.10) in which the zero solution $u = v = w = \theta = P_1 = 0$ is unique, for free bounding surfaces we exclude any other solution by requiring the usual 'average velocity conditions' (see [33])

$$\int_\Omega u \mathrm{d}\Omega = \int_\Omega v \mathrm{d}\Omega = 0. \tag{4.11}$$

4.3 Linear stability analysis

Since we have assumed that the viscosity is an analytic function of the temperature and pressure, for sufficiently small disturbances we can expand μ in the following manner:

$$\mu(\bar{p}, \bar{T} + \theta)\Delta\mathbf{u} = \left[\sum_{n=0}^{+\infty} \frac{1}{n!} \frac{\partial^n \mu}{\partial T^n}(\bar{p}, \bar{T})\theta^n\right] \Delta\mathbf{u} \approx \mu(z)\Delta\mathbf{u}$$

and

$$2\mathbf{d}\cdot\nabla\mu(\bar{p},\bar{T}+\theta) = 2\mathbf{d}\cdot\left\{\sum_{n=0}^{+\infty}\frac{1}{n!}\nabla\left[\frac{\partial^n\mu}{\partial T^n}(\bar{p},\bar{T})\theta^n\right]\right\}$$
$$\approx \mu'(z)\left[\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\mathbf{i}+\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\mathbf{j}+2\frac{\partial w}{\partial z}\mathbf{k}\right],$$

where

$$\mu(z) = \mu(\bar{p},\bar{T}) \tag{4.12}$$

and the prime denotes differentiation with respect to z. Thus linearizing (4.10) we obtain

$$\begin{cases} \mathbf{u}_t = -\nabla P_1 + \mu(z)\Delta\mathbf{u} + \mu'(z)\Big[\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\mathbf{i} \\ \qquad + \left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\mathbf{j} + 2\frac{\partial w}{\partial z}\mathbf{k}\Big] + R\theta\mathbf{k} \\ \mathrm{div}\,\mathbf{u} = 0 \\ Pr\theta_t - Rw = \Delta\theta. \end{cases} \tag{4.13}$$

We can easily eliminate the pressure P_1 and the dependent variables u and v. The curl of equation $(4.13)_1$ gives

$$\frac{\partial\boldsymbol{\omega}}{\partial t} = \mu(z)\Delta\boldsymbol{\omega} + \mu'(z)\frac{\partial\boldsymbol{\omega}}{\partial z} + \left[R\nabla\theta - \mu'(z)\Delta\mathbf{u} - 2\mu''(z)\mathbf{d}\cdot\mathbf{k}\right]\times\mathbf{k} \tag{4.14}$$

where the vorticity $\boldsymbol{\omega} = \nabla\times\mathbf{u}$. The curl of equation (4.14) in turn gives, after use of equation $(4.13)_2$,

$$\frac{\partial}{\partial t}\Delta\mathbf{u} = 2\mu'(z)\Delta\frac{\partial\mathbf{u}}{\partial z} + \mu(z)\Delta\Delta\mathbf{u} + \mu''(z)\frac{\partial^2\mathbf{u}}{\partial z^2} - \mu''(z)\Delta w\mathbf{k} + \mu''(z)\nabla\frac{\partial w}{\partial z}$$
$$+ R\Delta\theta\mathbf{k} - R\nabla\frac{\partial\theta}{\partial z} + \mu'(z)\Delta\boldsymbol{\omega}\times\mathbf{k} + \mu''(z)(\Delta\mathbf{u} - \Delta w\mathbf{k})$$
$$+ 2\mu'''(z)\left(\mathbf{d}\cdot\mathbf{k} - \frac{\partial w}{\partial z}\mathbf{k}\right) + \mu''(z)\frac{\partial\boldsymbol{\omega}}{\partial z}\times\mathbf{k}.$$

In particular

$$\frac{\partial}{\partial t}\Delta w = 2\mu'(z)\Delta\frac{\partial w}{\partial z} + \mu(z)\Delta\Delta w + \mu''(z)\frac{\partial^2 w}{\partial z^2} - \mu''(z)\Delta_1 w + R\Delta_1\theta$$

where by

$$\Delta_1 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

we denote the horizontal Laplacian. From the equation of continuity $(4.13)_2$ one can readily deduce that

$$\Delta_1 u = -\frac{\partial^2 w}{\partial x \partial z} - \frac{\partial \zeta}{\partial y} \tag{4.15}$$

and

$$\Delta_1 v = -\frac{\partial^2 w}{\partial y \partial z} + \frac{\partial \zeta}{\partial x}, \tag{4.16}$$

where

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

is the vertical component of vorticity. This is given by the vertical component of equation (4.14), namely

$$\frac{\partial \zeta}{\partial t} = \mu(z)\Delta\zeta + \mu'(z)\frac{\partial \zeta}{\partial z}. \tag{4.17}$$

From (4.7) and (4.8) the boundary conditions for ζ are

$$\zeta = 0 \quad \text{at } z = 0, 1, \quad \text{for rigid surfaces,}$$

$$\frac{\partial \zeta}{\partial z} = 0 \quad \text{at } z = 0, 1, \quad \text{for free surfaces.}$$

So u and v can be found by solving the Poisson equations (4.15), (4.16) when w has been found by solving the system

$$\begin{cases} \dfrac{\partial}{\partial t}\Delta w = 2\mu'(z)\Delta\dfrac{\partial w}{\partial z} + \mu(z)\Delta\Delta w + \mu''(z)\dfrac{\partial^2 w}{\partial z^2} \\ \qquad -\mu''(z)\Delta_1 w + R\Delta_1\theta \\ Pr\theta_t - Rw = \Delta\theta \end{cases} \tag{4.18}$$

and ζ by solving the diffusion equation (4.17).

Since the coefficients in equations (4.18) depend only on z, the equations admit solutions which depend on x, y and t exponentially. We consider therefore solutions of the form:

$$\begin{cases} w(x,y,z,t) = W(z)\exp[\mathrm{i}(a_x x + a_y y) + ct] \\ \theta(x,y,z,t) = \Theta(z)\exp[\mathrm{i}(a_x x + a_y y) + ct], \end{cases} \tag{4.19}$$

in which it is understood that the real parts of these expressions must be taken to obtain physical quantities. The wave speed c may be complex, i.e. $c = c_r + \mathrm{i}c_i$, and the expressions (4.19) thus represent waves which travel in the direction $(a_x, a_y, 0)$ with phase speed $c_\mathrm{i}/(a_x^2 + a_y^2)^{1/2}$ and which grow or decay in time like $\exp(c_{\rm r}t)$. Such a wave is stable if $c_{\rm r} \le 0$, unstable if $c_{\rm r} > 0$, and neutrally stable if $c_{\rm r} = 0$.

If we now let $\mathrm{D} = \mathrm{d}/\mathrm{d}z$ and $a = (a_x^2 + a_y^2)^{1/2}$, then on substituting the expressions (4.19) into equations (4.18) we obtain the system of ordinary differential equations

$$\begin{cases} c(\mathrm{D}^2 - a^2)W = 2\mu'(z)\mathrm{D}(\mathrm{D}^2 - a^2)W + \mu(z)(\mathrm{D}^2 - a^2)^2 W \\ \qquad\qquad + \mu''(z)(\mathrm{D}^2 + a^2)W - Ra^2\Theta \\ cPr\Theta - RW = (\mathrm{D}^2 - a^2)\Theta, \end{cases} \tag{4.20}$$

to which we add the boundary conditions

$$W = \mathrm{D}W = \Theta = 0 \quad \text{at } z = 0, 1, \quad \text{for rigid surfaces,} \tag{4.21}$$

or

$$W = \mathrm{D}^2 W = \Theta = 0 \quad \text{at } z = 0, 1, \quad \text{for free surfaces.} \tag{4.22}$$

Denoting by the superscript * the complex conjugate, multiplying $(4.20)_1$ by W^*, $(4.20)_2$ by $a^2\Theta^*$, summing and integrating over the interval $[0, 1]$, we have

$$\begin{aligned} c\int_0^1 [|\mathrm{D}W|^2 + a^2(|W|^2 + Pr|\Theta|^2)]\mathrm{d}z &= a^2 R\int_0^1 (\Theta W^* + W\Theta^*)\mathrm{d}z \\ &- \int_0^1 \mu(z)\left(|(\mathrm{D}^2 + a^2)W|^2 + 4a^2|\mathrm{D}W|^2\right)\mathrm{d}z \\ &- a^2\int_0^1 (|\mathrm{D}\Theta|^2 + a^2|\Theta|^2)\mathrm{d}z. \end{aligned} \tag{4.23}$$

The right hand side of (4.23) is real and then taking the imaginary part of (4.23) we find

$$c_i = 0.$$

Therefore the linearized equations for Bénard convection satisfy the principle of exchange of stabilities even when the fluid viscosity depends analytically on temperature and pressure. Thus, to find the instability boundary, the lowest value of $\mathscr{R} = R^2$ for which $c > 0$, we solve (4.20) for the smallest eigenvalue $R_L(a)$ with $c > 0$ (see [87]), that is we find the least eigenvalue $R_L(a)$ of the characteristic-value problem which gives the neutrally stable states

$$\begin{cases} 2\mu'(z)\mathrm{D}(\mathrm{D}^2 - a^2)W + \mu(z)(\mathrm{D}^2 - a^2)^2 W \\ \qquad\qquad + \mu''(z)(\mathrm{D}^2 + a^2)W = Ra^2\Theta \\ (\mathrm{D}^2 - a^2)\Theta + RW = 0 \end{cases} \tag{4.24}$$

with boundary conditions (4.21) or (4.22).

We now prove that for marginal stable disturbances

$$\frac{1}{R_L(a)} = \max_{(W,\Theta)\in\mathcal{H}} \frac{\mathcal{I}(W,\Theta)}{\mathcal{D}(W,\Theta)}, \tag{4.25}$$

$$\mathcal{I}(W,\Theta) = a^2 \int_0^1 (W\Theta^* + \Theta W^*)\mathrm{d}z, \tag{4.26}$$

$$\mathcal{D}(W,\Theta) = \int_0^1 \mu(z)\left(|(\mathrm{D}^2 + a^2)W|^2 + 4a^2|\mathrm{D}W|^2\right)\mathrm{d}z + a^2\left(\int_0^1 |\mathrm{D}\Theta|^2\mathrm{d}z + a^2\int_0^1 |\Theta|^2\mathrm{d}z\right) \tag{4.27}$$

and $\mathcal{H}$ denotes the set of the kinematically admissible disturbances:

$$\mathcal{H} = \left\{(W,\Theta) \in \mathrm{H}^2(0,1) \times \mathrm{H}^1(0,1) : W = \mathrm{D}W = \Theta = 0 \ \text{at}\ z = 0,1\right\}$$

for rigid boundaries, or

$$\mathcal{H} = \left\{(W,\Theta) \in \mathrm{H}^2(0,1) \times \mathrm{H}^1(0,1) : W = \mathrm{D}^2W = \Theta = 0 \ \text{at}\ z = 0,1\right\}$$

for free surfaces.

By (4.26) and (4.27), (4.23) becomes

$$c\int_0^1 [|\mathrm{D}W|^2 + a^2(|W|^2 + Pr|\Theta|^2)]\mathrm{d}z = \left[R\frac{\mathcal{I}(W,\Theta)}{\mathcal{D}(W,\Theta)} - 1\right]\mathcal{D}(W,\Theta) \tag{4.28}$$

by which we readily deduce that if

$$R \le \left[\max_{(W,\Theta)\in\mathcal{H}} \frac{\mathcal{I}(W,\Theta)}{\mathcal{D}(W,\Theta)}\right]^{-1},$$

then the modes of two-dimensional wave number a are linearly stable. Furthermore it is easy to check that the Euler-Lagrange equations associated with the variational problem (4.25) coincide with equations (4.24) giving the neutrally stable states and, since the maximum of the functional $\mathcal{I}/\mathcal{D}$ is the reciprocal of the least positive eigenvalue of the characteristic value problem (4.24) with boundary conditions (4.21) or (4.22), the equality in (4.25) holds true. Therefore the modes of two-dimensional wave number a are linearly stable if and only if $R \le R_L(a)$. Next we introduce the so-called critical Rayleigh number

$$\mathscr{R}_c = \min_{a>0} R_L^2(a)^1, \tag{4.29}$$

and note that if $\mathscr{R} \le \mathscr{R}_c$ then all modes are stable, while if $\mathscr{R} > \mathscr{R}_c$ there exists at least one unstable mode. Thus the conduction solution m_0 is linearly stable if and only if $\mathscr{R} \le \mathscr{R}_c$.

[1]For any eigenfunction $(\bar{W},\bar{\theta})$ of the characteristic-value problem (4.24) with boundary

4.4 Nonlinear stability

Let now $\|\cdot\|$ denote the $L^2(\Omega)$ norm. In order to establish a nonlinear stability result we commence by multiplying $(4.10)_1$ by $\mathbf{u}$, $(4.10)_3$ by θ, and we then integrate over Ω to find:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{u}\|^2 = R\int_\Omega w\theta \mathrm{d}\Omega - 2\int_\Omega \mu(\bar{p},\bar{T}+\theta)\mathbf{d}\cdot\mathbf{d}\mathrm{d}\Omega, \tag{4.30}$$

$$\frac{Pr}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\theta\|^2 = R\int_\Omega w\theta \mathrm{d}\Omega - \|\nabla\theta\|^2. \tag{4.31}$$

Hence by summing (4.30) and (4.31) we get

$$\frac{\mathrm{d}E}{\mathrm{d}t} = 2R\int_\Omega w\theta \mathrm{d}\Omega - 2\int_\Omega \mu(\bar{p},\bar{T}+\theta)\mathbf{d}\cdot\mathbf{d}\mathrm{d}\Omega - \|\nabla\theta\|^2, \tag{4.32}$$

where

$$E = \frac{1}{2}\|\mathbf{u}\|^2 + \frac{Pr}{2}\|\theta\|^2$$

is the sum of the kinetic and thermal energies associated with the perturbations.

We now state and prove a maximum principle (see Temam [89]) which will be very useful for our nonlinear stability analysis.

Lemma 4.1. *Let the disturbances* $\mathbf{u}$, P_1, θ *satisfy* (4.10) *with boundary conditions*

$$w = \theta = 0 \quad at\ z = 0, 1. \tag{4.33}$$

Then, if

$$|\theta(\mathbf{x},0)| \le \Theta_0 \quad a.e.\ \mathbf{x}\in\Omega \tag{4.34}$$

for constant $\Theta_0 \ge \dfrac{R}{Pr}$, *it follows that*

$$|\theta(\mathbf{x},t)| \le \Theta_0 \quad a.e.\ \mathbf{x}\in\Omega, \quad a.e.\ t\ge 0.$$

Proof. We start by defining the truncation operators that associate with a function $\psi:\Omega\to\mathbb{R}$, the functions ψ_+ and ψ_-

$$\psi_+(\mathbf{x}) = \max\{\psi(\mathbf{x}),0\}, \quad \psi_-(\mathbf{x}) = \max\{-\psi(\mathbf{x}),0\}, \quad \mathbf{x}\in\Omega.$$

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conditions (4.21) or (4.22) $\mathcal{I}(\bar{W},\bar{\theta})/\mathcal{D}(\bar{W},\bar{\theta})$ is a positive continuous function of the wave number a such that

$$\lim_{a\to 0^+}\frac{\mathcal{I}(\bar{W},\bar{\theta})}{\mathcal{D}(\bar{W},\bar{\theta})} = \lim_{a\to+\infty}\frac{\mathcal{I}(\bar{W},\bar{\theta})}{\mathcal{D}(\bar{W},\bar{\theta})} = 0,$$

then it admits maximum in $]0,+\infty[$. Consequently $R_L(a)$ is a positive continuous function such that $R_L(a)\to+\infty$ as $a\to 0^+$ and as $a\to+\infty$ and it admits minimum in $]0,+\infty[$.

Since $\mathbf{u}$, P_1, θ satisfy equations (4.10) with boundary conditions (4.33), the functions

$$T = T_2 + (T_1 - T_2)\left(1 - \frac{z}{d} + \frac{Pr}{R}\theta\right), \quad \mathbf{v} = \frac{\mu_0}{\rho_0 d}\mathbf{u},$$

$$P = \frac{\mu_0 P_1}{\alpha(T_1 - T_2)\rho_0 d^2} - \rho_0 g z\left(\frac{z}{2d} - 1\right) + P_0, \quad p = -\rho_0 g(z - d) + p_0$$

satisfy the boundary value problem (4.1)-(4.2).

We now prove that

$$T = \hat{T} + \tilde{T}$$

with

$$T_1^* = T_1 - (T_1 - T_2)\frac{Pr}{R}\Theta_0 \le \hat{T}(\mathbf{x}, t) \le T_2 + (T_1 - T_2)\frac{Pr}{R}\Theta_0 = T_2^*$$

almost everywhere in Ω for almost every $t \ge 0$ and $\tilde{T}(\cdot, t) \to 0$ in $\mathrm{L}^2(\Omega)$ as $t \to +\infty$.

Since $\theta \in \mathrm{H}^1(\Omega)$, it is clear that $(T - T_2^*)_+$ and $(T - T_1^*)_-$ are also in $\mathrm{H}^1(\Omega)$. Multiplying $(4.1)_4$ by $(T - T_2^*)_+$ and integrating over Ω we obtain, by taking into account the periodicity of the perturbations and the Poincaré inequality,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|(T - T_2^*)_+\|^2 + \kappa\frac{\pi^2}{d^2}\|(T - T_2^*)_+\|^2 \le 0 \tag{4.35}$$

by which we deduce that $\|(T - T_2^*)_+(\cdot, t)\|$ decreases exponentially

$$\|(T - T_2^*)_+(\cdot, t)\| \le \|(T - T_2^*)_+(\cdot, 0)\| \exp\left(-\kappa\frac{\pi^2}{d^2}t\right).$$

Similarly we prove that

$$\|(T - T_1^*)_-(\cdot, t)\| \le \|(T - T_1^*)_-(\cdot, 0)\| \exp\left(-\kappa\frac{\pi^2}{d^2}t\right).$$

Thus, setting

$$\tilde{T} = (T - T_2^*)_+ - (T - T_1^*)_- \quad \text{and} \quad \hat{T} = T - \tilde{T},$$

we see that

$$T_1^* \le \hat{T}(\mathbf{x}, t) \le T_2^* \quad \text{a.e. } \mathbf{x} \in \Omega, \text{ a.e. } t \ge 0$$

and

$$\|\tilde{T}(\cdot, t)\| \le \{\|(T - T_1^*)_-\| + \|(T - T_2^*)_+\|\}_{t=0} \exp\left(-\kappa\frac{\pi^2}{d^2}t\right).$$

Then

$$\theta = \hat{\theta} + \tilde{\theta}$$

with

$$-\Theta_0 \le \hat{\theta} = \frac{R}{Pr(T_1 - T_2)} \left[\hat{T} - T_2 - (T_1 - T_2)\left(1 - \frac{z}{d}\right)\right] \le \Theta_0$$

almost everywhere in Ω for almost every $t \ge 0$, and

$$\tilde{\theta} = \frac{R}{Pr(T_1 - T_2)} \tilde{T}.$$

But

$$(\theta - \Theta_0)_+ = \tilde{\theta}_+ = \frac{R}{Pr(T_1 - T_2)} (T - T_2^*)_+$$

and

$$(\theta + \Theta_0)_- = \tilde{\theta}_- = \frac{R}{Pr(T_1 - T_2)} (T - T_1^*)_- \, .$$

Therefore

$$\begin{aligned} \|(\theta - \Theta_0)_+(\cdot, t)\| &= \frac{R}{Pr(T_1 - T_2)} \|(T - T_2^*)_+(\cdot, t)\| \\ &\le \frac{R}{Pr(T_1 - T_2)} \|(T - T_2^*)_+(\cdot, 0)\| \exp\left(-\kappa \frac{\pi^2}{d^2} t\right) \\ &= \|(\theta - \Theta_0)_+(\cdot, 0)\| \exp\left(-\kappa \frac{\pi^2}{d^2} t\right) \end{aligned}$$

and, similarly,

$$\|(\theta + \Theta_0)_-(\cdot, t)\| \le \|(\theta + \Theta_0)_-(\cdot, 0)\| \exp\left(-\kappa \frac{\pi^2}{d^2} t\right).$$

For (4.34) we observe that $\|(\theta - \Theta_0)_+(\cdot, t)\|$ and $\|(\theta + \Theta_0)_-(\cdot, t)\|$ are decreasing functions of time that vanish at $t = 0$ and, consequently, they vanish for all later time $t > 0$. Thus $\tilde{\theta} = 0$ and the proof is completed. □

As an immediate consequence of Lemma 4.1, if the initial disturbance to the temperature field $\bar{T}$ satisfies the inequality

$$|\theta_0(\mathbf{x})| \le \Theta_0 \quad \text{a.e. } \mathbf{x} \in \Omega \tag{4.36}$$

for $\Theta_0 \ge R/Pr$ such that, by the analyticity of μ, we can write

$$\mu(\bar{p}, \bar{T} + \theta_0) = \sum_{n=0}^{+\infty} \frac{1}{n!} \frac{\partial^n \mu}{\partial T^n}(\bar{p}, \bar{T}) \theta_0^n \quad \text{a.e. } \mathbf{x} \in \Omega,$$

then

$$|\theta(\mathbf{x},t)| \leq \Theta_0 \quad \text{a.e. } \mathbf{x} \in \Omega, \quad \text{a.e. } t \geq 0 \tag{4.37}$$

and further

$$\mu(\bar{p}, \bar{T}+\theta) = \sum_{n=0}^{+\infty} \frac{1}{n!}\frac{\partial^n \mu}{\partial T^n}(\bar{p},\bar{T})\theta^n \quad \text{a.e. } \mathbf{x} \in \Omega, \quad \text{a.e. } t \geq 0. \tag{4.38}$$

Therefore for initial thermal disturbances satisfying (4.36), by (4.37) and (4.38), we have

$$\begin{aligned} \frac{\mathrm{d}E}{\mathrm{d}t} &= -\left(1 - R\frac{\mathscr{I}}{\mathscr{D}}\right)\mathscr{D} - 2\int_\Omega \sum_{n=1}^{+\infty} \frac{1}{n!}\frac{\partial^n \mu}{\partial T^n}(\bar{p},\bar{T})\theta^n \mathbf{d}\cdot\mathbf{d}\mathrm{d}\Omega \\ &\leq -\left[1 - \frac{R}{R_E(a)}\right]\mathscr{D} + 2\int_\Omega \sum_{n=1}^{+\infty}\frac{1}{n!}\left|\frac{\partial^n \mu}{\partial T^n}(z)\right|\Theta_0^n \mathbf{d}\cdot\mathbf{d}\mathrm{d}\Omega \\ &\leq -\left[1 - \frac{R}{R_E(a)}\right]\mathscr{D} + 2M\|\mathbf{d}\|^2, \end{aligned} \tag{4.39}$$

where

$$\mathscr{I} = 2\int_\Omega w\theta \mathrm{d}\Omega,$$

$$\mathscr{D} = 2\int_\Omega \mu(z)\mathbf{d}\cdot\mathbf{d}\mathrm{d}\Omega + \|\nabla\theta\|^2,$$

$$\frac{1}{R_E(a)} = \max_{\mathscr{W}} \frac{\mathscr{I}}{\mathscr{D}}, \tag{4.40}$$

$\mathscr{W}$ being the set of the kinematically admissible fields:

$$\begin{aligned}\mathscr{W} = \big\{&(\mathbf{u},\theta) \in \left(\mathrm{H}^1(\Omega)\right)^4 : \mathbf{u},\ \theta \text{ periodic in } x \text{ and } y \text{ of periods} \\ &2\pi/a_x,\ 2\pi/a_y, \operatorname{div}\mathbf{u} = 0,\ \ \mathbf{u} = \mathbf{0} \ \text{ and } \ \theta = 0 \ \text{ at } \ z = 0,1\big\}\end{aligned}$$

for rigid boundary conditions,

$$\begin{aligned}\mathscr{W} = \big\{&(\mathbf{u},\theta) \in \left(\mathrm{H}^1(\Omega)\right)^4 : \mathbf{u},\ \theta \text{ periodic in } x \text{ and } y \text{ of periods } 2\pi/a_x,\ 2\pi/a_y, \\ &\mathbf{u} \text{ satisfies } (4.11),\ \operatorname{div}\mathbf{u} = 0,\ \frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = w = \theta = 0 \ \text{ at } \ z = 0,1\big\}\end{aligned}$$

for stress-free boundary conditions, and

$$M = \max_{z\in[0,1]} \sum_{n=1}^{+\infty} \frac{1}{n!}\left|\frac{\partial^n \mu}{\partial T^n}(z)\right|\Theta_0^n.$$

By following Rionero [73] we prove the existence of the maximum of the functional $\mathscr{I}/\mathscr{D}$ in $\mathscr{W}$.

Theorem 4.2. *The functional $\mathscr{I}/\mathscr{D}$ admits maximum in $\mathscr{W}$.*

Proof. Taking into account the periodicity and the boundary conditions, by Poincaré and Wirtinger inequalities we have

$$\begin{aligned}\mathscr{D}(\mathbf{u},\theta) = 2\int_\Omega \mu(z)\mathbf{d}\cdot\mathbf{d}\mathrm{d}\Omega + \|\nabla\theta\|^2 \geq \mu_{\min}\|\nabla\mathbf{u}\|^2 + \|\nabla\theta\|^2 \\ \geq \mu_{\min}\pi_0^2\|\mathbf{u}\|^2 + \pi^2\|\theta\|^2 \quad \forall(\mathbf{u},\theta)\in\mathscr{W},\end{aligned}$$

where

$$\mu_{\min} = \min_{z\in[0,1]}\mu(z), \quad \mu_{\max} = \max_{z\in[0,1]}\mu(z) \quad \text{and} \quad \pi_0^2 = \min\{a_x^2, a_y^2, \pi^2\}.$$

Then, by Cauchy inequality, the functional $\mathscr{I}/\mathscr{D}$ is bounded from above

$$\begin{aligned}\frac{\mathscr{I}(\mathbf{u},\theta)}{\mathscr{D}(\mathbf{u},\theta)} &\leq \frac{\|\mathbf{u}\|^2+\|\theta\|^2}{\mu_{\min}\pi_0^2\|\mathbf{u}\|^2+\pi^2\|\theta\|^2} \\ &\leq \max\left\{\frac{1}{\mu_{\min}\pi_0^2},\frac{1}{\pi^2}\right\} \quad \forall(\mathbf{u},\theta)\in\mathscr{W}.\end{aligned}$$

Let now

$$\frac{1}{R_E(a)} = \sup_{(\mathbf{u},\theta)\in\mathscr{W}}\frac{\mathscr{I}(\mathbf{u},\theta)}{\mathscr{D}(\mathbf{u},\theta)} \tag{4.41}$$

and

$$\{\mathbf{u}_n,\theta_n\}_{n\in\mathbb{N}}\subset\mathscr{W}, \quad \mathscr{D}(\mathbf{u}_n,\theta_n)=1 \ \forall n\in\mathbb{N}$$

be a maximizing sequence, viz

$$\lim_{n\to+\infty}\mathscr{I}(\mathbf{u}_n,\theta_n) = \frac{1}{R_E(a)}. \tag{4.42}$$

We now observe that since $\mathscr{W}$ is a closed linear subspace of $(\mathrm{H}^1(\Omega))^4$ it is also weakly closed (see [50] page 134). Furthermore

$$\begin{aligned}\frac{1}{2}\min\left\{\mu_{\min}\pi_0^2,\mu_{\min},1\right\}\|(\mathbf{u},\theta)\|_{\mathscr{W}}^2 \leq \mathscr{D}(\mathbf{u},\theta) \\ \leq \max\left\{\mu_{\max},1\right\}\|(\mathbf{u},\theta)\|_{\mathscr{W}}^2 \quad \forall(\mathbf{u},\theta)\in\mathscr{W},\end{aligned} \tag{4.43}$$

that is in $\mathscr{W}$ the positive definite functional $\mathscr{D}$ defines a norm which is equivalent to that induced by the standard $(\mathrm{H}^1(\Omega))^4$-norm

$$\|(\mathbf{u},\theta)\|_{\mathscr{W}} = \left(\|\mathbf{u}\|^2+\|\nabla\mathbf{u}\|^2+\|\theta\|^2+\|\nabla\theta\|^2\right)^{1/2}.$$

Therefore by the previous observations and by Rellich-Kondrachov compact embedding Theorem there exists $(\mathbf{u}^*,\theta^*)\in\mathscr{W}$ such that, except for subsequences,

$$(\mathbf{u}_n,\theta_n)\rightharpoonup(\mathbf{u}^*,\theta^*) \quad \text{weakly in } (\mathrm{H}^1(\Omega))^4$$

and

$$(\mathbf{u}_n, \theta_n) \to (\mathbf{u}^*, \theta^*) \quad \text{strongly in } (\mathrm{L}^2(\Omega))^4. \tag{4.44}$$

We now prove that $\{(\mathbf{u}_n, \theta_n)\}_n$ is a Cauchy sequence with respect to the norm defined by $\mathscr{D}$ in $\mathscr{W}$. Obviously

$$\begin{cases} \mathscr{I}\left(\frac{\mathbf{u}_n + \mathbf{u}_m}{2}, \frac{\theta_n + \theta_m}{2}\right) = \frac{1}{2}\mathscr{I}(\mathbf{u}_n, \theta_n) + \frac{1}{2}\mathscr{I}(\mathbf{u}_m, \theta_m) \\ \qquad\qquad -\mathscr{I}\left(\frac{\mathbf{u}_n - \mathbf{u}_m}{2}, \frac{\theta_n - \theta_m}{2}\right) \\ \mathscr{D}\left(\frac{\mathbf{u}_n - \mathbf{u}_m}{2}, \frac{\theta_n - \theta_m}{2}\right) = \frac{1}{2}\mathscr{D}(\mathbf{u}_n, \theta_n) + \frac{1}{2}\mathscr{D}(\mathbf{u}_m, \theta_m) \\ \qquad\qquad -\mathscr{D}\left(\frac{\mathbf{u}_n + \mathbf{u}_m}{2}, \frac{\theta_n + \theta_m}{2}\right). \end{cases} \tag{4.45}$$

Let $\epsilon > 0$. By (4.41), (4.42) and (4.45) there exists $\nu_\epsilon \in \mathbb{N}$ such that

$$\frac{1}{R_E(a)}\left(1 - \frac{\epsilon}{8}\right) < \mathscr{I}(\mathbf{u}_n, \theta_n) < \frac{1}{R_E(a)}\left(1 + \frac{\epsilon}{8}\right) \quad \forall n \geq \nu_\epsilon,$$

$$\begin{aligned} \mathscr{D}\left(\frac{\mathbf{u}_n + \mathbf{u}_m}{2}, \frac{\theta_n + \theta_m}{2}\right) &\geq R_E(a)\mathscr{I}\left(\frac{\mathbf{u}_n + \mathbf{u}_m}{2}, \frac{\theta_n + \theta_m}{2}\right) \\ &> 1 - \frac{\epsilon}{8} - R_E(a)\mathscr{I}\left(\frac{\mathbf{u}_n - \mathbf{u}_m}{2}, \frac{\theta_n - \theta_m}{2}\right) \quad \forall n, m \geq \nu_\epsilon \end{aligned}$$

and

$$\mathscr{D}\left(\frac{\mathbf{u}_n - \mathbf{u}_m}{2}, \frac{\theta_n - \theta_m}{2}\right) < \frac{\epsilon}{8} + R_E(a)\mathscr{I}\left(\frac{\mathbf{u}_n - \mathbf{u}_m}{2}, \frac{\theta_n - \theta_m}{2}\right) \quad \forall n, m \geq \nu_\epsilon.$$

On the other hand by Hölder inequality and (4.44) there exists $\nu'_\epsilon \in \mathbb{N}$ such that

$$\mathscr{I}\left(\frac{\mathbf{u}_n - \mathbf{u}_m}{2}, \frac{\theta_n - \theta_m}{2}\right) < \frac{\epsilon}{8R_E(a)} \quad \forall n, m \geq \nu'_\epsilon,$$

and hence

$$\mathscr{D}(\mathbf{u}_n - \mathbf{u}_m, \theta_n - \theta_m) < \epsilon \quad \forall n, m \geq \max\{\nu_\epsilon, \nu'_\epsilon\}.$$

Since the norm defined by $\mathscr{D}$ in $\mathscr{W}$ is equivalent to $\|(\cdot,\cdot)\|_{\mathscr{W}}$ and since $(\mathscr{W}, \|(\cdot,\cdot)\|_{\mathscr{W}})$ is a Banach space, $(\mathbf{u}_n, \theta_n)$ converges strongly to $(\mathbf{u}_*, \theta_*)$ in $(\mathrm{H}^1(\Omega))^4$ and $\mathscr{D}(\mathbf{u}_*, \theta_*) = 1$.

Finally (4.44) and the continuity of the functional $\mathscr{I}$ in $(\mathrm{L}^2(\Omega))^4$ yield

$$\frac{1}{R_E(a)} = \lim_{n\to+\infty} \mathscr{I}(\mathbf{u}_n, \theta_n) = \mathscr{I}(\mathbf{u}^*, \theta^*).$$

The proof is thus completed. □

By assuming

$$R < R_E(a)$$

and by choosing Θ_0 such that

$$M < \left[1 - \frac{R}{R_E(a)}\right] \mu_{\min},$$

from (4.39), by Poincaré and Wirtinger inequalities, we deduce the following energy inequality

$$\frac{\mathrm{d}E}{\mathrm{d}t} \leq -\left[1 - \frac{R}{R_E(a)}\right] \nu_a E(t) \tag{4.46}$$

where

$$\nu_a = 2\min\left\{\pi_0^2\left[\mu_{\min} - \frac{MR_E(a)}{R_E(a) - R}\right], \frac{\pi^2}{Pr}\right\}.$$

Integrating (4.46) we have

$$E(t) \leq E(0)\exp\left\{-\left[1 - \frac{R}{R_E(a)}\right]\nu_a t\right\}. \tag{4.47}$$

The number $R_E(a)$ is found from the variational problem (4.40) and the Euler-Lagrange equations corresponding to this are

$$\begin{cases} -\nabla\chi = \mu(z)\Delta\mathbf{u} + \mu'(z)\Big[\Big(\dfrac{\partial u}{\partial z} + \dfrac{\partial w}{\partial x}\Big)\mathbf{i} \\ \qquad + \Big(\dfrac{\partial v}{\partial z} + \dfrac{\partial w}{\partial y}\Big)\mathbf{j} + 2\dfrac{\partial w}{\partial z}\mathbf{k}\Big] + R\theta\mathbf{k} \\ \operatorname{div}\mathbf{u} = 0 \\ \Delta\theta + Rw = 0, \end{cases} \tag{4.48}$$

where χ is a Lagrange multiplier associated with the divergence constraint. This eigenvalue problem is exactly the same as the one of linear stability theory and hence the critical Rayleigh numbers for the linear and nonlinear stability problems coincide. Finally, by Lemma 4.1 and by (4.47) we may state the following

Theorem 4.3. *Assume that*

$$\mathscr{R} < \mathscr{R}_c$$

with $\mathscr{R}_c$ *given by* (4.29)*, and*

$$|\theta_0(\mathbf{x})| \leq \Theta_0 \quad a.e.\ \mathbf{x} \in \mathbb{R}^2 \times [0,1]$$

for constant $\Theta_0 \geq R/Pr$ *such that*

$$M = \max_{z\in[0,1]} \sum_{n=1}^{+\infty} \frac{1}{n!}\left|\frac{\partial^n \mu}{\partial T^n}(z)\right| \Theta_0^n < \left(1 - \sqrt{\frac{\mathscr{R}}{\mathscr{R}_c}}\right)\mu_{\min}.$$

Then the conduction solution m_0 is nonlinearly stable with respect to the energy of the perturbations E(t), and

$$E(t) \le E(0)\exp\left[-\left(1-\sqrt{\frac{\mathscr{R}}{\mathscr{R}_c}}\right)\nu t\right],$$

where

$$\nu = 2\min\left\{\pi_0^2\left[\mu_{\min} - M\left(1-\sqrt{\frac{\mathscr{R}}{\mathscr{R}_c}}\right)^{-1}\right], \frac{\pi^2}{Pr}\right\}.$$

Remark 4.1. For temperature-dependent viscous fluids studied by Capone and Gentile in [8] and by Richardson and Straughan in [72] it is was found that the critical Rayleigh number depends on the choice of the reference temperature. Here we observe that the critical Rayleigh number will obviously depend on the choice of the reference pressure as well as of the reference temperature since the function $\mu(z)$ defined in (4.12) varies according to the choice of the reference state. We have chosen the values of pressure and temperature at the top of the fluid layer as reference state because we think this choice could be more convenient in the practical applications.

Remark 4.2. The results of this chapter may be condensed in the sentence: by using the generalization of the Oberbeck-Boussinesq equations we have derived in section 2.4 which is valid at small values of the dimensionless quantity $\alpha(T_1 - T_2)$, the nonlinear energy stability result agrees with the linear one even when viscosity is an analytic function of both temperature and pressure.

4.5 Numerical results

We now consider the pressure-temperature-viscosity relationship (3.12), nondimensionalize it as indicated in (4.9) and obtain the dimensionless viscosity

$$\mu(z) = \exp[\Gamma(z-1)]$$

with $\Gamma = \gamma(T_1 - T_2) - \beta\rho_0 g d$. The equations giving the marginal stable disturbances are then

$$\begin{cases} 2\Gamma \mathrm{D}(\mathrm{D}^2 - a^2)W + (\mathrm{D}^2 - a^2)^2 W \\ \qquad\qquad +\Gamma^2(\mathrm{D}^2 + a^2)W = \exp[-\Gamma(z-1)]Ra^2\Theta \\ (\mathrm{D}^2 - a^2)\Theta + RW = 0 \end{cases}$$

with boundary conditions (4.21) or (4.22). By employing the Galerkin-type method developed by Chandrasekhar [11] we find approximations to the critical Rayleigh number for different values of the dimensionless parameter Γ both for rigid (Table 4.1) and free (Table 4.2) bounding surfaces. For rigid boundaries we used "beam functions" (see [26]) and sines for free surfaces. We observe that for $\Gamma = 0$, in particular for constant viscosity ($\beta = 0$ and $\gamma = 0$), we obtain the classical results (see for instance [17]).

Table 4.1: Approximations to the critical Rayleigh and wave numbers against Γ in the rigid case.

Γ	$\mathscr{R}_c$	a_c
-2	5026.42	3.072
-1.5	3790.86	3.084
-1	2885.93	3.093
-0.5	2217.33	3.100
0	1707.76	3.117
0.5	1344.88	3.100
1	1061.67	3.093
1.5	845.855	3.084
2	680.252	3.072

Table 4.2: Approximations to the critical Rayleigh and wave numbers against Γ in the stress-free case.

Γ	$\mathscr{R}_c$	a_c
-2	1991.74	2.134
-1.5	1480.22	2.171
-1	1114.29	2.198
-0.5	850.114	2.216
0	657.51	2.221
0.5	515.62	2.216
1	409.926	2.198
1.5	330.281	2.171
2	269.552	2.134

As concerns the nonlinear energy stability analysis, Theorem 4.3 may be re-stated as follows

Theorem 4.4. *Assume that*

$$\mathscr{R} < \mathscr{R}_c$$

with $\mathscr{R}_c$ given by (4.29),

$$\gamma(T_1 - T_2) < \ln\left\{1 + \left(1 - \sqrt{\frac{\mathscr{R}}{\mathscr{R}_c}}\right)\exp(-|\Gamma|)\right\}$$

and

$$|\theta_0(\mathbf{x})| \leq \Theta_0 \in \left[\frac{\sqrt{\mathscr{R}}}{Pr}, \frac{\sqrt{\mathscr{R}}}{\gamma(T_1 - T_2)Pr}\ln\left\{1 + \left(1 - \sqrt{\frac{\mathscr{R}}{\mathscr{R}_c}}\right)\exp(-|\Gamma|)\right\}\right[$$

almost everywhere in $\mathbb{R}^2 \times [0,1]$. Then the conduction solution m_0 is nonlinearly stable with respect to the energy of the perturbations E(t), and

$$E(t) \leq E(0)\exp\left[-\left(1 - \sqrt{\frac{\mathscr{R}}{\mathscr{R}_c}}\right)\nu t\right],$$

where

$$\nu = 2\min\left\{\pi_0^2 A, \frac{\pi^2}{Pr}\right\},$$

$$A = \begin{cases} \exp(-\Gamma) - \dfrac{\exp\left[\dfrac{\gamma(T_1 - T_2)Pr}{\sqrt{\mathscr{R}}}\Theta_0\right] - 1}{1 - \sqrt{\dfrac{\mathscr{R}}{\mathscr{R}_c}}} & \text{if } \Gamma \geq 0 \\[4ex] 1 - \exp(-\Gamma)\dfrac{\exp\left[\dfrac{\gamma(T_1 - T_2)Pr}{\sqrt{\mathscr{R}}}\Theta_0\right] - 1}{1 - \sqrt{\dfrac{\mathscr{R}}{\mathscr{R}_c}}} & \text{if } \Gamma < 0. \end{cases}$$