Chapter 3

Steklov operators

In this chapter we consider Steklov operators in spaces of continuous functions on the real line and on a bounded interval. We study the connections of these operators with some second-order degenerate parabolic problems establishing a general Voronovskaja-type formula. We also need a quantitative version of Voronovskaja's formula in order to apply the quantitative estimates in Chapter 1.

The choice of Steklov operators is motivated by the fact that these operators can be used in different setting, such as spaces of continuous functions or weighted spaces of continuous functions both on bounded than unbounded real intervals.

The results in this chapter have been obtained in collaboration with I. Rasa (Cluj-Napoca, Romania) and published in [33], [34].

3.1 Steklov operators on the real line

In this section we point out some general properties of Steklov operators and we construct a sequence which can be canonically associated with an assigned second-order differential operator A.

Let $L^1_{\mathrm{loc}}(\mathbb{R})$ be the space of all locally integrable real functions and for every $b > 0$ define the integral mean operator $M_b : L^1_{\mathrm{loc}}(\mathbb{R}) \to C(\mathbb{R})$ by putting

$$M_b f(x) := \frac{1}{2b} \int_{x-b}^{x+b} f(t)\, dt \,, \qquad f \in L^1_{\mathrm{loc}}(\mathbb{R}) \,, \quad x \in \mathbb{R} \,. \tag{3.1.1}$$

Then, for every $n \geq 1$, the n-th Steklov operator $S_{n,b} : L^1_{\mathrm{loc}}(\mathbb{R}) \to C(\mathbb{R})$ is defined by setting

$$S_{n,b} := M_b^n \,, \tag{3.1.2}$$

where, as usual, M_b^0 is the identity operator and $M_b^n = M_b \circ M_b^{n-1}$ if $n \geq 1$.

Observe that, for every $f \in L^1_{\text{loc}}(\mathbb{R})$ and $x \in \mathbb{R}$,

$$\begin{aligned} S_{0,b}f(x) &= f(x) , \\ S_{n,b}f(x) &= M_b\left(S_{n-1,b}f\right)(x) = \frac{1}{2b}\int_{x-b}^{x+b} S_{n-1,b}f(t)dt . \end{aligned}$$

The preceding definition is meaningful also in the case where b is a bounded continuous strictly positive real function on $\mathbb{R}$; in this case the integration is extended over the interval $[x-b(x), x+b(x)]$ and (3.1.1) becomes

$$M_b f(x) := \frac{1}{2b(x)}\int_{x-b(x)}^{x+b(x)} f(t)\,dt , \qquad f \in L^1_{\text{loc}}(\mathbb{R}) , \quad x \in \mathbb{R} . \tag{3.1.3}$$

In this section we shall be mainly interested in studying some properties of Steklov operators in the space $C(\overline{\mathbb{R}})$ of all continuous real functions on $\mathbb{R}$ which admit finite limits at the points $\pm\infty$. The space $C(\overline{\mathbb{R}})$ is endowed with the uniform norm and obviously every function in $C(\overline{\mathbb{R}})$ is bounded and uniformly continuous.

Moreover as usual, we shall denote by $C^2(\overline{\mathbb{R}})$ the space of all functions $f \in C(\overline{\mathbb{R}})$ which are twice differentiable and such that $f'' \in C(\overline{\mathbb{R}})$. Observe that $C^2(\overline{\mathbb{R}})$ is obviously dense in $C(\overline{\mathbb{R}})$ with respect to the uniform norm.

Observe that if $f \in C(\mathbb{R})$ and $\lim_{x\to+\infty} f(x) = \ell \in \mathbb{R}$, then we also have $\lim_{x\to+\infty} S_{n,b}f(x) = \ell$. Indeed, this easily follows from an inductive argument on the integer $n \geq 0$ using the equality $S_{n,b}f(x) = S_{n-1,b}f(\xi)$ which holds for some $\xi \in [x-b(x), x+b(x)]$ and using the boundedness of the function b which implies that $\lim_{x\to+\infty}(x-b(x)) = +\infty$.

Hence Steklov operators may be regarded as linear operators from $C(\overline{\mathbb{R}})$ into $C(\overline{\mathbb{R}})$ and in this case they become positive linear contractions with respect to the uniform norm.

Our aim is to use these operators for the investigation of some degenerate second-order differential operators.

Namely let $a \in C(\overline{\mathbb{R}})$ be a strictly positive function such that

$$\text{id} \cdot a \in C(\overline{\mathbb{R}}) . \tag{3.1.4}$$

Consider the differential operator $A : D(A) \to C(\overline{\mathbb{R}})$ defined by

$$Au(x) := \frac{1}{6}\, a(x)^2\, u''(x) , \qquad u \in D(A) , \quad x \in \mathbb{R} , \tag{3.1.5}$$

where

$$D(A) := \left\{ u \in C(\overline{\mathbb{R}}) \cap C^2(\mathbb{R}) \mid a^2\, u'' \in C(\overline{\mathbb{R}}) \right\} . \tag{3.1.6}$$

Remark 3.1.1 Observe that A is a second-order elliptic differential operator, which is degenerate since a vanishes at the endpoints of the real line.

Moreover, the endpoints $\pm\infty$ are natural endpoints and hence the operator $(A, D(A))$ generates a C_0-semigroup of contractions on $C(\overline{\mathbb{R}})$ on the maximal domain $D(A)$.

Furthermore, every function $u \in D(A)$ also satisfies Ventcel's boundary conditions

$$\lim_{x\to\pm\infty} a(x)^2\, u''(x) = 0 \ .$$

Indeed if $u \in D(A) \setminus D_V(A)$, we should have for example $\lim\limits_{x\to+\infty} Au(x) = \ell \neq 0$. Then $\lim\limits_{x\to+\infty} \dfrac{1}{6x^2}\, a^2(x)x^2u''(x) = \ell$ and therefore $\lim\limits_{x\to+\infty} u''(x) \neq 0$; consequently $\lim\limits_{x\to+\infty} \dfrac{1}{x^2}u(x) \neq 0$, contradicting the condition $u \in C(\overline{\mathbb{R}})$. At the point $-\infty$ we can reason similarly.

Hence we conclude that in this case the maximal domain coincides with Ventcel's domain. For these and further results we also refer to the Chapter II, [48, Chapter VI, Section 4] and [69]. □

We shall be interested in studying the connections between the differential operator A and sequences of Steklov type operators.

Namely, we define the functions $b_n := a/n$ and consider the operators $L_n : C(\overline{\mathbb{R}}) \to C(\overline{\mathbb{R}})$ defined by

$$L_n f(x) := S_{n,b_n} f(x) \ , \qquad f \in C(\overline{\mathbb{R}}) \ , \quad x \in \mathbb{R} \ . \tag{3.1.7}$$

Since b_n is bounded, the operator L_n is well-defined as an operator acting on $C(\overline{\mathbb{R}})$; furthermore, since a vanishes at $\pm\infty$, we have that L_n interpolates every function $f \in C(\overline{\mathbb{R}})$ at $\pm\infty$, in the sense that $\lim_{x\to\pm\infty} L_n f(x) = \lim_{x\to\pm\infty} f(x)$.

Moreover, observe that $L_n = M_{b_n}^n$ for every $n \in \mathbb{N}$ and consequently, for every $k \geq 1$, we also have

$$L_n^k = M_{b_n}^{kn} = S_{kn,b_n} \ . \tag{3.1.8}$$

Using (3.1.7), the operators L_n can be extended in a natural way to the space of all continuous functions on $\mathbb{R}$, and for this extension we can state the following properties, for every $x \in \mathbb{R}$:

i) $L_n\mathbf{1}(x) = S_{n,b_n}\mathbf{1}(x) = 1$,

ii) $L_n\mathrm{id}(x) = S_{n,b_n}\mathrm{id}(x) = x$,

iii) $L_n(\mathrm{id}^2)(x) = S_{n,b_n}(\mathrm{id}^2)(x)$
$= x^2 + \frac{1}{3}\sum_{i=0}^{n-1} S_{i,b_n}(b_n^2)(x) = x^2 + \frac{1}{3n}\left(\frac{1}{n}\sum_{i=0}^{n-1} S_{i,b_n}(a^2)(x)\right)$.

Since S_{i,b_n} is a linear contraction, we can conclude that

$$\lim_{n\to+\infty} L_n(\mathrm{id}^2)(x) = \lim_{n\to+\infty} S_{n,b_n}(\mathrm{id}^2)(x) = x^2$$

uniformly with respect to $x \in \mathbb{R}$.

We need some further preliminary results in order to state some deeper properties of the operators L_n.

Proposition 3.1.2 *For every $f \in C(\overline{\mathbb{R}})$, $k \geq 1$ and $x \in \mathbb{R}$,*

$$\|S_{k,b_n}(f) - f\| \leq k\|S_{1,b_n}(f) - f\| . \tag{3.1.9}$$

PROOF. We argue by induction on the integer $k \geq 1$. If $k = 1$ then (3.1.9) is obviously true. Now, assume that (3.1.9) holds for $k \geq 1$. We have

$$\begin{aligned} |S_{k+1,b_n}f(x) - f(x)| &\leq |S_{1,b_n}(S_{k,b_n}f - f)(x)| + |S_{1,b_n}f(x) - f(x)| \\ &\leq \|S_{k,b_n}(f) - f\| + \|S_{1,b_n}(f) - f\| \\ &\leq k\|S_{1,b_n}(f) - f\| + \|S_{1,b_n}(f) - f\| \\ &\leq (k+1)\|S_{1,b_n}(f) - f\| \end{aligned}$$

and this completes the induction argument. □

As a consequence of the above result, we can state the following estimate.

Proposition 3.1.3 *For every $f \in C^2(\overline{\mathbb{R}})$ we have*

$$\|S_{k,b_n}f - f\| \leq \frac{\|a^2\|\|f''\|}{6}\frac{k}{n^2} . \tag{3.1.10}$$

PROOF. Let $f \in C^2(\overline{\mathbb{R}})$; for every $x, t \in \mathbb{R}$ we can write

$$f(t) - f(x) = f'(x)\,(t-x) + \frac{1}{2}f''(\xi_t)(t-x)^2$$

with ξ_t between t and x.

Then

$$\begin{aligned} S_{1,b}f(x) - f(x) &= \frac{1}{2b}\int_{x-b}^{x+b}(f(t) - f(x))dt \\ &= f'(x)\frac{1}{2b}\int_{x-b}^{x+b}(t-x)dt + \frac{1}{2b}\int_{x-b}^{x+b}\frac{1}{2}f''(\xi_t)(t-x)^2dt \\ &= \frac{1}{2b}\int_{x-b}^{x+b}\frac{1}{2}f''(\xi_t)(t-x)^2dt \end{aligned}$$

and

$$\begin{aligned} |S_{1,b}f(x) - f(x)| &\leq \frac{1}{2}\sup_{t\in[x-b,x+b]}|f''(t)|\frac{1}{2b}\int_{x-b}^{x+b}(t-x)^2dt \\ &= \frac{b^2}{3!}\sup_{t\in[x-b,x+b]}|f''(t)| . \end{aligned}$$

Letting $b := b_n(x) = a(x)/n$ and taking into account that f'' and a are bounded we can write

$$|S_{1,b_n} f(x) - f(x)| \le \frac{1}{n^2} \frac{a(x)^2}{3!} \sup_{t \in [x-b_n(x), x+b_n(x)]} |f''(t)| \, , \qquad (3.1.11)$$

and since $f \in C(\overline{\mathbb{R}})$ and $a \in C(\overline{\mathbb{R}})$

$$|S_{1,b_n} f(x) - f(x)| \le \frac{\|a^2\| \|f''\|}{6} \frac{1}{n^2} \, .$$

Finally, from Proposition 3.1.2, we get

$$\|S_{k,b_n} f - f\| \le k \, \|S_{1,b_n} f - f\| \le \frac{\|a^2\| \|f''\|}{6} \frac{k}{n^2} \, .$$

□

Under the following further hypothesis on a

$$\exists \, \delta > 0, \; \exists \, M > 0 \quad : \quad \sup_{\substack{x,y \in \mathbb{R} \\ |x-y| \le \delta}} \frac{a^2(x)}{a^2(y)} \le M \, , \qquad (3.1.12)$$

we can have (3.1.10) in terms of the operator A,

Proposition 3.1.4 *If condition* (3.1.12) *holds, then for every* $f \in D(A)$, *we have*

$$\|S_{k,b_n} f - f\| \le M \|Af\| \frac{k}{n^2} \, .$$

Proof. From (3.1.11) we have

$$|S_{1,b_n} f(x) - f(x)| \le \frac{1}{n^2} \frac{a(x)^2}{3!} \sup_{t \in [x-b_n(x), x+b_n(x)]} |f''(t)| \qquad (3.1.13)$$

$$= \frac{1}{n^2} \frac{a(x)^2}{3!} |f''(\xi)| = \frac{1}{n^2} \frac{|a^2(\xi) f''(\xi)|}{6} \frac{a^2(x)}{a^2(\xi)} \le \frac{\|Af\|}{n^2} \left(\frac{a(x)}{a(\xi)} \right)^2 \, ,$$

for some $\xi \in [x - a(x)/n, x + a(x)/n]$, that is $|\xi - x| \le \frac{\|a\|}{n}$. If we choose n such that $|\xi - x| \le \delta$ from condition (3.1.12) we have $\left(\frac{a(x)}{a(\xi)} \right)^2 \le M$ and from (3.1.13)

$$|S_{1,b_n} f(x) - f(x)| \le M \frac{\|Af\|}{n^2} \, .$$

Finally, from Proposition 3.1.2, we get

$$\|S_{k,b_n} f - f\| \le k \, \|S_{1,b_n} f - f\| \le M \|Af\| \, \frac{k}{n^2} \, .$$

□

Corollary 3.1.5 *For every $f \in C^2(\overline{\mathbb{R}})$,*

$$\|L_n(f) - f\| \le \frac{\|a^2\| \|f''\|}{6} \frac{1}{n} .$$

Corollary 3.1.6 *If condition* (3.1.12) *holds, then for every $f \in D(A)$, we have*

$$\|L_n(f) - f\| \le M \|Af\| \frac{1}{n} .$$

From the above results, we also obtain the following approximation properties of the sequence $(L_n)_{n\ge1}$.

Theorem 3.1.7 *For every $f \in C(\overline{\mathbb{R}})$, we have*

$$\lim_{n\to+\infty} L_n(f) = f \qquad \textit{uniformly on } \mathbb{R} .$$

Proof. Indeed, it is clear that the uniform convergence of $(L_n(f))_{n\ge1}$ to f holds true for every $f \in C^2(\overline{\mathbb{R}})$. Since $C^2(\overline{\mathbb{R}})$ is dense in $C(\overline{\mathbb{R}})$ and $(L_n)_{n\ge1}$ is a sequence of positive contractions, the proof is complete. □

Our next aim is to obtain a quantitative Voronovskaja-type formula for Steklov operators. We begin with some properties of independent interest.

Proposition 3.1.8 *For every $f \in C^2(\overline{\mathbb{R}})$ we have*

$$\left\| \frac{1}{n} \sum_{k=0}^{n-1} S_{k,b_n}(f) - f \right\| \le \frac{\|a^2\| \|f''\|}{12} \frac{1}{n} .$$

Proof. Let $f \in C^2(\overline{\mathbb{R}})$ using Proposition 3.1.3, for every $x \in \mathbb{R}$ we can write

$$\begin{aligned} \left| \frac{1}{n} \sum_{k=0}^{n-1} S_{k,b_n} f(x) - f(x) \right| &= \frac{1}{n} \left| \sum_{k=0}^{n-1} (S_{k,b_n} f(x) - f(x)) \right| \\ &\le \frac{1}{n} \sum_{k=0}^{n-1} |S_{k,b_n} f(x) - f(x)| \\ &\le \frac{1}{n} \sum_{k=0}^{n-1} \frac{\|a^2\| \|f''\|}{6} \frac{k}{n^2} = \frac{\|a^2\| \|f''\|}{6n^3} \sum_{k=0}^{n-1} k \\ &= \frac{\|a^2\| \|f''\|}{6n^3} \frac{n(n-1)}{2} \end{aligned}$$

and this completes the proof. □

Proposition 3.1.9 *If condition* (3.1.12) *holds, then for every* $f \in D(A)$, *we have*

$$\left\| \frac{1}{n} \sum_{k=0}^{n-1} S_{k,b_n}(f) - f \right\| \le M \frac{\|Af\|}{2} \frac{1}{n} .$$

Proposition 3.1.10 *For every* $f \in C(\overline{\mathbb{R}})$ *we have*

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{k=0}^{n-1} S_{k,b_n}(f) = f$$

uniformly on $\mathbb{R}$.

PROOF. If follows from the density of $C^2(\overline{\mathbb{R}})$ into $C(\overline{\mathbb{R}})$ and from Proposition 3.1.8 □

Finally, we shall need the following lemma.

Lemma 3.1.11 *We have*

1. $S_{n,b_n}\mathbf{1}(x) - 1 = 0$,

2. $S_{n,b_n}(\mathrm{id} - x)(x) = 0$,

3. $S_{n,b_n}(\mathrm{id} - x)^2(x) = \dfrac{1}{3n}\left(\dfrac{1}{n}\displaystyle\sum_{i=0}^{n-1} S_{i,b_n}(a^2)(x)\right)$,

4. $|S_{n,b_n}((\mathrm{id} - x)^4)(x)| \le \dfrac{1}{n^2}\dfrac{\|a^2\|^2}{3}$.

PROOF. We have already evaluated S_{n,b_n} at the functions $\mathbf{1}$, id and id^2, then a straightforward computations gives *1*, *2* and *3*.
Now, we also observe that

$$S_{n,b_n}(\mathrm{id}^3)(x) = x^3 + \sum_{k=0}^{n-1} S_{k,b_n}(\mathrm{id} \cdot b_n^2)(x) ,$$

$$S_{n,b_n}(\mathrm{id}^4)(x) = x^4 + 2\sum_{k=0}^{n-1} S_{k,b_n}(\mathrm{id}^2 \cdot b_n^2)(x) + \frac{1}{5}\sum_{k=0}^{n-1} S_{k,b_n}(b_n^4)(x) .$$

At this point we can easily evaluate $S_{n,b_n}\left((\mathrm{id}-x)^4\right)(x)$ and obtain

$$\begin{aligned}
S_{n,b_n}((\mathrm{id}-x)^4)(x) &= S_{n,b_n}(\mathrm{id}^4)(x) - 4xS_{n,b_n}(\mathrm{id}^3)(x) + 6x^2 S_{n,b_n}(\mathrm{id}^2)(x) \\
&\quad -4x^3 S_{n,b_n}(\mathrm{id})(x) + x^4 \\
&= x^4 + 2\sum_{k=0}^{n-1} S_{k,b_n}(\mathrm{id}^2\cdot b_n^2)(x) + \frac{1}{5}\sum_{k=0}^{n-1} S_{k,b_n}(b_n^4)(x) \\
&\quad -4x\left(x^3 + \sum_{k=0}^{n-1} S_{k,b_n}(\mathrm{id}\cdot b_n^2)(x)\right) \\
&\quad +6x^2\left(x^2 + \frac{1}{3}\sum_{k=0}^{n-1} S_{k,b_n}(b_n^2)(x)\right) - 4x^3x + x^4 \\
&= 2\sum_{k=0}^{n-1} S_{k,b_n}(\mathrm{id}^2\cdot b_n^2)(x) - 4x\sum_{k=0}^{n-1} S_{k,b_n}(\mathrm{id}\cdot b_n^2)(x) \\
&\quad +2x^2\sum_{k=0}^{n-1} S_{k,b_n}(b_n^2)(x) + \frac{1}{5}\sum_{k=0}^{n-1} S_{k,b_n}(b_n^4)(x) \\
&= \frac{2}{n^2}\sum_{k=0}^{n-1} S_{k,b_n}(\mathrm{id}^2\cdot a^2)(x) - 4x\frac{1}{n^2}\sum_{k=0}^{n-1} S_{k,b_n}(\mathrm{id}\cdot a^2)(x) \\
&\quad +2x^2\frac{1}{n^2}\sum_{k=0}^{n-1} S_{k,b_n}(a^2)(x) + \frac{1}{5n^2}\frac{1}{n^2}\sum_{k=0}^{n-1} S_{k,b_n}(a^4)(x)\ .
\end{aligned}$$

Consequently,

$$S_{n,b_n}((\mathrm{id}-x)^4)(x) = \frac{2}{n^2}\sum_{k=0}^{n-1} S_{k,b_n}((\mathrm{id}-x)^2\cdot a^2)(x) + \frac{1}{5n^4}\sum_{k=0}^{n-1} S_{k,b_n}(a^4)(x)\ .$$

As regards the first term, for every $k = 1, \ldots, n-1$, we have

$$\begin{aligned}
\left|S_{k,b_n}((\mathrm{id}-x)^2\cdot a^2)(x)\right| &\leq \left\|a^2\right\| \left|S_{k,b_n}((\mathrm{id}-x)^2)(x)\right| \\
&= \left\|a^2\right\| \frac{1}{3n^2}\sum_{i=0}^{k-1} S_{i,b_n}(a^2)(x) \leq \left\|a^2\right\|^2 \frac{k}{3n^2}
\end{aligned}$$

and consequently

$$\left|\frac{2}{n^2}\sum_{k=0}^{n-1} S_{k,b_n}((\mathrm{id}-x)^2\cdot a^2)(x)\right| \leq \frac{2}{3n^4}\left\|a^2\right\|^2 \sum_{k=1}^{n-1} k = \frac{n(n-1)}{3n^4}\left\|a^2\right\|^2\ ;$$

since $a^4 \in C(\overline{\mathbb{R}})$ and S_{k,b_n} are positive contractions on $C(\overline{\mathbb{R}})$ with respect to the uniform norm, we can write

$$
\begin{aligned}
|S_{n,b_n}((\mathrm{id}-x)^4)(x)| &\le \frac{n(n-1)}{3n^4}\,\left\|a^2\right\|^2 + \frac{1}{5n^4}\sum_{k=0}^{n-1}\|a^4\| \\
&= \left(\frac{1}{n^2}-\frac{1}{n^3}\right)\frac{\left\|a^2\right\|^2}{3} + \frac{1}{n^3}\frac{\|a^4\|}{5} \\
&= \frac{1}{n^2}\frac{\left\|a^2\right\|^2}{3} + \frac{1}{n^3}\left(\frac{\|a^4\|}{5} - \frac{\left\|a^2\right\|_\infty^2}{3}\right) \le \frac{1}{n^2}\frac{\left\|a^2\right\|^2}{3}\,.
\end{aligned}
$$

□

Theorem 3.1.12 (Voronovskaja-type formula)
For every $f \in C^2(\overline{\mathbb{R}})$ we have

$$\lim_{n\to\infty} n\,(L_n f - f) = \frac{a^2}{6} f''\,.$$

Proof. Let $f \in C^2(\overline{\mathbb{R}})$; we apply Theorem 2.2.2, taking $h_n = \frac{1}{n}$. The operators $\mathcal{A}_n f(x)$ became $n\frac{1}{2}L_n((\mathrm{id}-x)^2)(x)f''(x) = \frac{1}{6n}\sum_{k=0}^{n-1} S_{k,b_n}(a^2)(x)f''(x)$ which converges uniformly with respect $x \in \mathbb{R}$ to $\frac{1}{6}a^2(x)f''(x) = Af(x)$ from Proposition 3.1.10 and since $f \in C^2(\overline{\mathbb{R}})$. On the other hand $L_n\mathbf{1} - \mathbf{1} = 0$ and from Lemma 3.1.11 hypothesis (2.2.2,3) is also satisfied. Finally since $f \in C^2(\overline{\mathbb{R}})$, the second-order derivative is bounded and uniformly continuous. □

In order to consider a quantitative version of the above Voronovskaja-type formula we need to introduce the following space

$$C^{2,\alpha}(\overline{\mathbb{R}}) := \left\{ f \in C^2(\overline{\mathbb{R}}) \mid f'' \in C^\alpha(\mathbb{R}) \right\}\,.$$

Theorem 3.1.13 *If $a^2 \in C^2(\overline{\mathbb{R}})$ for every $f \in C^{2,\alpha}(\overline{\mathbb{R}})$ we have*

$$\|n(L_n f - f) - Af\| \le C_a \frac{M_f}{n^{\alpha/2}}\,, \tag{3.1.14}$$

where M_f is the seminorm defined by

$$M_f := L_{f''} + \|f''\| \tag{3.1.15}$$

and C_a is a constant depending on a defined by

$$C_a := \|a^2\| \max\left\{1, \frac{\|(a^2)''\|}{72\sqrt{n}}\right\}\,. \tag{3.1.16}$$

Proof. Let $f \in C^{2,\alpha}(\overline{\mathbb{R}})$ and let $\mathcal{A}_{S_{n,b_n}}$ be the operator (2.2.2) obtained by taking $L = S_{n,b_n}$ and which can be evaluated from Lemma 3.1.11,

$$\mathcal{A}_{S_{n,b_n}} f(x) = \frac{1}{6}\frac{1}{n}\left(\frac{1}{n}\sum_{i=0}^{n-1} S_{i,b_n}(a^2)(x)\right) f''(x)\ ,$$

we have

$$\begin{aligned} |n(L_n f(x) - f(x)) - Af| \quad &\leq \quad |n(L_n f(x) - f(x) - \mathcal{A}_{S_{n,b_n}} f(x)| \\ &\quad + |n\mathcal{A}_{S_{n,b_n}} f(x) - Af(x)|\ , \qquad (3.1.17) \end{aligned}$$

in regard to the first term of the the righthand side using Theorem 2.2.1 end Lemma 3.1.11 we have

$$\begin{aligned} &|n(L_n f(x) - f(x) - \mathcal{A}_{S_{n,b_n}} f(x)| \\ &\quad \leq \frac{L_{f''}}{2}\left(\frac{1}{3n^2}\sum_{i=0}^{n-1} S_{i,b_n}(a^2)(x)\right)^{\alpha/2}\left(\frac{1}{3}\left(\frac{1}{n}\sum_{i=0}^{n-1} S_{i,b_n}(a^2)(x)\right) + \frac{\|a^2\|}{\sqrt{3}}\right) \\ &\quad \leq \frac{L_{f''}}{2}\left(\frac{\|a^2\|}{3n}\right)^{\alpha/2}\frac{1+\sqrt{3}}{3}\|a^2\|\ . \end{aligned}$$

As regards the second term of (3.1.17) from Proposition 3.1.8, since $a^2 \in C^2(\overline{\mathbb{R}})$, we have

$$\begin{aligned} |n\mathcal{A}_{S_{n,b_n}} f(x) - Af(x)| &= \frac{1}{6}|f''(x)|\left|\frac{1}{n}\sum_{i=0}^{n-1} S_{i,b_n}(a^2)(x) - a(x)\right| \\ &\leq \frac{1}{6}\|f''\|\frac{\|a^2\|\|(a^2)''\|}{12}\frac{1}{n} \\ &\leq \frac{1}{72n}\|f''\|\|a^2\|\|(a^2)''\|\ . \end{aligned}$$

Collecting the above inequalities we have

$$\begin{aligned} &|n(L_n f(x) - f(x)) - Af| \\ &\quad \leq \frac{L_{f''}}{2}\left(\frac{\|a^2\|}{3n}\right)^{\alpha/2}\frac{1+\sqrt{3}}{3}\|a^2\| + \frac{1}{72n}\|f''\|\|a^2\|\|(a^2)''\| \\ &\quad \leq \|a^2\|\left(\frac{L_{f''}}{n^{\alpha/2}} + \frac{\|f''\|\|(a^2)''\|}{72n}\right) \leq \frac{\|a^2\|}{n^{\alpha/2}}\left(L_{f''} + \|f''\|\frac{\|(a^2)''\|}{72\sqrt{n}}\right) \\ &\quad \leq C_a\frac{M_f}{n^{\alpha/2}} \end{aligned}$$

where M_f is the seminorm defined by (3.1.15) and C_a is a constant depending on a defined by (3.1.16). □

Observe that for n large enough C_a is equal to $\|a^2\|$.

At this point we deepen the connection with the differential operator A. We shall need the following core property of the operator $(A, D(A))$.

Proposition 3.1.14 *The space $C^2(\overline{\mathbb{R}})$ is a core for $(A, D(A))$.*

PROOF. Let $u \in D(A)$ and $0 < \varepsilon < 1$. We show the existence of a function $v \in C^2(\overline{\mathbb{R}})$ such that

$$|u(x) - v(x)| \le \varepsilon\ , \quad |Au(x) - Av(x)| \le \varepsilon\ , \qquad x \in \mathbb{R}\ .$$

We argue only in the interval $[0, +\infty[$ since the same argument can be applied to $]-\infty, 0]$. Since $u \in C(\overline{\mathbb{R}})$ and

$$\lim_{x\to+\infty} Au(x) = 0\ , \qquad \lim_{x\to+\infty} a(x) = 0\ ,$$

we can find $c > 0$ such that

$$|u(x) - \ell| < \varepsilon\ , \quad |Au(x)| < \varepsilon\ , \quad a(x)^2 < \varepsilon\ , \qquad x \ge c\ ,$$

where $\ell := \lim_{x\to+\infty} u(x)$.

We observe that $\liminf_{x\to+\infty} u'(x) \le 0 \le \limsup_{x\to+\infty} u'(x)$, otherwise we could not have $u \in C(\overline{\mathbb{R}})$; consequently, we can choose $x_0 > c$ such that $|u'(x_0)| \le \varepsilon$. Now, we consider $\delta_\varepsilon > 0$ such that

$$\frac{a(x)^2}{a(x_0)^2} \le 2\ , \qquad x \in [x_0, x_0 + \delta_\varepsilon]$$

and define

$$h_1 := \min\left\{\sqrt{\varepsilon}, \frac{\varepsilon}{|u''(x_0)| + 1}, \delta_\varepsilon\right\}\ , \quad h_2 := 3\sqrt{\varepsilon}\ .$$

We put, for simplicity,

$$x_1 := x_0 + h_1\ , \quad x_3 := x_1 + h_2\ ;$$

finally, we consider $x_1 < x_2 < x_3$, and define

$$M := -\frac{2u'(x_0)}{h_2} - u''(x_0)\,\frac{h_1}{h_2}\ ,$$

and the functions $w_2, w_1, w : [x_0, x_3] \to \mathbb{R}$ by setting, for every $x \in [x_0, x_3]$,

$$w_2(x) := \begin{cases} u''(x_0)\,\dfrac{x_1 - x}{x_1 - x_0}\ , & x \in [x_0, x_1]\ , \\[2ex] M\,\dfrac{x - x_1}{x_2 - x_1}\ , & x \in]x_1, x_2[\ , \\[2ex] M\,\dfrac{x_3 - x}{x_3 - x_2}\ , & x \in [x_2, x_3]\ , \end{cases}$$

$$w_1(x) := u'(x_0) + \int_{x_0}^{x} w_2(t)\,dt\ ,$$

$$w(x) := u(x_0) + \int_{x_0}^{x} w_1(t)\,dt\ .$$

We easily get $w(x_0) = u(x_0)$, $w'(x_0) = w_1(x_0) = u'(x_0)$ and $w''(x_0) = w_2(x_0) = u''(x_0)$ and further

$$w''(x_3) = w_2(x_3) = 0$$

and

$$\begin{aligned} w'(x_3) &= u'(x_0) + \int_{x_0}^{x_1} w_2(t)dt + \int_{x_1}^{x_2} w_2(t)dt + \int_{x_2}^{x_3} w_2(t)dt \\ &= u'(x_0) + \frac{1}{2}(x_1 - x_0)u''(x_0) + \frac{1}{2}(x_2 - x_1)M + \frac{1}{2}(x_3 - x_2)M \\ &= u'(x_0) + \frac{1}{2}(x_1 - x_0)u''(x_0) + \frac{1}{2}(x_3 - x_1)M \end{aligned}$$

which yields $w'(x_3) = 0$ from the definition of M.

Moreover, we observe that $h_1 \le \sqrt{\varepsilon}$, $h_1|u''(x_0)| \le \varepsilon$, $h_2|M| \le 3\varepsilon$ and $|M| \le \sqrt{\varepsilon}$. Hence we obtain

$$\sup_{x\in[x_0,x_1]} |w'(x)| \le |w'(x_0)| + (x_1 - x_0) \sup_{x\in[x_0,x_1]} |w''(x)| \le |u'(x_0)| + h_1|u''(x_0)|$$

and consequently

$$\begin{aligned} \sup_{x\in[x_1,x_3]} |w'(x)| &\le |w'(x_1)| + (x_3 - x_1) \sup_{x\in[x_1,x_3]} |w''(x)| \\ &\le |u'(x_0)| + h_1|u''(x_0)| + h_2|M|\ ; \end{aligned}$$

therefore, in any case

$$\sup_{x\in[x_0,x_3]} |w'(x)| \le 5\varepsilon$$

and this implies

$$\sup_{x\in[x_0,x_3]} |w(x) - w(x_0)| \le (h_1 + h_2) \sup_{x\in[x_0,x_3]} |w'(x)| \le 20\varepsilon\sqrt{\varepsilon}\ .$$

We conclude that

$$\begin{aligned} \sup_{x\in[x_0,x_3]} |w(x) - u(x)| &\le \sup_{x\in[x_0,x_3]} |w(x) - w(x_0)| + \sup_{x\in[x_0,x_3]} |u(x) - u(x_0)| \\ &\le 20\varepsilon\sqrt{\varepsilon} + 2\varepsilon\ , \end{aligned}$$

and

$$\begin{aligned} \sup_{x\in[x_0,x_3]} |a(x)^2\, w''(x)| &\le \sup_{x\in[x_0,x_1]} |a(x)^2\, w''(x)| + \sup_{x\in[x_1,x_3]} |a(x)^2\, w''(x)| \\ &\le \sup_{x\in[x_0,x_1]} |a(x)^2\, u''(x_0)| + \sup_{x\in[x_1,x_3]} |a(x)^2\, M| \\ &\le \sup_{x\in[x_0,x_1]} \frac{a(x)^2}{a(x_0)^2}\, |a(x_0)^2\, u''(x_0)| + \varepsilon\,\sqrt{\varepsilon} \\ &\le 12\varepsilon + \varepsilon\,\sqrt{\varepsilon}\ . \end{aligned}$$

At this point it is clear that the function $v : [0, +\infty[\to \mathbb{R}$ defined by setting

$$v(x) := \begin{cases} u(x) \, , & 0 \le x < x_0 \, , \\ w(x) \, , & x_0 \le x \le x_3 \, , \\ w(x_3) \, , & x_3 < x \, , \end{cases}$$

is in $C^2([0, +\infty])$ and satisfies the required properties. □

We have the following main result.

Theorem 3.1.15 *The operator* $(A, D(A))$ *generates a* C_0*-semigroup* $(T(t))_{t \ge 0}$ *of positive contractions in* $C(\overline{\mathbb{R}})$ *and, for every* $t \ge 0$ *and for every sequence* $(k(n))_{n \ge 1}$ *of positive integers satisfying* $\lim_{n \to +\infty} k(n)/n = t$*, we have*

$$T(t) = \lim_{n \to +\infty} L_n^{k(n)} \qquad \textit{strongly on } C(\overline{\mathbb{R}}) \, , \tag{3.1.18}$$

moreover if $a^2 \in C^2(\overline{\mathbb{R}})$*, for every* $f \in C^{2,\alpha}(\overline{\mathbb{R}})$ *we have*

$$\begin{aligned} \left\| T(t)u - L_n^{k(n)} u \right\| \le & t \, \frac{C_a M_f}{n^{\alpha/2}} + \left(\left| \frac{k(n)}{n} - t \right| + \sqrt{\frac{2}{\pi}} \, \frac{\sqrt{k(n)}}{n} \right) \\ & \times \left(\|Af\| + \frac{C_a M_f}{n^{\alpha/2}} \right) \end{aligned} \tag{3.1.19}$$

and choosing $k(n) = [nt]$

$$\left\| T(t)u - L_n^{[nt]} u \right\| \le t \, \frac{C_a M_f}{n^{\alpha/2}} + \frac{1}{\sqrt{n}} \left(\frac{1}{\sqrt{n}} + \sqrt{\frac{2t}{\pi}} \right) \left(\|Af\| + \frac{C_a M_f}{n^{\alpha/2}} \right) . \tag{3.1.20}$$

Proof. We already know that the operator $(A, D(A))$ generates a C_0-semigroup $(T(t))_{t \ge 0}$ of contractions and hence, for every $\lambda > 0$, the range $(\lambda - A)(D(A))$ coincides with $C(\overline{\mathbb{R}})$. Moreover, by Proposition 3.1.14, $C^2(\overline{\mathbb{R}})$ is a core for $(A, D(A))$ and therefore $(\lambda - A)(C^2(\overline{\mathbb{R}}))$ is dense in $C(\overline{\mathbb{R}})$. Hence we can apply Trotter's approximation theorem [70] and obtain that the closure of the operator arising from the Voronovskaja-formula (Theorem 3.1.12) generates a C_0-semigroup represented by (3.1.18). Finally, this closure coincides with $(A, D(A))$ by the core property of $C^2(\overline{\mathbb{R}})$. The positivity of the semigroup is a consequence of the representation (3.1.18).

At this point we can apply Theorem 1.1.2. From Theorem 3.1.13 follows that the seminorms are given by $\psi_n(f) = C_a \frac{M_f}{n^{\alpha/2}}$ and $\varphi_n(f) = \|Af\| + C_a \frac{M_f}{n^{\alpha/2}}$, then taking into account that the growth bound of $(T(t))_{t \ge 0}$ is equal to 0 and every $T(t)$ is a linear contraction, i.e. $\omega = 0$ and $M = 1$, the estimates (3.1.19) and (3.1.20) follow directly from (1.1.10) and (1.1.11), and this completes the proof. □

3.2 Steklov operators on bounded intervals

In this section we show how Steklov operators can be considered even in spaces of continuous functions on a bounded interval. For the sake of simplicity, we shall consider only the case of the interval $[0, 1]$. Also in this case we can construct a suitable sequence of positive Steklov type operators which can be associated with the differential operator $A : D(A) \to C([0, 1])$ defined by

$$Au(x) := \frac{1}{6}\, a(x)^2\, u''(x)\ , \qquad u \in D(A)\ , \qquad x \in [0,1]\ , \tag{3.2.1}$$

on the domain

$$D(A) := \{u \in C([0,1]) \cap C^2(]0,1[) \ \mid\ a^2 u'' \in C([0,1])\}\ , \tag{3.2.2}$$

where $a \in C([0, 1])$ is strictly positive on $]0, 1[$ and

$$a(x) = O(x(1-x))\ , \qquad \text{as } x \to 0, 1 \tag{3.2.3}$$

(hence $a(0) = a(1) = 0$).

It is straightforward to check that 0 and 1 are natural endpoints and therefore the maximal domain $D(A)$ coincides with the following Ventcel's domain

$$D_V(A) := \{u \in C([0,1]) \cap C^2(]0,1[) \ \mid\ \lim_{x\to 0,1} a^2(x) u''(x) = 0\}\ .$$

Condition (3.2.3) also ensures that

$$b_n(x) := \frac{a(x)}{n} \le \frac{1}{2} - \left| x - \frac{1}{2} \right|$$

for n large enough and consequently for such integers the integral mean operators $M_{b_n} : L^1_{\mathrm{loc}}(0, 1) \to C(]0, 1[)$

$$M_{b_n} f(x) = \frac{1}{2b_n(x)} \int_{x-b_n(x)}^{x+b_n(x)} f(t)\, dt \tag{3.2.4}$$

are well-defined together with all their iterates. Observe that if $f \in C([0, 1])$ then we can use the convention $M_{b_n} f(0) = f(0)$ and $M_{b_n} f(1) = f(1)$ and consider M_{b_n} as an operator acting on $C([0, 1])$.

Now, consider the operators $L_n := S_{n,b_n} : C([0, 1]) \to C([0, 1])$.

We have the following properties:

i) $L_n\mathbf{1}(x) = 1$,

ii) $L_n\mathrm{id}(x) = x$,

iii) $L_n(\mathrm{id}^2)(x) = x^2 + \frac{1}{3}\sum_{i=0}^{n-1} S_{i,b_n}(b_n^2)(x) = x^2 + \frac{1}{3n}\left(\frac{1}{n}\sum_{i=0}^{n-1} S_{i,b_n}(a^2)(x)\right)$.

Since $|M_{b_n}(a^2)(x)| \le \|a^2\|$ we have that $|\frac{1}{n}\sum_{k=0}^{n-1} S_{k,b_n}(a^2)(x)| \le n\|a^2\|/n = \|a^2\|$, and therefore

$$\lim_{n\to\infty} L_n(\mathrm{id}^2) = \mathrm{id}^2$$

uniformly on $[0,1]$. From the classical first Korovkin's theorem [9, Theorem 4.2.7], we obtain the following result.

Theorem 3.2.1 *For every $f \in C([0,1])$,*

$$\lim_{n\to+\infty} L_n(f) = f \qquad \textit{uniformly with respect to } x \in [0,1]\ .$$

As in the previous section, we can state the following results.

Proposition 3.2.2 *We have the following properties:*

1. *For every $f \in C^2([0,1])$, there exists a constant $C_f > 0$, depending on f, such that for every $n \ge 1$*

$$\|S_{k,b_n}(f) - f\| \le C_f \frac{k}{n^2}\ .$$

2. *For every $f \in C([0,1])$ we have*

$$\lim_{n\to+\infty} \frac{1}{n}\sum_{k=0}^{n-1} S_{k,b_n}(f) = f$$

uniformly on $[0,1]$.

3. *We have*

$$|S_{n,b_n}((\mathrm{id}-x)^4)(x)| \le \frac{1}{n^2}\frac{\|a^2\|^2}{3} \qquad \textit{for every } x \in [0,1]\ .$$

Proof. As regards the first property, we can reason as in Proposition 3.1.4 and obtain

$$C_f = \frac{1}{6}\,\|\alpha^2\|\|f''\|\ .$$

Property 2) follows from the density of $C^2([0,1])$ in $C([0,1])$ and the analogous result in the previous section.

Finally, property 3) can be shown in the same way as in Lemma 3.1.11. □

At this point we can state the following Voronovskaja-type formula and its quantitative version; the proof is similar to that of Theorem 3.1.12 and 3.1.13 and for the sake of brevity we shall omit it.

Theorem 3.2.3 (Voronovskaja-type formula)
For every $f \in C^2([0,1])$ *we have*

$$\lim_{n\to\infty} n\,(L_n(f) - f) = \frac{a^2}{6} f'' \,.$$

Theorem 3.2.4 (Quantitative Voronovskaja formula for Steklov operators)
If $a^2 \in C^2([0,1])$ *for every* $f \in C^{2,\alpha}([0,1])$ *we have*

$$\|n\,(L_n(f) - f) - Af\| \le C_a \frac{M_f}{n^{\alpha/2}} \,.$$

where M_f *is the seminorm defined by*

$$M_f := L_{f''} + \|f''\|$$

and C_a *is a constant depending on* a *defined by*

$$C_a := \|a^2\| \max\left\{1, \frac{\|(a^2)''\|}{72\sqrt{n}}\right\} \,.$$

We already know that the operator $(A, D(A))$ generates a C_0-semigroup $(T(t))_{t\ge 0}$ of contractions (see, e.g., [44] or [48, Chapter VI, Section 4]) and hence, for every $\lambda > 0$, the range $(\lambda - A)(D(A))$ coincides with $C([0,1])$. If $C^2([0,1])$ is a core for $(A, D(A))$, then $(\lambda - A)(C^2([0,1]))$ is dense in $C([0,1])$ and we can apply Trotter's approximation theorem, which yields the representation of the semigroup $(T(t))_{t\ge 0}$ in terms of iterates of the operators L_n with the same arguments of the preceding section. In some particular cases, it can be easily proved that $C^2([0,1])$ is a core for $(A, D(A))$.

Theorem 3.2.5 *Assume that*

$$a(x) = Cx(1-x)\,, \qquad 0 \le x \le 1\,,$$

for a suitable constant $C > 0$. *Then, the space* $C^2([0,1])$ *is a core for* $(A, D(A))$ *and, for every* $t \ge 0$ *and for every sequence* $(k(n))_{n\ge 1}$ *of positive integers satisfying* $\lim_{n\to+\infty} k(n)/n = t$, *we have*

$$T(t) = \lim_{n\to+\infty} L_n^{k(n)} \qquad \text{strongly on } C([0,1])\,, \tag{3.2.5}$$

moreover if $f \in C^{2,\alpha}([0,1])$ *we have*

$$\left\|T(t)f - L_n^{k(n)} f\right\| \le t\,\frac{C_a M_f}{n^{\alpha/2}} + \left(\left|\frac{k(n)}{n} - t\right| + \sqrt{\frac{2}{\pi}}\,\frac{\sqrt{k(n)}}{n}\right) \left(\|Af\| + \frac{C_a M_f}{n^{\alpha/2}}\right) \tag{3.2.6}$$

and choosing $k(n) = [nt]$

$$\left\|T(t)f - L_n^{[nt]} f\right\| \le t\,\frac{C_a M_f}{n^{\alpha/2}} + \frac{1}{\sqrt{n}} \left(\frac{1}{\sqrt{n}} + \sqrt{\frac{2t}{\pi}}\right) \left(\|Af\| + \frac{C_a M_f}{n^{\alpha/2}}\right) \,. \tag{3.2.7}$$

Proof. The core property is well-known (see e.g. [26, Lemma 1.2]). Hence, we can apply Trotter's approximation theorem [70] and obtain the representation (3.2.5).

At this point estimates (3.2.6) and (3.2.7) follows from Theorem 1.1.2 taking into account of quantitative version of voronovskaja's formula obtained in Theorem 3.2.4 and this completes the proof. □

3.3 Steklov operators in weighted spaces

In this section we consider Steklov operators on the space

$$C_w(\overline{\mathbb{R}}) := \{f \in C(\mathbb{R}) \mid f \cdot w \in C(\overline{\mathbb{R}})\} ,$$

where $w : \mathbb{R} \to \mathbb{R}$ is a strictly positive continuous real function which tends to 0 at the points $\pm\infty$. The space $C_w(\overline{\mathbb{R}})$ is endowed with the norm

$$\|f\|_w := \sup_{x\in\mathbb{R}} |f(x) \cdot w(x)| \quad (= \|f \cdot w\|) , \qquad f \in C_w(\overline{\mathbb{R}}) .$$

In order to present a unified treatment we consider weight functions w having the form

$$w(x) := \frac{1}{1+|x|^p} , \qquad x \in \mathbb{R} ,$$

with $p \geq 2$ fixed, even if some partial results can be established in a more general setting.

Let $a \in C(\overline{\mathbb{R}})$ be a strictly positive function satisfying

$$\mathrm{id} \cdot a \in C(\overline{\mathbb{R}}) \tag{3.3.1}$$

and consider the differential operator $A_w : D(A_w) \to C_w(\overline{\mathbb{R}})$ defined by

$$A_w u(x) := \frac{1}{6}\, a(x)^2\, u''(x) , \qquad u \in D(A_w) , \quad x \in \mathbb{R} , \tag{3.3.2}$$

on the following maximal domain

$$D(A_w) := \{u \in C_w(\overline{\mathbb{R}}) \cap C^2(\mathbb{R}) \mid A_w u \in C_w(\overline{\mathbb{R}})\} .$$

We need to assume the following additional condition on a

$$\exists\, \delta > 0,\ \exists\, M > 0\ : \qquad \sup_{\substack{x,y\in\mathbb{R}\\ |x-y|<\delta}} \frac{a(x)^2}{(1+|x|)\, a(y)^2} \leq M , \tag{3.3.3}$$

which is obviously satisfied if there exist $q \geq 1$ and $C_1, C_2 > 0$ such that, for every $x \in \mathbb{R}$,

$$\frac{C_1}{1+|x|^{q+1}} \leq a(x)^2 \leq \frac{C_2}{1+|x|^q} .$$

We can define $b_n := a/n$ and consider the operator S_{n,b_n}.

Different properties of Steklov operators are based on the behavior of the function $\omega_n : \mathbb{R} \to \mathbb{R}$ defined by setting, for every $x \in \mathbb{R}$,

$$\omega_n(x) := \frac{w(x)}{2b_n(x)} \int_{x-b_n(x)}^{x+b_n(x)} \frac{1}{w(t)}\, dt .$$

We have the following result.

Proposition 3.3.1 *There exists a constant $C_\omega > 0$ such that*

$$\|\omega_n - \mathbf{1}\| \le \frac{C_\omega}{n^2} .$$

PROOF. The property is obviously true on a neighborhood $[-\delta, \delta]$ of 0, hence by symmetry we can prove it only in the interval $[\delta, +\infty[$. Since $p \ge 2$, we have $1/w \in C^2(]0, +\infty[)$ and for every $x, t > 0$ we can write

$$\frac{1}{w(t)} - \frac{1}{w(x)} = \left(\frac{1}{w(x)}\right)'(t-x) + \frac{1}{2}\left(\frac{1}{w(\xi_t)}\right)''(t-x)^2$$

for a suitable ξ_t in the interval with endpoints t and x.

Now, let $x \ge \delta$ and set for simplicity $b = b_n(x) = a(x)/n$; for n enough large we can assume that the interval of integration in the definition of the n-th Steklov operator is contained in $]0, +\infty[$ and consequently

$$\begin{aligned}
\left|S_{1,b}\left(\frac{1}{w}\right)(x) - \frac{1}{w(x)}\right| &= \left|\frac{1}{2b}\int_{x-b}^{x+b}\left(\frac{1}{w(t)} - \frac{1}{w(x)}\right)dt\right| \\
&= \Bigg|\left(\frac{1}{w(x)}\right)'\frac{1}{2b}\int_{x-b}^{x+b}(t-x)dt \\
&\quad + \frac{1}{2b}\int_{x-b}^{x+b}\frac{1}{2}\left(\frac{1}{w(\xi_t)}\right)''(t-x)^2dt\Bigg| \\
&= \left|\frac{1}{2b}\int_{x-b}^{x+b}\frac{1}{2}\left(\frac{1}{w(\xi_t)}\right)''(t-x)^2dt\right| \\
&\le \frac{1}{2b}\int_{x-b}^{x+b}\left|\frac{1}{2}p(p-1)\xi_t^{p-2}(t-x)^2\right| dt \\
&\le \frac{1}{2b}\frac{1}{2}p(p-1)(x+b)^{p-2}\int_{x-b}^{x+b}(t-x)^2dt \\
&= \frac{b^2}{6}p(p-1)(x+b)^{p-2} .
\end{aligned}$$

Hence, we can conclude that

$$\begin{aligned}
|\omega_n(x) - 1| &= \left|w(x)S_{1,b_n}\left(\frac{1}{w}\right)(x) - w(x)\frac{1}{w(x)}\right| \\
&= w(x)\left|S_{1,b}\left(\frac{1}{w}\right)(x) - \frac{1}{w(x)}\right| \\
&\le \frac{1}{1+x^p}\frac{1}{n^2}\frac{a(x)^2}{6}p(p-1)\left(x + \frac{a(x)}{n}\right)^{p-2} \\
&\le \frac{1}{1+x^p}\frac{1}{6n^2}\|a\|^2 p(p-1)\left(x + \|a\|\right)^{p-2} \\
&\le \frac{C_\omega}{n^2}
\end{aligned}$$

and this completes the proof. □

From the preceding result, we have $\omega_n \le 1 + C_\omega/n^2$ and consequently we can find a constant $C_w > 0$ such that, for every $n \ge 1$,

$$\|\omega_n\|^n \le C_w \ . \tag{3.3.4}$$

Therefore, taking into account that, for every $f \in C_w(\overline{\mathbb{R}})$ and $x \in \mathbb{R}$, we have

$$|w(x)\, S_{1,b_n} f(x)| = \left| w(x)\, \frac{1}{2b_n(x)} \int_{x-b_n(x)}^{x+b_n(x)} f(t)\, w(t)\, \frac{1}{w(t)}\, dt \right| \le \|\omega_n\|\ \|f\|_w \ .$$

Arguing by induction on the integer $n \ge 1$ we get, for every $f \in C_w(\overline{\mathbb{R}})$,

$$\|S_{n,b_n}(f)\|_w \le \|\omega_n\|^n\ \|f\|_w \le C_w\, \|f\|_w \tag{3.3.5}$$

and we conclude that the operators S_{n,b_n} are equibounded.

Remark 3.3.2 The operators S_{n,b_n} map the space $C_w(\overline{\mathbb{R}})$ into itself.

Indeed, let $f \in C_w(\overline{\mathbb{R}})$; on a neighborhood of $+\infty$ we have

$$w(x)\, S_{1,b_n} f(x) = \frac{w(x)}{2b_n(x)} \int_{x-b_n(x)}^{x+b_n(x)} f(t)\, dt = w(x)\, f(\xi_x) = \frac{w(x)}{w(\xi_x)}\, w(\xi_x)\, f(\xi_x) \ ,$$

for a suitable $\xi_x \in]x - b_n(x), x + b_n(x)[$; taking the limit as $x \to +\infty$ we have $\xi_x \to +\infty$ and consequently $w(\xi_x)\, f(\xi_x)$ tends to a finite limit; moreover $\lim_{x\to+\infty} w(x)/w(\xi_x) = 1$ since

$$\frac{1 + (x - b_n(x))^p}{1 + x^p} \le \frac{w(x)}{w(\xi_x)} \le \frac{1 + (x + b_n(x))^p}{1 + x^p} \ .$$

Now, a simple induction argument yields the existence of a finite limit of $S_{n,b_n} f(x)$ at the point $+\infty$.

A similar argument can be applied on a neighborhood of $-\infty$ and hence the property is completely established. □

As a consequence, we may now consider the operators $L_{w,n} : C_w(\overline{\mathbb{R}}) \to C_w(\overline{\mathbb{R}})$ defined by

$$L_{w,n} f(x) := S_{n,b_n} f(x) \ , \qquad f \in C_w(\overline{\mathbb{R}}) \ , \quad x \in \mathbb{R} \ . \tag{3.3.6}$$

Observe that the operators $L_{w,n}$ are equibounded and satisfy $\|L_{w,n}\| \le C_w$ for every $n \ge 1$.

Proposition 3.3.3 *For every $f \in C_w(\overline{\mathbb{R}})$, $k \ge 1$ and $x \in \mathbb{R}$,*

$$\|S_{k,b_n}(f) - f\|_w \le \|S_{1,b_n}(f) - f\|_w \sum_{i=0}^{k-1} \|\omega_n\|^i \ . \tag{3.3.7}$$

PROOF. We argue by induction on the integer $k \geq 1$. If $k = 1$ then (3.3.7) is obviously true. Now, assume that (3.3.7) holds for $k \geq 1$. We have

$$\begin{aligned}
&|w(x)(S_{k+1,b_n} f(x) - f(x))| \\
&\quad \leq |w(x) S_{1,b_n}(S_{k,b_n} f - f)(x)| + |w(x)(S_{1,b_n} f(x) - f(x))| \\
&\quad \leq \|\omega_n\| \, \|S_{k,b_n}(f) - f\|_w + \|S_{1,b_n}(f) - f\|_w \\
&\quad \leq \|\omega_n\| \sum_{i=0}^{k-1} \|\omega_n\|^i \|S_{1,b_n}(f) - f\|_w + \|S_{1,b_n}(f) - f\|_w \\
&\quad \leq \sum_{i=0}^{k} \|\omega_n\|^i \|S_{1,b_n}(f) - f\|_w
\end{aligned}$$

and this completes the induction argument. □

In the sequel we need to introduce the space

$$C_w^2(\overline{\mathbb{R}}) := \{ f \in C_w(\overline{\mathbb{R}}) \cap C^2(\mathbb{R}) \mid f'' \in C_w(\overline{\mathbb{R}}) \} .$$

Proposition 3.3.4 *For every $f \in C_w^2(\overline{\mathbb{R}})$ and for every $x \in \mathbb{R}$*

$$\|S_{1,b_n}(f) - f\|_w \leq C \frac{\|a^2\| \|f''\|_w}{n^2} , \tag{3.3.8}$$

where $C > 0$ is a suitable constant depending on a and the weight w.

PROOF. Let $f \in C_w^2(\overline{\mathbb{R}})$; for every $x, t \in \mathbb{R}$, we can write

$$f(t) - f(x) = f'(x)\,(t - x) + \frac{1}{2} f''(\xi_t)(t - x)^2$$

with ξ_t in the interval with endpoints t and x.

Then

$$\begin{aligned}
S_{1,b_n} f(x) - f(x) &= \frac{1}{2b_n(x)} \int_{x-b_n(x)}^{x+b_n(x)} (f(t) - f(x)) dt \\
&= f'(x) \frac{1}{2b_n(x)} \int_{x-b_n(x)}^{x+b_n(x)} (t - x) dt + \frac{1}{2b_n(x)} \int_{x-b_n(x)}^{x+b_n(x)} \frac{1}{2} f''(\xi_t)(t - x)^2 dt \\
&= \frac{1}{2b_n(x)} \int_{x-b_n(x)}^{x+b_n(x)} \frac{1}{2} f''(\xi_t)(t - x)^2 dt
\end{aligned}$$

and

$$\begin{aligned}
|S_{1,b_n(x)} f(x) - f(x)| &\leq \frac{1}{2} \sup_{t \in [x-b_n(x), x+b_n(x)]} |f''(t)| \frac{1}{2b_n(x)} \int_{x-b_n(x)}^{x+b_n(x)} (t - x)^2 dt \\
&= \frac{b_n(x)^2}{6} \sup_{t \in [x-b_n(x), x+b_n(x)]} |f''(t)| .
\end{aligned}$$

Observe that the functions $w\,f''$ and a are bounded and consequently, since

$$\sup_{t\in[x-b_n(x),x+b_n(x)]} \frac{|w(x)|}{|w(t)|} \leq \frac{1+(|x|+\|a\|)^p}{1+|x|^p} \; ,$$

the function $\sup_{t\in[x-b_n(x),x+b_n(x)]} \frac{|w(x)|}{|w(t)|}$ is bounded as well. Hence, we can write

$$\begin{aligned} &|S_{1,b_n}f(x)-f(x)|_w \leq \frac{1}{n^2}\frac{a(x)^2}{6}w(x) \sup_{t\in[x-b_n(x),x+b_n(x)]} |f''(t)| \\ &\leq \frac{1}{n^2}\frac{a(x)^2}{6} \sup_{t\in[x-b_n(x),x+b_n(x)]} \frac{|w(x)|}{|w(t)|} \sup_{t\in[x-b_n(x),x+b_n(x)]} |w(t)f''(t)| \\ &\leq C\frac{\|a^2\|\|f''\|_w}{6n^2} \; , \end{aligned}$$

where $C := \sup_{x\in\mathbb{R}} \frac{1+(|x|+\|a\|)^p}{1+|x|^p}$. □

Proposition 3.3.5 *For every $f \in C^2_w(\overline{\mathbb{R}})$ we have*

$$\left\| \frac{1}{n}\sum_{k=0}^{n-1} S_{k,b_n}(f) - f \right\|_w \leq C\frac{\|a^2\|\|f''\|_w}{n};,$$

where $C > 0$ is a suitable constant depending on a and the weight w.

PROOF. We can apply Propositions 3.3.3, 3.3.4 and 3.3.1 and obtain

$$\begin{aligned}
\left\| \frac{1}{n}\sum_{k=0}^{n-1} S_{k,b_n}(f) - f \right\|_w &\leq \frac{1}{n}\sum_{k=0}^{n-1} \|S_{k,b_n(x)}(f) - f\|_w \\
&\leq \frac{1}{n}\|S_{1,b_n}(f) - f\|_w \sum_{k=0}^{n-1}\sum_{i=0}^{k-1} \|\omega_n\|^i \\
&\leq C\frac{\|a^2\|\|f''\|_w}{6n^2}\frac{1}{n}\sum_{k=0}^{n-1}\sum_{i=0}^{k-1}(1 + C_\omega\frac{1}{n^2})^i \\
&= C\frac{\|a^2\|\|f''\|_w}{6C_\omega}\left(\frac{n}{C_\omega}\left(\left(1+\frac{C_\omega}{n^2}\right)^n - 1\right) - 1\right) \\
&= C\frac{\|a^2\|\|f''\|_w}{6C_\omega}\left(\frac{n}{C_\omega}\left(\sum_{k=0}^{n}\binom{n}{k}\left(\frac{C_\omega}{n^2}\right)^k - 1\right) - 1\right) \\
&= C\frac{\|a^2\|\|f''\|_w}{6C_\omega} \\
&\quad\times\left(\frac{n}{C_\omega}\left(1 + \frac{C_\omega}{n} + \frac{n-1}{2}\frac{C_\omega^2}{n^3} + \sum_{k=3}^{n}\binom{n}{k}\left(\frac{C_\omega}{n^2}\right)^k - 1\right) - 1\right) \\
&= C\frac{\|a^2\|\|f''\|_w}{6C_\omega}\left(\frac{n-1}{2}\frac{C_\omega}{n^2} + \frac{n}{C_\omega}\sum_{k=3}^{n}\binom{n}{k}\left(\frac{C_\omega}{n^2}\right)^k\right) \\
&\leq C\frac{\|a^2\|\|f''\|_w}{6}\left(\frac{1}{2n} + \sum_{k=3}^{n}\binom{n}{k}\frac{C_\omega^{k-2}}{n^{2k-1}}\right),
\end{aligned}$$

we have $\sum_{k=3}^{n}\binom{n}{k}\frac{C_\omega^{k-2}}{n^{2k-1}} \leq \frac{C_1}{2n}$, where C_1 depend on the weight w, and the proof is complete.
□

If condition (3.1.12) on a holds we can establish some quantitative results similar to the unweighted case.

Proposition 3.3.6 *If condition* (3.1.12) *holds, then we have the following properties:*

1. *For every $f \in D(A_w)$ we have*

$$\|S_{1,b_n}(f) - f\|_w \leq C\frac{\|Af\|_w}{n^2},$$

where $C > 0$ is a suitable constant depending on a and the weight w.

2. *For every $f \in D(A_w)$ we have*

$$\left\| \frac{1}{n}\sum_{k=0}^{n-1} S_{k,b_n}(f) - f \right\|_w \leq C\frac{\|Af\|_w}{n},$$

where $C > 0$ is a suitable constant depending on a and the weight w.

At this point, we can state the following main results.

Theorem 3.3.7 *For every $f \in C_w(\overline{\mathbb{R}})$, we have*

$$\lim_{n\to+\infty} L_{w,n}(f) = f$$

with respect to the norm $\|\cdot\|_w$.

PROOF. Since $C^2_w(\overline{\mathbb{R}})$ is dense in $C_w(\overline{\mathbb{R}})$ we can apply Propositions 3.3.3, 3.3.4 and 3.3.1 and obtain

$$\begin{aligned}
\|L_{w,n}(f) - f\|_w &\leq \|S_{1,b_n}(f) - f\|_w \sum_{i=0}^{n} \|\omega_n\|^i \\
&\leq C\frac{\|a^2\|\|f''\|_w}{6n^2} \sum_{i=0}^{n-1} \|\omega_n\|^i \\
&\leq C\frac{\|a^2\|\|f''\|_w}{6n^2} \sum_{i=0}^{n-1} \left(1 + \frac{C_\omega}{n^2}\right)^i \\
&= C\frac{\|a^2\|\|f''\|_w}{6} \frac{1}{C_\omega} \left(\left(1 + \frac{C_\omega}{n^2}\right)^n - 1\right) .
\end{aligned}$$

Thus, it is enough to take the limit as $n \to +\infty$ and the proof is complete. □

Theorem 3.3.8 (Weighted Voronovskaja-type formula)
For every $f \in C^2_w(\overline{\mathbb{R}})$ we have

$$\lim_{n\to\infty} \left\| n\left(L_{w,n}(f) - f\right) - \frac{a^2}{6} f''\right\|_w = 0 .$$

PROOF. Let $f \in C^2_w(\overline{\mathbb{R}})$; we apply Theorem 2.2.2, taking $h_n = \frac{1}{n}$. The operators $\mathcal{A}_n f(x)$ become $\frac{n}{2} L_n((\mathrm{id} - x)^2)(x) f''(x) = \frac{1}{6n}\sum_{k=0}^{n-1} S_{k,b_n}(a^2)(x) f''(x)$ which converge with respect the weighted norm to $\frac{1}{6}a^2(x)f''(x) = Af(x)$ taking into account Proposition 3.1.10 and since $f \in C^2_w(\overline{\mathbb{R}})$. On the other hand $L_n\mathbf{1} - \mathbf{1} = 0$ and from Lemma 3.1.11 $nL_n((\mathrm{id} - x)^4)$ converges to zero even with respect the weighted norm. Finally since $f \in C^2_w(\overline{\mathbb{R}})$, the seconorder derivative is bounded uniformly continuous with respect the weighted norm. □

In order to obtain a quantitative estimate of the Voronovskaja'a formula we need to consider the class of functions

$$C^{2,\alpha}_w(\overline{\mathbb{R}}) := \{f \in C^2_w(\overline{\mathbb{R}}) \mid wf'' \in C^\alpha(\mathbb{R})\} .$$

Theorem 3.3.9 (Weighted quantitative Voronovskaja-type formula)

If $a^2 \in C^2(\overline{\mathbb{R}})$ *for every* $f \in C_w^{2,\alpha}(\overline{\mathbb{R}})$ *we have*

$$\left\| n\left(L_{w,n}(f) - f\right) - \frac{a^2}{6} f'' \right\|_w \leq C_a \frac{M_f}{n^{\alpha/2}} \, ,$$

where M_f *is the seminorm defined by*

$$M_f = L_{wf''} + \|f''\|_w \tag{3.3.9}$$

and

$$C_a := \|a^2\| \max\left\{ \frac{4}{3}, \frac{\|(a^2)''\|}{72\sqrt{n}} \right\} \, . \tag{3.3.10}$$

Proof. Let $f \in C_w^{2,\alpha}(\overline{\mathbb{R}})$; for every $x, t \in \mathbb{R}$, we can write

$$f(t) - f(x) = f'(x)\,(t-x) + \frac{1}{2} f''(x)(t-x)^2 + \eta(t,x)(t-x)^2 \tag{3.3.11}$$

where $\eta : \mathbb{R}^2 \to \mathbb{R}$ satisfies $\eta(t,x) = \frac{1}{2}\left(f''(\xi(t)) - f''(x)\right)$ for some $\xi(t) \in [x,t]$.

Then

$$\begin{aligned}
n\left(L_{w,n}f(x) - f(x)\right) &= n\, S_{n,b_n}(f - f(x)\cdot \mathbf{1})(x) \\
&= n\, S_{n,b_n}\left(f'(x)\,(\mathrm{id} - x) + \frac{1}{2} f''(x)(\mathrm{id}-x)^2 + \eta(\mathrm{id},x)(\mathrm{id}-x)^2 \right)(x) \\
&= n\, f'(x)\, S_{n,b_n}(\mathrm{id}-x)(x) + n\frac{1}{2} f''(x)\, S_{n,b_n}\left((\mathrm{id}-x)^2\right)(x) \\
&\quad + n\, S_{n,b_n}\left(\eta(\mathrm{id},x)(\mathrm{id}-x)^2\right)(x) \\
&= n\frac{1}{2} f''(x)\, S_{n,b_n}\left((\mathrm{id}-x)^2\right)(x) + n\, S_{n,b_n}\left(\eta(\mathrm{id},x)(\mathrm{id}-x)^2\right)(x) \, ,
\end{aligned}$$

and moreover

$$\begin{aligned}
&|w(x)(n\left(L_{w,n}f(x) - f(x)\right) - Af(x))| \\
&\quad \leq \left| w(x) \left(n\frac{1}{2} f''(x)\, S_{n,b_n}\left((\mathrm{id}-x)^2\right)(x) - Af(x) \right) \right| \\
&\qquad + \left| w(x) n\, S_{n,b_n}\left(\eta(\mathrm{id},x)(\mathrm{id}-x)^2\right)(x) \right| \, .
\end{aligned}$$

We can write

$$\begin{aligned}
w(x)\eta(t,x) &= w(x)(f''(\xi) - f''(x)) \\
&= w(\xi)f''(\xi) - w(x)f''(x) + w(x)f''(\xi) - w(\xi)f''(\xi) \\
&= \left(w(\xi)f''(\xi) - w(x)f''(x)\right) + f''(\xi)w(\xi)\left(\frac{w(x)}{w(\xi)} - 1\right) \, ,
\end{aligned}$$

and then

$$
\begin{aligned}
&|w(x)(n\,(L_{w,n}f(x)-f(x))-Af(x))| && (3.3.12)\\
&\quad\le \left|w(x)\left(n\frac{1}{2}f''(x)\,S_{n,b_n}\left((\mathrm{id}-x)^2\right)(x)-Af(x)\right)\right| && (3.3.13)\\
&\qquad+\left|n\,S_{n,b_n}\left(\left((w\circ\xi)(f''\circ\xi)-w(x)f''(x)\right)(\mathrm{id}-x)^2\right)(x)\right|\\
&\qquad+\left|n\,S_{n,b_n}\left((f''\circ\xi)(w\circ\xi)\left(\frac{w(x)}{w\circ\xi}-1\right)(\mathrm{id}-x)^2\right)(x)\right|\ .
\end{aligned}
$$

As regards the first addend, since $a^2\in C^2(\overline{\mathbb{R}})$, from Proposition 3.1.8 we have

$$
\begin{aligned}
&\left|w(x)f''(x)\left(\frac{1}{2}n\,S_{n,b_n}((\mathrm{id}-x)^2)(x)-\frac{1}{6}a^2(x)\right)\right|\\
&\quad=\left|\frac{w(x)f''(x)}{6}\right|\left|\frac{1}{n}\sum_{k=0}^{n-1}S_{k,b_n}(a^2)(x)-a^2(x)\right|\\
&\quad\le\frac{\|f''\|_w}{6}\frac{\|a^2\|\|(a^2)''\|}{12n}\\
&\quad=\frac{\|f''\|_w\|a^2\|\|(a^2)''\|}{72n}\ .
\end{aligned}
$$

As regards the second term in (3.3.12) since $f\in C^{2,\alpha}(\overline{\mathbb{R}})$ and $|w(\xi)f''(\xi)-w(x)f''(x)|\le L_{wf''}|\xi-x|^\alpha\le L_{wf''}|t-x|^\alpha$, we have

$$
\begin{aligned}
&\left|n\,S_{n,b_n}\left(\left((w\circ\xi)(f''\circ\xi)-w(x)f''(x)\right)(\mathrm{id}-x)^2\right)(x)\right|\\
&\quad\le L_{wf''}nS_{n,b_n}\left(|\mathrm{id}-x|^{2+\alpha}\right)(x)\ .
\end{aligned}
$$

For every $\delta>0$ we have

$$
|t-x|^{2+\alpha}\le\delta^\alpha\left(\delta^2+\frac{(t-x)^4}{\delta^2}\right)\ ;
$$

choosing $n\delta^2=\|a^2\|$ and taking into account Lemma 3.1.11 we obtain

$$
\begin{aligned}
&\left|w(x)n\,S_{n,b_n}\left(\eta(\mathrm{id},x)(\mathrm{id}-x)^2\right)(x)\right|\le\delta^\alpha\left(n\delta^2+\frac{n}{\delta^2}S_{n,b_n}\left((\mathrm{id}-x)^4\right)(x)\right)\\
&\le\frac{L_{wf''}}{n^{\alpha/2}}(\|a^2\|+\frac{n^2}{\|a^2\|}S_{n,b_n}\left((\mathrm{id}-x)^4\right)(x))\\
&\le\frac{4}{3}\frac{L_{wf''}}{n^{\alpha/2}}\|a^2\|\ .
\end{aligned}
$$

As regards the last term in (3.3.12) using the Cauchy-Schwartz inequality

$$
\begin{aligned}
&\left|n\,S_{n,b_n}\left((f''\circ\xi)(w\circ\xi)\left(\frac{w(x)}{w\circ\xi}-1\right)(\mathrm{id}-x)^2\right)(x)\right|\\
&\quad\le n\|f''\|_w\left(S_{n,b_n}\left(\left(\frac{w(x)}{w\circ\xi}-1\right)^2\right)(x)\right)^{1/2}\left(S_{n,b_n}(\mathrm{id}-x)^4)(x)\right)^{1/2}\ .
\end{aligned}
$$

In order to estimate the term $S_{n,b_n}\left(\left(\frac{w(x)}{w\circ\xi}-1\right)^2\right)(x)$ we observe that $S_{1,b_n}\left(\left(\frac{w(x)}{w\circ\xi}-1\right)^2\right)(x)=\left(\frac{w(x)}{w(\xi(t_0))}-1\right)^2$ for some $t_0\in[x-b_n(x),x+b_n(x)]$; since $\xi_0:=\xi(t_0)\in[x,t_0]$, it follows $\xi_0\in[x-b_n(x),x+b_n(x)]$. Consequently we have

$$\begin{aligned}\left(\frac{w(x)}{w(\xi_0)}-1\right)^2&=\left(\frac{1+|\xi_0|^p}{1+|x|^p}-1\right)^2\le\left(\frac{1+|x+\|\alpha\|/n|^p}{1+|x|^p}-1\right)^2\\&=\left(\frac{|x+\|\alpha\|/n|^p-|x|^p}{1+|x|^p}\right)^2\le\frac{C}{n^2}\end{aligned}$$

in the case $x\ge 0$; if $x\le 0$ we can argue similarly. From Proposition 3.1.2, since $\frac{w(x)}{w\circ\xi(x)}-1=0$ we have

$$\left\|S_{n,b_n}\left(\frac{w(x)}{w\circ\xi}-1\right)^2\right\|\le\frac{C}{n}\,,$$

and from Lemma 3.1.11 the last term in (3.3.12) can be estimated as follows

$$\left|n\,S_{n,b_n}\left((f''\circ\xi)(w\circ\xi)\left(\frac{w(x)}{w\circ\xi}-1\right)(\mathrm{id}-x)^2\right)(x)\right|\le\|f''\|_w\frac{C\|a^2\|}{\sqrt{3}}\frac{1}{\sqrt{n}}\,.$$

Finally we have

$$\begin{aligned}&|n(L_{w,n}(f)(x)-f(x)-A_wf''(x)|\\&\quad\le\frac{L_{wf''}}{n^{\alpha/2}}\frac{4\|a^2\|}{3}+\frac{\|f''\|_w\|a^2\|\|(a^2)''\|}{72n}+\|f''\|_w\frac{C\|a^2\|}{\sqrt{3}}\frac{1}{\sqrt{n}}\\&\quad\le C_a\frac{M_f}{n^{\alpha/2}}\,,\end{aligned}$$

where M_f is the seminorm defined by (3.3.9) and C_a is a constant defined by (3.3.10). □

The following lemma will allow us to state the core property of $C^2_w(\overline{\mathbb{R}})$ for $(A_w,D(A_w))$. Under additional assumptions and in different settings, the core property has been considered also in [59, 13, 14, 8].

Lemma 3.3.10 *For every $u\in C^2(\mathbb{R})$ and $h\ge 0$, the following statements are equivalent:*

a) $(wu)''(x)=O(x^h)$ (respectively, $(wu)''(x)=o(x^h)$) as $x\to\pm\infty$;

b) $wu''(x)=O(x^h)$ (respectively, $wu''(x)=o(x^h)$) as $x\to\pm\infty$;

c) $(wu')'(x)=O(x^h)$ (respectively, $(wu')'(x)=o(x^h)$) as $x\to\pm\infty$.

PROOF. Assume that $(wu)''(x) = O(x^h)$ as $x \to \pm\infty$. Then, for every $x \in \mathbb{R}$, $x \neq 0$,

$$\begin{aligned}
(wu'')(x) &= (wu)''(x) + 2(wu)'(x)w(x)\,(1/w)'(x) + (wu)(x)\,w(x)\,(1/w)''(x) \\
&= (wu)''(x) + 2(wu)'(x)\,\frac{px}{1/|x|^{p-2} + x^2} \\
&\quad + (wu)(x)\,p(p-1)\frac{1}{1/|x|^{p-2} + x^2} \\
&\approx (wu)''(x) + (wu)'(x)\,\frac{x}{1+x^2} + (wu)(x)\,\frac{1}{1+x^2}\,.
\end{aligned}$$

Condition $(wu)''(x) = O(x^h)$ implies $(wu)'(x) = O(x^{h+1})$ which in turn yields $(wu)(x) = O(x^{h+2})$; hence $(wu'')(x) = O(x^h)$ as $x \to \pm\infty$.

Now, let $(wu'')(x) = O(x^h)$ as $x \to \pm\infty$. We have

$$\begin{aligned}
(wu)''(x) &= (wu'')(x) + 2(wu')(x)\frac{w'(x)}{w(x)} + (wu)(x)\frac{w''(x)}{w(x)} \\
&= (wu'')(x) + 2(wu')(x)\frac{w'(x)}{w(x)} \\
&\quad + (wu)(x)\left(2\left(\frac{w'(x)}{w(x)}\right)^2 - w(x)\left(\frac{1}{w(x)}\right)''\right) \\
&= (wu'')(x) - 2(wu')(x)\,\frac{px}{1/|x|^{p-2} + x^2} \\
&\quad + (wu)(x)\left(2\left(\frac{px}{1/|x|^{p-2} + x^2}\right)^2 - p(p-1)\frac{1}{1/|x|^{p-2} + x^2}\right) \\
&\approx (wu'')(x) + (wu')(x)\frac{x}{1+x^2} + (wu)(x)\,\frac{1}{1+x^2}\,;
\end{aligned}$$

as before, from $(wu'')(x) = O(x^h)$, we obtain $u''(x) = O(x^{h+p})$ and in turn $u'(x) = O(x^{h+p+1})$ and $u(x) = O(x^{h+p+2})$; hence $(wu)''(x) = O(x^h)$.

The equivalence between a) and c) can be proved similarly. □

Remark 3.3.11 We observe that the Ventcel domain of A_w

$$D_V(A_w) := \{u \in C_w(\overline{\mathbb{R}}) \cap C^2(\mathbb{R}) \mid \lim_{x\to\pm\infty} w(x)A_w u(x) = 0\}$$

coincides with the maximal domain already defined.

Indeed let $u \in D(A_w)$ and by contradiction assume that $u \notin D_V(A_w)$, for example $\lim_{x\to+\infty} w(x)A_w u(x) = \ell \neq 0$. Then

$$\lim_{x\to+\infty} \frac{1}{1+x^p}\,\frac{1}{6x^2}\,a^2(x)x^2u''(x) = \ell$$

and therefore $\lim_{x\to+\infty} \frac{1}{x^{p+2}} u''(x) \neq 0$; consequently $\lim_{x\to+\infty} \frac{1}{x^{p+4}} u(x) \neq 0$, contradicting the condition $w\, u \in C(\overline{\mathbb{R}})$. The same reasoning holds at the point $-\infty$. □

Proposition 3.3.12 *The space $C^2_w(\overline{\mathbb{R}})$ is a core for $(A_w, D(A_w))$.*

PROOF. We consider the canonical isometry $\Gamma : C_w(\overline{\mathbb{R}}) \to C(\overline{\mathbb{R}})$ introduced in [8] and defined by setting, for every $f \in C_w(\overline{\mathbb{R}})$,

$$\Gamma(f) = f \cdot w \, .$$

Denote by $(A, D(A))$ the differential operator obtained by (3.3.2) with $w = 1$ on the domain

$$D(A) := \left\{ u \in C(\overline{\mathbb{R}}) \cap C^2(\mathbb{R}) \mid a^2\, u'' \in C(\overline{\mathbb{R}}) \right\} \, .$$

From [33, Proposition 2.10] we know that $C^2(\overline{\mathbb{R}})$ is a core for $(A, D(A))$.

Now, we show that $\Gamma(D(A_w)) = D(A)$. Let $v = \Gamma u \in D(A)$; we have $u \in C_w(\overline{\mathbb{R}}) \cap C^2(\mathbb{R})$ and further

$$\begin{aligned}
|\Gamma(Au)(x)| &= \left| \left(w \frac{a^2}{6} u'' \right)(x) \right| \\
&\leq \frac{a(x)^2}{6} |v''(x)| + \frac{C_1}{1+|x|} \frac{a(x)^2}{6} |v'(x)| + \frac{a(x)^2}{6} \frac{C_2}{1+x^2} |v(x)| \\
&\leq |Av(x)| + \frac{C_1}{1+|x|} \frac{a(x)^2}{6} |v'(x)| + \frac{C_3}{1+x^2} |v(x)| \, .
\end{aligned}$$

Using Taylor's formula we can write $v'(x) = \frac{v(x+\delta)-v(x)}{\delta} - \frac{v''(\xi)}{2}\delta$ for some $\xi \in]x, x+\delta[$, where δ is given by (3.3.3), and consequently

$$\begin{aligned}
\left| \frac{1}{1+|x|} \frac{a(x)^2}{6} v'(x) \right| &= \left| \frac{1}{1+|x|} \frac{a(x)^2}{6} \frac{v(x+\delta)-v(x)}{\delta} - \frac{1}{1+|x|} \frac{a(x)^2}{6} v''(\xi) \frac{\delta}{2} \right| \\
&\leq \frac{1}{1+|x|} \frac{1}{3\delta} \|a^2\| \|v\| + \left| \frac{1}{1+|x|} \frac{a(x)^2}{a(\xi)^2} \frac{a(\xi)^2}{6} v''(\xi) \frac{\delta}{2} \right| \\
&\leq \frac{C_4}{1+|x|} \|v\| + \frac{\delta}{2} M \left| \frac{a(\xi)^2}{6} v''(\xi) \right| \\
&= \frac{C_4}{1+|x|} \|v\| + \frac{\delta}{2} M |Av(\xi)|
\end{aligned}$$

which implies

$$|\Gamma(Au)(x)| \leq |Av(x)| + \frac{C_4}{1+|x|} \|v\| + \frac{\delta}{2} M |Av(\xi)| + \frac{C_3}{1+x^2} \|v\|. \quad (3.3.14)$$

Since $\|v\| < \infty$ and $\lim_{x\to\pm\infty} Av(x) = 0$ (see Remark 3.3.11), we have $\lim_{x\to\pm\infty} \Gamma(Au)(x) = 0$ and $u \in D(A_w)$.

Conversely let $u \in D(A_w)$, we have $\Gamma(u) \in C(\overline{\mathbb{R}}) \cap C^2(\mathbb{R})$ and further

$$\begin{aligned}
|A(\Gamma u)(x)| &= \left|\frac{a(x)^2}{6}(wu)''(x)\right| \\
&\leq \left|\frac{a(x)^2}{6}(wu'')(x)\right| + \frac{C_1}{1+|x|}\frac{a(x)^2}{6}|(wu')(x)| \\
&\quad +C_2\left|\frac{a(x)^2}{6}(wu)(x)\,\frac{1}{1+x^2}\right| \\
&\leq |\Gamma(Au)(x)| + \frac{C_1}{1+|x|}\frac{a(x)^2}{6}|(wu')(x)| + \frac{C_3}{1+x^2}|\Gamma u(x)|\ .
\end{aligned}$$

Again from Taylor's formula we have $u'(x) = \frac{u(x+\delta)-u(x)}{\delta} - \frac{u''(\xi)}{2}\,\delta$ where $\xi \in]x, x+\delta[$ and δ is given by (3.3.3), and consequently

$$\begin{aligned}
\left|\frac{1}{1+|x|}\frac{a(x)^2}{6}(wu')(x)\right| &\leq \left|\frac{1}{1+|x|}\frac{a(x)^2}{6}w(x)\frac{u(x+\delta)-u(x)}{\delta}\right| \\
&\quad +\left|\frac{1}{1+|x|}\frac{a(x)^2}{6}w(x)u''(\xi)\frac{\delta}{2}\right| \\
&\leq \frac{\|a^2\|}{6\delta}\frac{1}{1+|x|}\frac{w(x)}{w(x+\delta)}|\Gamma u(x+\delta)| \\
&\quad +\frac{\|a^2\|}{6\delta}\frac{1}{1+|x|}|\Gamma u(x)| \\
&\quad +\left|\frac{1}{1+|x|}\frac{a(x)^2}{a(\xi)^2}\,\frac{w(x)}{w(\xi)}\,w(\xi)\,\frac{a(\xi)^2}{6}\,u''(\xi)\frac{\delta}{2}\right| \\
&\leq \frac{C_4}{1+|x|}\|\Gamma(u)\| + C_5|\Gamma(Au)(\xi)|\ .
\end{aligned}$$

Then we have

$$\begin{aligned}
|A(\Gamma u)(x)| &\leq |\Gamma(Au)(x)| + \frac{C_4}{1+|x|}\|\Gamma(u)\| \\
&\quad +C_5|\Gamma(Au)(\xi)| + \frac{C_3}{1+x^2}\|\Gamma u\|\ ;
\end{aligned} \tag{3.3.15}$$

and taking the limit as $x \to \pm\infty$ we obtain $\lim_{x\to\pm\infty} A(\Gamma u)(x) = 0$ (see Remark 3.3.11) and consequently $\lim_{x\to\pm\infty} \Gamma u(x) = 0$.

Now, we observe that $\Gamma(C^2_w(\overline{\mathbb{R}})) = \Gamma(\{u \in C_w(\overline{\mathbb{R}}) \cap C^2(\mathbb{R}) \mid w\,u'' \in C(\overline{\mathbb{R}})\})$ and $C^2(\overline{\mathbb{R}}) = \{v \in C(\overline{\mathbb{R}}) \cap C^2(\mathbb{R}) \mid v'' \in C(\overline{\mathbb{R}})\}$ and taking into account that $wu'' \in C(\overline{\mathbb{R}})$ if and only if $(wu)'' = (\Gamma u)'' \in C(\overline{\mathbb{R}})$, we deduce the equality

$$\Gamma(C^2_w(\overline{\mathbb{R}})) = C^2(\overline{\mathbb{R}})\ .$$

Finally from (3.3.14), it immediately follows that if $\Gamma(u) \in D(A)$ then

$$\|\Gamma(Au)\| \le C\left(\|A(\Gamma u)\| + \|\Gamma(u)\|\right) \tag{3.3.16}$$

for some constant $C > 0$.

Taking into account that $C^2(\overline{\mathbb{R}})$ is a core for $(A, D(A))$, the proof is complete. □

Finally, we discuss the representation of the semigroup generated by A by means of iterates of weighted Steklov operators.

Proposition 3.3.13 *The operator $(A_w, D(A_w))$ generates a strongly continuous semigroup on $C_w(\overline{\mathbb{R}})$.*

Proof. Let $\lambda > 0$ and consider the problem

$$\lambda u - A(u) = f \qquad f \in C_w(\overline{\mathbb{R}})\ , \tag{3.3.17}$$

which is equivalent to

$$\lambda v - B(v) = g\ , \qquad g \in C(\overline{\mathbb{R}})\ , \tag{3.3.18}$$

where $v = uw$, $g = wf$, and

$$B(v) = wA\left(\frac{v}{w}\right)\ .$$

From $\left(\frac{v}{w}\right)' = \frac{v'}{w} - \frac{w'}{w^2}v$ and $\left(\frac{v}{w}\right)'' = \frac{v''}{w} - 2\frac{w'}{w^2}v' - \left(\frac{w'}{w^2}\right)'v$, we obtain

$$B(v) = \frac{a^2}{6}v'' - 2\frac{a^2}{6}\frac{w'}{w}v' - \frac{a^2}{6}w\left(\frac{w'}{w^2}\right)'v =: \alpha_B v'' + \beta_B v' + \gamma_B v\ .$$

Using Feller's classification of the endpoints (see [48]), we prove that B generates a C_0-semigroup on its maximal domain $D(B) := \{u \in C(\overline{\mathbb{R}}) \mid Bu \in C(\overline{\mathbb{R}})\}$. In our case the functions W, R and Q in [48, p. 391–393]) are given by

$$W_B(x) = \exp\left(-\int_{x_0}^{x}\frac{\beta_B}{\alpha_B}\right) = \exp\left(\int_{x_0}^{x}2\frac{w'}{w}\right) = \frac{w(x)^2}{w(x_0)^2}\ ,$$

$$R_B(x) = W_B(x)\int_{x_0}^{x}\frac{1}{\alpha_B W_B} = w(x)^2\int_{x_0}^{x}\frac{6}{a^2w^2}$$

and

$$Q_B(x) = \frac{1}{\alpha_B(x)W_B(x)}\int_{x_0}^{x}W_B = \frac{6}{a(x)^2w(x)^2}\int_{x_0}^{x}w^2\ .$$

Since $w(x) = \frac{1}{1+|x|^p}$, it can be readily seen that

$$Q_B(x) = \frac{6(1+|x|^p)^2}{a(x)^2}\int_0^x\frac{1}{(1+|t|^p)^2}\,dt$$

and hence Q_B is not integrable on $[0,+\infty[$ and on $]-\infty,0]$; moreover,

$$R_B(x) = \frac{6}{(1+|x|^p)^2}\int_0^x \frac{(1+|t|^p)^2}{a(t)^2}\,dt$$

and from the estimate

$$R_B(x) \geq \frac{C}{1+x^{2p}}\int_0^x t^{2p+2}dt = C_1\,\frac{x^{2p+3}}{1+x^{2p}}\ ,$$

it follows that R_B is not integrable on $[0,+\infty[$ and on $]-\infty,0]$ as well.

So the endpoint $-\infty$ and ∞ are both natural for the operator B, and consequently the operator $Bv-\gamma_B v$ generates a strongly continuous semigroup on its maximal domain on $C(\overline{\mathbb{R}})$. Observe also that the maximal domain coincides with the Ventcel domain in the case of natural endpoints, as an immediate consequence of the classical generation results by Clément and Timmermans [44] and Timmermans [69] (see Chapter II).

Taking into account that

$$\gamma_B(x) = \frac{a(x)^2}{6}\,\frac{p(p-1)|x|^{p-2}}{1+|x|^p}$$

we conclude that γ_B is bounded and hence $(B, D(B))$ generates a strongly continuous semigroup on $C(\overline{\mathbb{R}})$.

Finally, we observe that if $v \in C(\overline{\mathbb{R}})$ is a solution of (3.3.18), then $u := v/w \in C_w(\overline{\mathbb{R}})$ is a solution of (3.3.17). Moreover $u \in D(A)$ if and only if $u\cdot w \in D(B)$ since Γ is an isometry between $C_w(\overline{\mathbb{R}})$ and $C(\overline{\mathbb{R}})$, and from the generation property of $(B, D(B))$ we deduce that $(A_w, D(A_w)$ generates a C_0-semigroup on $C_w(\overline{\mathbb{R}})$. □

Finally we can state the following representation theorem.

Theorem 3.3.14 *The operator $(A_w, D(A_w))$ generates a positive C_0-semigroup $(T(t))_{t\geq 0}$ in $C_w(\overline{\mathbb{R}})$ satisfying $\|T(t)\| \leq e^{Ct}$ and, for every $t \geq 0$ and for every sequence $(k(n)_{n\geq 1})$ of positive integers satisfying $\lim_{n\to+\infty} k(n)/n = t$, we have*

$$T(t) = \lim_{n\to+\infty} L_n^{k(n)} \qquad \textit{strongly on } C(\overline{\mathbb{R}})\ ,$$

moreover if $a \in C^2(\overline{\mathbb{R}})$, for every $f \in C_w^{2,\alpha}(\mathbb{R})$ we have

$$\left\|T(t)u - L_n^{k(n)}u\right\|_w \leq t\,\frac{C_\alpha M_f}{n^{\alpha/2}} + \left(\left|\frac{k(n)}{n} - t\right| + \sqrt{\frac{2}{\pi}}\,\frac{\sqrt{k(n)}}{n}\right) \times \left(\|Af\|_w + \frac{C_\alpha M_f}{n^{\alpha/2}}\right) \qquad (3.3.19)$$

and choosing $k(n) = [nt]$

$$\left\|T(t)u - L_n^{[nt]}u\right\|_w \leq t\,\frac{C_\alpha M_f}{n^{\alpha/2}} + \frac{1}{\sqrt{n}}\left(\frac{1}{\sqrt{n}} + \sqrt{\frac{2t}{\pi}}\right)\left(\|Af\|_w + \frac{C_\alpha M_f}{n^{\alpha/2}}\right)\ . \qquad (3.3.20)$$

Proof. From Proposition 3.3.13 we know that the operator $(A_w, D(A_w))$ generates a C_0-semigroup $(T(t))_{t\geq 0}$ and hence, for every $\lambda > 0$, the range $(\lambda - A_w)(D(A_w))$ coincides with $C_w(\overline{\mathbb{R}})$. Moreover, by Proposition 3.3.12, $C^2_w(\overline{\mathbb{R}})$ is a core for $(A_w, D(A_w))$ and therefore $(\lambda - A_w)(C^2_w(\overline{\mathbb{R}}))$ is dense in $C_w(\overline{\mathbb{R}})$. We observe that we have

$$\|L_{w,n}\| \leq \|\omega_n\|^n_\infty \leq (1 + \|\omega_n - 1\|)^n \leq \left(1 + \frac{C}{n^2}\right)^n$$

as a consequence of (3.3.5) and Proposition 3.3.1, and consequently

$$\|L^k_{w,n}\| \leq \left(\left(1 + \frac{C}{n^2}\right)^{n^2}\right)^{k/n} \leq e^{Ck/n} .$$

Hence we can apply Trotter's approximation theorem [70] and obtain that the closure of the operator arising from the Voronovskaja's formula (Theorem 3.3.8) generates a C_0-semigroup represented by (3.3.14). Finally, this closure coincides with $(A_w, D(A_w))$ since $C^2_w(\overline{\mathbb{R}})$ is a core by Proposition 3.3.12. The positivity of the semigroup is a consequence of the representation (3.3.14).

At this point we can apply Theorem 1.1.2. From Theorem 3.3.9 follows that the seminorms are given by $\psi_n(f) = C_a\frac{M_f}{n^{\alpha/2}}$ and $\varphi_n(f) = \|Af\|_w + C_a\frac{M_f}{n^{\alpha/2}}$, then taking into account that the growth bound of $(T(t))_{t\geq 0}$ is equal to 0 and every $T(t)$ is a linear contraction, i.e. $\omega = 0$ and $M = 1$, the estimates (3.3.19) and (3.3.20) follow directly from (1.1.10) and (1.1.11), and this completes the proof. □

3.4 Steklov operators in weighted spaces on $[0,1]$

In this section we consider the weighted space of continuous functions on the interval $[0,1]$. We fix $a \in C([0,1])$ satisfying $a(0) = a(1) = 0$ and require that it is differentiable at 0 and 1, i.e.

$$\lim_{x\to 0,1} \frac{a(x)}{x(1-x)} \in \mathbb{R} \,. \tag{3.4.1}$$

We consider the weight function

$$w(x) := x^p(1-x)^q \,, \qquad x \in [0,1] \,, \qquad p,q \geq 2 \,. \tag{3.4.2}$$

In the sequel, we shall set $b_n(x) := a(x)/n$. From (3.4.1), it follows $[x - b_n(x), x + b_n(x)] \subset [0,1]$ for every $x \in [0,1]$ and large enough n; hence, we can define the functions $\omega_n : [0,1] \to \mathbb{R}$ by setting

$$\omega_n(x) := \frac{w(x)}{2b_n(x)} \int_{x-b_n(x)}^{x+b_n(x)} \frac{1}{w(t)}\, dt \,, \qquad x \in]0,1[\,, \tag{3.4.3}$$

and

$$\omega_n(0) := \lim_{x\to 0^+} \omega_n(x) \,, \qquad \omega_n(1) := \lim_{x\to 1^-} \omega_n(x) \,. \tag{3.4.4}$$

Remark 3.4.1 Let $n \in \mathbb{N}$ be large enough; then the limits in (3.4.4) exist and are finite at the points 0 and 1.

Since the discussion is at all similar on neighborhoods of the endpoints 0 and 1, in the sequel we shall limit ourselves to consider only a neighborhood of 0, where the weight function can be taken of the form

$$w(x) := x^p \,.$$

We observe that, for every $x \in]0,1[$,

$$\begin{aligned}
\omega_n(x) &= x^p \frac{1}{2b_n(x)} \int_{x-b_n(x)}^{x+b_n(x)} \frac{1}{t^p}\, dt \qquad (3.4.5)\\
&= \frac{1}{1-p}\frac{1}{2b_n(x)} \left[\frac{1}{(x+b_n(x))^{p-1}} - \frac{1}{(x-b_n(x))^{p-1}} \right] x^p \\
&= \frac{1}{p-1}\frac{1}{2b_n(x)} \frac{(x+b_n(x))^{p-1} - (x-b_n(x))^{p-1}}{(x^2-b_n(x)^2)^{p-1}}\, x^p \\
&= \frac{1}{p-1}\frac{x^{2p-1}}{2b_n(x)\, x^{2p-2}} \frac{(1+b_n(x)/x)^{p-1} - (1-b_n(x)/x)^{p-1}}{(1-b_n(x)^2/x^2)^{p-1}} \\
&= \frac{x}{2(p-1)b_n(x)} \frac{(1+b_n(x)/x)^{p-1} - (1-b_n(x)/x)^{p-1}}{(1-b_n(x)^2/x^2)^{p-1}} \,.
\end{aligned}$$

Since a is differentiable at 0, we have

$$\lim_{x\to 0^+} \omega_n(x) \quad = \quad \begin{cases} 1\,, & a'(0)=0\,,\\ \dfrac{1}{2(p-1)}\,\dfrac{\left(1+\frac{\gamma}{n}\right)^{p-1}-\left(1-\frac{\gamma}{n}\right)^{p-1}}{\frac{\gamma}{n}\left(1-\frac{\gamma^2}{n^2}\right)^{p-1}}\,, & a'(0)\neq 0\,,\end{cases}$$

where $\gamma := a'(0)$. □

In the next Proposition 3.4.2 we estimate ω_n.

Proposition 3.4.2 *There exists a constant* $C>0$ *such that*

$$\|\omega_n - \mathbf{1}\| \le \frac{C}{n^2}\,.$$

Proof. For n large enough, we have $0 \le b_n(x)/x \le 1/2$ for every $x \in]0,1[$ and hence, using the Taylor's expansions of $(1\pm y)^{p-1}$ at 0, we get the existence of $\xi_x, \theta_x, \eta_x \in]0,1/2[$ such that

$$\begin{aligned} \left(1+\frac{b_n(x)}{x}\right)^{p-1} \quad &= \quad 1+(p-1)\frac{b_n(x)}{x}+\frac{1}{2}\,(p-1)(p-2)\,\frac{b_n(x)^2}{x^2}\\ &\qquad +\frac{1}{6}\,(p-1)(p-2)(p-3)\,(1+\xi_x)^{p-4}\,\frac{b_n(x)^3}{x^3}\,,\\ \left(1-\frac{b_n(x)}{x}\right)^{p-1} \quad &= \quad 1-(p-1)\frac{b_n(x)}{x}+\frac{1}{2}\,(p-1)(p-2)\,\frac{b_n(x)^2}{x^2}\\ &\qquad -\frac{1}{6}\,(p-1)(p-2)(p-3)\,(1-\theta_x)^{p-4}\,\frac{b_n(x)^3}{x^3}\,,\\ \left(1-\frac{b_n(x)^2}{x^2}\right)^{p-1} \quad &= \quad 1-(p-1)\,(1-\eta_x)^{p-2}\,\frac{b_n(x)^2}{x^2}\,. \end{aligned}$$

Consequently, using (3.4.5),

$$\begin{aligned}
\omega_n(x) - 1 &= \frac{1}{2(p-1)b_n(x)/x} \, \frac{(1+b_n(x)/x)^{p-1} - (1-b_n(x)/x)^{p-1}}{(1-b_n(x)^2/x^2)^{p-1}} \\
&\quad - \frac{(1-b_n(x)^2/x^2)^{p-1}}{(1-b_n(x)^2/x^2)^{p-1}} \\
&= \frac{1}{(1-b_n(x)^2/x^2)^{p-1}} \times \\
&\quad \times \Big(1 + \frac{1}{12}\,(p-2)(p-3)\,\big((1+\xi_x)^{p-4} + (1-\theta_x)^{p-4}\big)\,\frac{b_n(x)^2}{x^2} \\
&\quad -1 + (p-1)\,(1-\eta_x)^{p-2}\,\frac{b_n(x)^2}{x^2}\Big) \\
&= \frac{b_n(x)^2/x^2}{(1-b_n(x)^2/x^2)^{p-1}} \,\Big(\frac{1}{12}\,(p-2)(p-3) \times \\
&\quad \times \ \big((1+\xi_x)^{p-4} + (1-\theta_x)^{p-4}\big) + (p-1)\,(1-\eta_x)^{p-2}\Big) \\
&= \frac{1}{n^2}\, \frac{a(x)^2/x^2}{(1-a(x)^2/(n^2\,x^2))^{p-1}}\, \varphi(x)\ , \qquad (3.4.6)
\end{aligned}$$

where

$$\varphi(x) := \frac{1}{12}\,(p-2)(p-3)\,\big((1+\xi_x)^{p-4} + (1-\theta_x)^{p-4}\big) + (p-1)\,(1-\eta_x)^{p-2}\ .$$

Since φ is bounded as well as $a(x)/x$, we get the desired result. □

Now, we consider the weighted space

$$C_w([0,1]) := \{f \in C(]0,1[) \mid \exists \lim_{x\to 0,1} w(x)\, f(x) \in \mathbb{R}\}$$

endowed with the norm

$$\|f\|_w = \sup_{x\in[0,1]} |f(x)\, w(x)|\ , \qquad f \in C_w([0,1])\ .$$

As in the preceding section we define the operators $L_{w,n} : C_w([0,1]) \to C_w([0,1])$ by setting, for every $f \in C_w([0,1])$, $L_{w,n}(f) := S_{n,b_n}(f)$.

We also need to define the subspace

$$C^2_w([0,1]) := \{f \in C_w([0,1]) \cap C^2(]0,1[) \mid f'' \in C_w([0,1])\}\ .$$

If n is large enough, then for every $f \in C_w([0,1])$, the function $L_{w,n}(f)$ is well-defined by (3.4.1). Moreover we have,

$$\|L_{w,n}(f)\|_w = \|S_{n,b_n}(f)\|_w \le \|\omega_n\|^n\ \|f\|_w \le C_w\, \|f\|_w\ , \qquad (3.4.7)$$

and hence $L_{w,n}(f) \in C_\omega([0,1])$.

The proofs of the following properties are at all similar to the unbounded weighted case and therefore we shall omit them.

Proposition 3.4.3 *We have the following properties:*

1. *For every* $f \in C(0,1)$, $k \geq 1$ *and* $x \in \mathbb{R}$,

$$\|S_{k,b_n}(f) - f\|_w \leq \|S_{1,b_n}(f) - f\|_w \sum_{i=0}^{k} \|\omega_n\|^i \ . \tag{3.4.8}$$

2. *For every* $f \in C_w^2([0,1])$, *there exists a constant* $C_f > 0$, *depending on* f, *such that for every* $x \in (0,1)$

$$\|S_{1,b_n}(f) - f\|_w \leq \frac{C_f}{n^2} \ . \tag{3.4.9}$$

3. *For every* $f \in C_w([0,1])$ *we have*

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{k=0}^{n-1} S_{k,b_n}(f) = f$$

with respect to the norm $\|\cdot\|_w$.

As a consequence, also in the present setting we can state the following results. The proof is omitted since it is at all similar to the unweighted case.

Theorem 3.4.4 *For every* $f \in C_w([0,1])$, *we have*

$$\lim_{n\to+\infty} L_{w,n}(f) = f$$

uniformly with respect to the weighted norm $\|\cdot\|_w$.

Theorem 3.4.5 (Weighted Voronovskaja-type formula)
For every $f \in C_w^2([0,1])$ *we have*

$$\lim_{n\to\infty} \left\| n\left(L_{w,n}(f) - f\right) - \frac{a^2}{6} f'' \right\|_w = 0 \ .$$

In order to obtain a quantitative estimate of the Voronovskaja'a formula we need to consider the class of functions

$$C_w^{2,\alpha}([0,1]) := \{f \in C_w^2([0,1]) \mid wf'' \in C^\alpha([0,1])\} \ .$$

Theorem 3.4.6 (Weighted quantitative Voronovskaja-type formula)

If $a^2 \in C^2([0,1])$ *for every* $f \in C_w^{2,\alpha}([0,1])$ *we have*

$$\left\| n\left(L_{w,n}(f) - f\right) - \frac{a^2}{6} f'' \right\|_w \leq C_a \frac{M_f}{n^{\alpha/2}} \ ,$$

where M_f is the seminorm defined by

$$M_f = L_{wf''} + \|f''\|_w \tag{3.4.10}$$

and

$$C_a := \|a^2\| \max\left\{\frac{4}{3}, \frac{\|(a^2)''\|}{72\sqrt{n}}\right\} . \tag{3.4.11}$$

Finally, we shall be concerned with the core property and the representation of the semigroup generated by the differential operator arising from the Voronovskaja-type formula.

The differential operator is $A_w u(x) := a(x)^2\, u''(x)/6$ on the following Ventcel's domain

$$D_V(A_w) := \{u \in C_w([0,1]) \cap C^2(]0,1[) \mid \lim_{x\to 0,1} w(x)\, A_w u(x) = 0\} .$$

As in the preceding section, we consider only first-order degeneracy of the function a at the endpoints. Different generation results in the space of continuous functions vanishing at the endpoints are available in [8] (see also [6] and the references given there).

Theorem 3.4.7 *Assume that*

$$a(x) = Cx(1-x) , \qquad 0 \le x \le 1 ,$$

for a suitable constant $C > 0$. Then, the space $C^2([0,1])$ (and hence $C^2_w([0,1])$) is a core for $(A_w, D_V(A_w))$; moreover, the operator $(A_w, D_V(A_w))$ generates a C_0-semigroup $(T(t))_{t\ge 0}$ in $C_w([0,1])$ and, for every $t \ge 0$ and for every sequence $(k(n))_{n\ge 1}$ of positive integers satisfying $\lim_{n\to+\infty} k(n)/n = t$, we have

$$T(t) = \lim_{n\to+\infty} L_{w,n}^{k(n)} \qquad \textit{strongly on } C_w([0,1]) , \tag{3.4.12}$$

moreover for every $f \in C^{2;\alpha}_w([0,1])$ we have

$$\left\|T(t)u - L_n^{k(n)}u\right\|_w \le t\,\frac{C_\alpha M_f}{n^{\alpha/2}} + \left(\left|\frac{k(n)}{n} - t\right| + \sqrt{\frac{2}{\pi}}\,\frac{\sqrt{k(n)}}{n}\right) \times \left(\|Af\|_w + \frac{C_\alpha M_f}{n^{\alpha/2}}\right) \tag{3.4.13}$$

and choosing $k(n) = [nt]$

$$\left\|T(t)u - L_n^{[nt]}u\right\|_w \le t\,\frac{C_\alpha M_f}{n^{\alpha/2}} + \frac{1}{\sqrt{n}}\left(\frac{1}{\sqrt{n}} + \sqrt{\frac{2t}{\pi}}\right)\left(\|Af\|_w + \frac{C_\alpha M_f}{n^{\alpha/2}}\right) . \tag{3.4.14}$$

Proof. We reason only in the interval $[0,1/2]$ and assume $a(x)=Cx$. Let $u\in D_V(A_w)$ and fix $\varepsilon>0$. We have $u''(x)=o(1/x^{p+2})$ as $x\to 0$ and hence $u'(x)=o(1/x^{p+1})$ and $u(x)=o(1/x^p)$, that is $\lim_{x\to 0} w(x)\,x\,u'(x)=0$ and $\lim_{x\to 0} w(x)\,u(x)=0$. Hence, we can choose $\delta>0$ such that $\delta<1/2$ and

$$\begin{aligned}
|w(x)\,u(x)|<\varepsilon\,, &\qquad x\in[0,\delta]\,,\\
|w(x)\,x\,u'(x)|<\varepsilon\,, &\qquad x\in]0,\delta]\,,\\
|w(x)\,x^2\,u''(x)|<\varepsilon\,, &\qquad x\in]0,\delta]\,.
\end{aligned}$$

Let $v:[0,1/2]\to\mathbb{R}$ as follows

$$v(x):=\begin{cases} u(\delta)+u'(\delta)(x-\delta)+u''(\delta)\,\dfrac{(x-\delta)^2}{2}\,, & x\in[0,\delta[\,,\\ u(x)\,, & x\in\left[\delta,\dfrac{1}{2}\right]\,.\end{cases}$$

Then, $v\in C^2([0,1/2])$ and, for every $x\in]0,\delta]$, we have

$$\begin{aligned}
|w(x)\,Av(x)| &= \left|w(x)\,\frac{C^2\,x^2}{6}\,v''(x)\right| = \left|\frac{C^2\,x^{p+2}}{6}\,u''(\delta)\right|\\
&\le \left|\frac{C^2\,\delta^{p+2}}{6}\,u''(\delta)\right| < \frac{C^2}{6}\,\varepsilon
\end{aligned}$$

and consequently

$$|w(x)(Au(x)-Av(x))|\le \frac{C^2}{6}\,\varepsilon+\frac{C^2}{6}\,\varepsilon=\frac{C^2}{3}\,\varepsilon\,,\qquad x\in\left[0,\frac{1}{2}\right]\,.$$

Finally, for every $x\in[0,\delta]$,

$$\begin{aligned}
|w(x)(u(x)-v(x))| &\le |w(x)\,u(x)|+|w(x)\,u(\delta)|+|w(x)\,u'(\delta)\,(x-\delta)|\\
&\quad +\left|w(x)\,u''(\delta)\,\frac{(x-\delta)^2}{2}\right|\\
&\le \varepsilon+|w(\delta)\,u(\delta)|+|w(\delta)\,\delta\,u'(\delta)|+\left|w(\delta)\,\frac{\delta^2}{2}\,u''(\delta)\right|\\
&\le \varepsilon+\varepsilon+\varepsilon+\frac{\varepsilon}{2}=\frac{7}{2}\,\varepsilon\,.
\end{aligned}$$

The same inequality obviously extends to the interval $[0,1/2]$ and since ε is arbitrary, this completes the proof of the core property.

In order to apply Trotter's approximation theorem, we need to establish the stability estimate $\|L_{w,n}^k\|\le Me^{ck/n}$ for every $n,k\ge 1$ for a suitable constant $c\ge 0$.

Indeed, from (3.4.7) and Proposition 3.4.2, we get

$$\|L_{w,n}\|\le\|\omega_n\|^n\le(1+\|\omega_n-\mathbf{1}\|)^n\le\left(1+\frac{C}{n^2}\right)^n\,,$$

where C is the constant in Proposition 3.4.2, and consequently

$$\|L_{w,n}^k\| \le \left(\left(1+\frac{C}{n^2}\right)^{n^2}\right)^{k/n} \le e^{Ck/n} \, .$$

Hence, we can apply Trotter's approximation theorem [70] and obtain that the closure $(\bar{A}, D(\bar{A}))$ of the restriction of A_w to $C^2([0,1])$ generates a C_0-semigroup $(T(t))_{t\ge 0}$ on $C_w([0,1])$ which can be represented by (3.4.12). Moreover, the stability estimate also gives $\|T(t)\| \le e^{Ct}$ for every $t \ge 0$.

Now, we show that $(A_w, D_V(A_w))$ is closed and this, together with the core property, will imply that $(\bar{A}, D(\bar{A})) = (A_w, D_V(A_w))$ and complete the proof.

Let $(u_n)_{n\ge 1}$ be a sequence in $D_V(A_w)$ and $u, v \in C_w([0,1])$ such that $(w \cdot u_n)_{n\ge 1}$ converges uniformly to $w \cdot u$ and $(w \cdot A(u_n))_{n\ge 1}$ converges uniformly to $w \cdot v$. Since α and w are continuous and strictly positive in $]0,1[$, they have a positive minimum in every interval $[a,b] \subset]0,1[$ and consequently $(u_n)_{n\ge 1}$ converges uniformly to u and $(A(u_n))_{n\ge 1}$ converges uniformly to v in $[a,b]$; from the classical theory, we have that $u \in C^2([a,b])$ and $Au = v$ in $[a,b]$. Since the interval $[a,b]$ is arbitrary, we get $u \in C^2(]0,1[)$ and $Au = v$ in $]0,1[$. Finally, from the uniform convergence of $(w \cdot A(u_n))_{n\ge 1}$ to $w \cdot v$ and the condition $\lim_{x\to 0,1} w(x)\, Au_n(x) = 0$, we also have $\lim_{x\to 0,1} w(x)\, Au(x) = 0$ and hence $u \in D_V(A_w)$.

At this point we can apply Theorem 1.1.2. From Theorem 3.4.6 follows that the seminorms are given by $\psi_n(f) = C_a \frac{M_f}{n^{\alpha/2}}$ and $\varphi_n(f) = \|Af\|_w + C_a \frac{M_f}{n^{\alpha/2}}$, then taking into account that the growth bound of $(T(t))_{t\ge 0}$ is equal to 0 and every $T(t)$ is a linear contraction, i.e. $\omega = 0$ and $M = 1$, the estimates (3.4.13) and (3.4.14) follow directly from (1.1.10) and (1.1.11), and this completes the proof. □

3.5 An extension to the multivariate case

In this final section we briefly consider a possible extension to the multivariate case; we establish some approximation results and a Voronovskaja's formula.

The following results have been published in [42].

For the sake of simplicity, we limit ourselves to the two-variables case since a similar construction can be extended in a straightforward way in more variables.

The case considered in this section is of particular interest since it involves a second-order partial differential operator on the whole space $\mathbb{R}^2$ whose coefficients of the second-order partial derivatives may be even unbounded or degenerate.

First, we define the mean integral operator over a rotated rectangle. Let $a, b : \mathbb{R}^2 \to \mathbb{R}$ be strictly positive continuous functions satisfying the following condition

$$a(x,y), b(x,y) \le c_1 + c_2(|x| + |y|) , \qquad (x,y) \in \mathbb{R}^2 , \tag{3.5.1}$$

for some suitable constants $c_1, c_2 > 0$ and let $\theta : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function.

Moreover, denote by $L^1_{\text{loc}}(\mathbb{R}^2)$ the space of locally integrable functions on $\mathbb{R}^2$.

The mean integral operator $M_{a,b,\theta} : L^1_{\text{loc}}(\mathbb{R}^2) \to C(\mathbb{R}^2)$ is defined by setting, for every $f \in L^1_{\text{loc}}(\mathbb{R}^2)$ and $(x,y) \in \mathbb{R}^2$,

$$M_{a,b,\theta} f(x,y) := \frac{1}{|R[x,y]|} \iint_{R[x,y]} f(\xi,\eta)\, d\xi\, d\eta , \tag{3.5.2}$$

where

$$\begin{aligned} R[x,y] \;:=\; & \{(\xi,\eta) \in \mathbb{R}^2 \mid \\ & |(\xi - x)\cos\theta(x,y) + (\eta - y)\sin\theta(x,y)| \le a(x,y) , \\ & |-(\xi - x)\sin\theta(x,y) + (\eta - y)\cos\theta(x,y)| \le b(x,y)\} \end{aligned} \tag{3.5.3}$$

is the rectangle with center (x,y), sides $2\,a(x,y)$ and $2\,b(x,y)$ and rotated anticlockwise of an angle $\theta(x,y)$; moreover $|R[x,y]| := 4\,a(x,y)\,b(x,y)$ denotes the Lebesgue area of $R[x,y]$.

Using the parallel variables to the sides of the rectangle $R[x,y]$, from (3.5.2) we easily get, for every $f \in L^1_{\text{loc}}(\mathbb{R}^2)$ and $(x,y) \in \mathbb{R}^2$,

$$M_{a,b,\theta} f(x,y) := \frac{1}{4\,a(x,y)\,b(x,y)} \int_{-a(x,y)}^{a(x,y)} \int_{-b(x,y)}^{b(x,y)} f \circ \varphi_{x,y}\,(\xi,\eta)\; d\xi\, d\eta , \tag{3.5.4}$$

where

$$\varphi_{x,y}(\xi,\eta) := (x + \xi\cos\theta(x,y) - \eta\sin\theta(x,y), y + \xi\sin\theta(x,y) + \eta\cos\theta(x,y))$$

defines the change of variables.

Denote by $C^{(b)}(\mathbb{R}^2)$ the space of all continuous bounded real functions on $\mathbb{R}^2$ and by $C_0(\mathbb{R}^2)$ the subspace consisting of all continuous functions vanishing at the point at infinity of $\mathbb{R}^2$. These spaces are endowed with the usual uniform norm

$$\|f\| := \sup_{(x,y)\in\mathbb{R}^2} |f(x,y)| , \qquad f \in C^{(b)}(\mathbb{R}^2) .$$

Moreover, we shall consider the function $w : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$w(x,y) := \frac{1}{1+x^2+y^2} , \qquad (x,y) \in \mathbb{R}^2 , \tag{3.5.5}$$

and the space

$$C_w^{(b)}(\mathbb{R}^2) := \left\{ f \in C(\mathbb{R}^2) \mid w\, f \in C^{(b)}(\mathbb{R}^2) \right\}$$

endowed with the norm

$$\|f\|_w := \sup_{(x,y)\in\mathbb{R}^2} |w(x,y)\, f(x,y)| .$$

Observe that $M_{a,b,\theta}$ maps $C^{(b)}(\mathbb{R}^2)$ into itself and is a positive contraction when considered as an operator on this space. Moreover, $M_{a,b,\theta}$ maps the space of compactly supported functions into $C_0(\mathbb{R}^2)$ and by continuity also $C_0(\mathbb{R}^2)$ into itself.

Now, we show that $M_{a,b,\theta}$ maps $C_w^{(b)}(\mathbb{R}^2)$ into itself. First, we put

$$r(x,y) := \sqrt{a(x,y)^2 + b(x,y)^2} , \tag{3.5.6}$$

for every $(x,y) \in \mathbb{R}^2$ and we observe that (see (3.5.3))

$$R[x,y] \subset [x - r(x,y), x + r(x,y)] \times [y - r(x,y), y + r(x,y)] . \tag{3.5.7}$$

Moreover, from (3.5.1), we have

$$r(x,y) \le \sqrt{2}\ (c_1 + c_2(|x| + |y|))$$

and hence, using (3.5.6) and (3.5.7),

$$\begin{aligned}
\frac{|M_{a,b,\theta} f(x,y)|}{1+x^2+y^2} &\leq \frac{1}{(1+x^2+y^2)\,|R[x,y]|} \times \\
&\quad \times \iint_{R[x,y]} \frac{f(\xi,\eta)}{1+\xi^2+\eta^2}\,(1+\xi^2+\eta^2) d\xi\, d\eta \\
&\leq \frac{1}{1+x^2+y^2} \sup_{(\xi,\eta)\in R[x,y]} (1+\xi^2+\eta^2)\, \|f\|_w \\
&\leq \frac{1+x^2+y^2+2r(x,y)^2+2(|x|+|y|)r(x,y)}{1+x^2+y^2}\, \|f\|_w \\
&\leq \left(1+\frac{c_3+c_4(x^2+y^2)}{1+x^2+y^2}\right) \|f\|_w \leq (1+c_5)\, \|f\|_w
\end{aligned} \tag{3.5.8}$$

for some suitable constant $c_3, c_4 > 0$.

Then $M_{a,b,\theta}$ is a bounded operator when considered on the space $C_w^{(b)}(\mathbb{R}^2)$ endowed with the norm $\|\cdot\|_w$.

We are now in a position to define our Steklov operators. For every $n \geq 1$, set $a_n := a/n$, $b_n := b/n$ and consider the n-th Steklov operator $S_n : C_w^{(b)}(\mathbb{R}^2) \to C_w^{(b)}(\mathbb{R}^2)$ defined by setting, for every $f \in C_w^{(b)}(\mathbb{R}^2)$ and $(x,y) \in \mathbb{R}^2$,

$$S_n f(x,y) := M^n_{a_n,b_n,\theta} f(x,y)\ , \tag{3.5.9}$$

where, as usual, $M^n_{a_n,b_n,\theta}$ denotes n-th iterate of the operator $M_{a_n,b_n,\theta}$.

We shall write M_n in place of $M_{a_n,b_n,\theta}$ if no confusion arises.

As a consequence of the properties of $M_{a,b,\theta}$, from (3.5.9) we have that S_n is well-defined as an operator on $C_w^{(b)}(\mathbb{R}^2)$ and maps the spaces $C^{(b)}(\mathbb{R}^2)$ and $C_0(\mathbb{R}^2)$ into themselves. When necessary, we shall consider the Steklov operators acting on these spaces too; we also observe that S_n is a positive contraction when acting on $C^{(b)}(\mathbb{R}^2)$ and $C_0(\mathbb{R}^2)$ endowed with the uniform norm.

In order to estimate the norm of S_n with respect to the weighted uniform norm in $C_w^{(b)}(\mathbb{R}^2)$, we use (3.5.8) taking into account that in this case we have to consider $r_n := \sqrt{a^2+b^2}/n$ in place of r and obtain

$$\begin{aligned}
\frac{|M_n f(x,y)|}{1+x^2+y^2} &\leq \frac{1+x^2+y^2+2r_n(x,y)^2+2(|x|+|y|)r_n(x,y)}{1+x^2+y^2}\, \|f\|_w \\
&\leq \left(1+\frac{C}{n}\right) \|f\|_w\ , \qquad (x,y)\in\mathbb{R}^2\ ,
\end{aligned} \tag{3.5.10}$$

for a suitable constant $C > 0$; this yields

$$\|S_n(f)\|_w = \|M_n^n(f)\|_w \leq \left(1+\frac{C}{n}\right)^n \|f\|_w \leq e^C\, \|f\|_w\ ,$$

and hence the sequence $(S_n)_{n\geq 1}$ is equibounded with respect to the weighted uniform norm in $C_w^{(b)}(\mathbb{R}^2)$; moreover, it is also a sequence of positive contractions on $C^{(b)}(\mathbb{R}^2)$ and $C_0(\mathbb{R}^2)$.

We have the following preliminary properties.

Lemma 3.5.1 *The following equalities hold, for every* $n \in \mathbb{N}$,

1) $S_n(\mathbf{1}) = \mathbf{1}$.

2) $S_n(\mathrm{pr}_i) = \mathrm{pr}_i$, $i = 1, 2$.

3) *For every* $(x, y) \in \mathbb{R}$, *we have*

$$\begin{aligned} S_n(\mathrm{pr}_1^2)(x,y) &= x^2 + \frac{1}{3n^2}\sum_{k=0}^{n-1} M_n^k(a^2\,\cos^2\theta + b^2\,\sin^2\theta)(x,y)\ , \\ S_n(\mathrm{pr}_2^2)(x,y) &= y^2 + \frac{1}{3n^2}\sum_{k=0}^{n-1} M_n^k(a^2\,\sin^2\theta + b^2\,\cos^2\theta)(x,y)\ , \\ S_n(\mathrm{pr}_1\,\mathrm{pr}_2)(x,y) &= x\,y + \frac{1}{3n^2}\sum_{k=0}^{n-1} M_n^k\left((a^2-b^2)\,\cos\theta\,\sin\theta\right)(x,y)\ . \end{aligned}$$

PROOF. It is obvious that $M_n\mathbf{1} = \mathbf{1}$ and hence property 1) is true. Moreover, from (3.5.4) we easily obtain $M_n\mathrm{pr}_i = \mathrm{pr}_i$, $i = 1, 2$, and this yields property 2). Finally a straightforward calculation based on (3.5.4) gives

$$\begin{aligned} M_n(\mathrm{pr}_1^2)(x,y) &= x^2 + \frac{1}{3n^2}\,(a(x,y)^2\,\cos^2\theta(x,y) \\ &\quad +b(x,y)^2\,\sin^2\theta(x,y))\ . \\ M_n(\mathrm{pr}_2^2)(x,y) &= y^2 + \frac{1}{3n^2}\,(a(x,y)^2\,\sin^2\theta(x,y) \\ &\quad +b(x,y)^2\,\cos^2\theta(x,y))\ . \\ M_n(\mathrm{pr}_1\,\mathrm{pr}_2)(x,y) &= x\,y + \frac{1}{3n^2}\,((a(x,y)^2 - b(x,y)^2)\,\times \\ &\quad \times\ \cos\theta(x,y)\,\sin\theta(x,y)) \end{aligned} \tag{3.5.11}$$

and an induction argument on the integer $n \geq 1$ yields property 3). □

The convergence of the sequence $(S_n)_{n\geq 1}$ of Steklov-type operators will be obtained studying the behavior of these operators on the subspace

$$C_w^{2,(b)}(\mathbb{R}^2) := \left\{ f \in C_w^{(b)}(\mathbb{R}^2) \cap C^2(\mathbb{R}^2) \mid \frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial y^2}, \frac{\partial^2 f}{\partial x\,\partial y} \in C^{(b)}(\mathbb{R}^2) \right\} \tag{3.5.12}$$

of all functions in $C_w^{(b)}(\mathbb{R}^2)$ with bounded second-order partial derivatives.

Proposition 3.5.2 *For every $f \in C_w^{2,(b)}(\mathbb{R}^2)$ there exists a constant $C_f \geq 0$, depending on f, such that*

$$\|M_n f - f\|_w \leq C_f \frac{1}{n^2} \,. \tag{3.5.13}$$

PROOF. For every $x, y, s, t \in \mathbb{R}$, there exist $\xi, \eta \in \mathbb{R}$ such that

$$\begin{aligned} f(s,t) - f(x,y) &= \frac{\partial f}{\partial x}(x,y)(s-x) + \frac{\partial f}{\partial y}(t-y) + \frac{\partial^2 f}{\partial x^2}(\xi,\eta)\frac{(s-x)^2}{2} \\ &\quad + \frac{\partial^2 f}{\partial y^2}(\xi,\eta)\frac{(t-y)^2}{2} + \frac{\partial^2 f}{\partial x \partial y}(\xi,\eta)(s-x)(t-y) \end{aligned}$$

and consequently

$$\begin{aligned} M_n f(x,y) - f(x,y) &= \frac{\partial f}{\partial x}(x,y) M_n(\mathrm{pr}_1 - x)(x,y) \\ &+ \frac{\partial f}{\partial y}(x,y) M_n(\mathrm{pr}_2 - y)(x,y) + M_n\left(\frac{\partial^2 f}{\partial x^2}(\xi,\eta)\frac{(\mathrm{pr}_1 - x)^2}{2}\right)(x,y) \\ &+ M_n\left(\frac{\partial^2 f}{\partial y^2}(\xi,\eta)\frac{(\mathrm{pr}_2 - y)^2}{2}\right)(x,y) \\ &+ M_n\left(\frac{\partial^2 f}{\partial x \partial y}(\xi,\eta)(\mathrm{pr}_1 - x)(\mathrm{pr}_2 - y)\right)(x,y) \,. \end{aligned}$$

Let $C := \max\{\|\partial^2 f/\partial x^2\|, \|\partial^2 f/y^2\|, 2\|\partial^2 f/\partial x \partial y\|\}$; then

$$\begin{aligned} |M_n f(x,y) - f(x,y)| &\leq \frac{C}{2}\left(M_n(\mathrm{pr}_1 - x)^2 + M_n(\mathrm{pr}_2 - y)^2\right. \\ &\quad \left. + M_n(\mathrm{pr}_1 - x)(\mathrm{pr}_2 - y)\right) \\ &= \frac{C}{6n^2}\left(a^2 + b^2 + (a^2 - b^2)\sin(2\theta)\right)(x,y) \,. \end{aligned}$$

Multiplying by $w(x,y)$ and taking the maximum, from (3.5.1) we obtain the desired estimate. □

At this point, we introduce the function $w_n : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$\omega_n(x,y) := w(x,y)\, M_n\left(\frac{1}{w}\right)(x,y)$$

(see (3.5.5)). Observe that w_n is bounded since $1/w \in C_w^{(b)}(\mathbb{R}^2)$ and consequently $M_n(1/w) \in C_w^{(b)}(\mathbb{R}^2)$ too.

Proposition 3.5.3 *For every $f \in C_w^{(b)}(\mathbb{R}^2)$ and $k \geq 1$,*

$$\|M_n^k(f) - f\|_w \leq \|M_n(f) - f\|_w \sum_{i=0}^{k-1} \|\omega_n\|^i \,. \tag{3.5.14}$$

Proof. We argue by induction on the integer $k \geq 1$. If $k = 1$ then (3.5.14) is obviously true. Now, assume that (3.5.14) holds for $k \geq 1$. We have

$$\begin{aligned}
&|w(x,y)(M_n^{k+1}f(x,y) - f(x,y))| \\
&\quad \leq |w(x,y)M_n(M_n^k f - f)(x,y)| + |w(x,y)(M_n f(x,y) - f(x,y))| \\
&\quad \leq \omega_n(x,y)\|M_n^k(f) - f\|_w + \|M_n(f) - f\|_w \\
&\quad \leq \omega_n(x,y)\sum_{i=0}^{k-1}\omega_n(x,y)^i\,\|M_n(f) - f\|_w + \|M_n(f) - f\|_w \\
&\quad \leq \sum_{i=1}^{k}\omega_n(x,y)^i\,\|M_n(f) - f\|_w + \|M_n(f) - f\|_w \\
&\quad \leq \sum_{i=0}^{k}\omega_n(x,y)^i\,\|S_{1,b_n}(f) - f\|_w
\end{aligned}$$

and this completes the induction argument. □

In order to deduce the convergence of Steklov operators from the above proposition, we need to estimate the convergence of the sequence $(w_n)_{n\geq 1}$.

Proposition 3.5.4 *There exists a constant $C > 0$ such that*

$$\|\omega_n - \mathbf{1}\| \leq \frac{C}{n^2}\,.$$

Proof. First, we observe that

$$\begin{aligned}
|\omega_n(x) - 1| &= \left| w(x)S_{1,b_n}\left(\frac{1}{w}\right)x - w(x)\frac{1}{w(x)}\right| \\
&= \left| w(x)\left[S_{1,b}\left(\frac{1}{w}\right)x - \frac{1}{w(x)}\right]\right| \\
&\leq \left\| M_n\left(\frac{1}{w}\right) - \frac{1}{w}\right\|_w\,.
\end{aligned}$$

Since $1/w \in C_w^{(b)}(\mathbb{R}^2) \cap C^2(\mathbb{R}^2)$ and its second-order partial derivatives are bounded, we can apply Proposition 3.5.2 and obtain a constant $C > 0$ such that

$$\left\| M_n\left(\frac{1}{w}\right) - \frac{1}{w}\right\|_w \leq \frac{C}{n^2}$$

and this completes the proof. □

Now, we are in a position to state the convergence property of the sequence $(S_n)_{n\geq 1}$.

Theorem 3.5.5 *For every $f \in C_w^{2,(b)}(\mathbb{R}^2)$, we have*

$$\lim_{n\to+\infty} \|S_n(f) - f\|_w = 0 \,.$$

PROOF. Let $f \in C_w^{2,(b)}(\mathbb{R}^2)$; from Propositions 3.5.2, 3.5.3 and 3.5.4 we obtain

$$\begin{aligned} \|S_n(f) - f\|_w &\le \|M_n(f) - f\|_w \sum_{i=0}^{n-1} \|\omega_n\|^i \le \frac{C_f}{n^2} \sum_{i=0}^{n-1} \|\omega_n\|^i \\ &\le \frac{C_f}{n^2} \sum_{i=0}^{n-1} \left(1 + C\frac{1}{n^2}\right)^i = \frac{C_f}{C}\left(\left(1+\frac{C}{n^2}\right)^n - 1\right) \end{aligned}$$

and consequently $\lim_{n\to+\infty} \|S_n(f) - f\|_w = 0$. □

As a consequence of Theorem 3.5.5, we have that the sequence $(S_n(f))_{n\ge1}$ converges to f uniformly on every compact subset of $\mathbb{R}^2$ whenever $f \in C_w^{(b)}(\mathbb{R}^2)$.

In the following result, we study the uniform convergence of the sequence of Steklov operators in the space $C_0(\mathbb{R}^2)$.

We observe that from Proposition 3.5.2, it follows that if $f \in C_w^{(b)}(\mathbb{R}^2) \cap C^2(\mathbb{R}^2)$ has bounded second-order partial derivatives, we have

$$|M_n f(x,y) - f(x,y)| \le \frac{C_f}{6n^2}\left(a^2 + b^2 + (a^2 - b^2)\sin(2\theta)\right)(x,y)\,,\ (x,y) \in \mathbb{R}^2$$

and hence if a, b are bounded, we also obtain

$$\|M_n(f) - f\| \le \frac{C_f}{n^2}\left(\|a^2\| + \|b^2\|\right)\,. \tag{3.5.15}$$

Theorem 3.5.6 *Assume that $a, b \in C^{(b)}(\mathbb{R}^2)$. Then, for every $f \in C_0(\mathbb{R}^2)$, we have*

$$\lim_{n\to+\infty} \|S_n f - f\| = 0\,. \tag{3.5.16}$$

PROOF. Let $f \in C^2(\mathbb{R}^2)$ with bounded second-order partial derivatives and observe that the second member of (3.5.15) tends uniformly to 0 as $n \to +\infty$. Moreover, in this case the operators M_n are positive contractions with respect to the uniform norm, and consequently

$$\|S_n(f) - f\| = \|M_n^n(f) - f\| \le n\|M_n(f) - f\|$$

which yields $\lim_{n\to+\infty} \|S_n(f) - f\| = 0$. The general case where $f \in C_0(\mathbb{R}^2)$ follows from a density argument. □

Now we establish a Voronovskaja-type formula for the operators S_n. We need some preliminary properties of independent interest which establishes the convergence of the mean of iterates of the operators M_n.

Proposition 3.5.7 *For every $f \in C_w^{2,(b)}(\mathbb{R}^2)$, we have*

$$\lim_{n\to+\infty} \left\| \frac{1}{n} \sum_{k=0}^{n-1} M_n^k(f) - f \right\|_w = 0 \,. \tag{3.5.17}$$

Proof. Let $f \in C_w^{2,(b)}(\mathbb{R}^2)$; from Propositions 3.5.3 and 3.5.4 it follows

$$\begin{aligned} \left\| \frac{1}{n} \sum_{k=0}^{n-1} M_n^k(f) - f \right\|_w &\le \frac{1}{n} \sum_{k=0}^{n-1} \left\| M_n^k(f) - f \right\|_w \\ &\le \frac{1}{n} \left\| M_n(f) - f \right\|_w \sum_{k=0}^{n-1} \sum_{i=0}^{k-1} \|\omega_n\|^i \le C_f \frac{1}{n} \frac{1}{n^2} \sum_{k=0}^{n-1} \sum_{i=0}^{k-1} (1 + C\frac{1}{n^2})^i \\ &= \frac{C_f}{C} \left(\frac{n}{C} \left(\left(1 + \frac{C}{n^2} \right)^n - 1 \right) - 1 \right) \,, \end{aligned}$$

and taking the limit as $n \to +\infty$, we have the validity of (3.5.17). □

Remark 3.5.8 If we consider the subspace

$$C_{0,w}(\mathbb{R}^2) := \left\{ f \in C_w^{(b)}(\mathbb{R}^2) \mid w\, f \in C_0(\mathbb{R}^2) \right\} \,, \tag{3.5.18}$$

we can observe that the preceding proposition is still true for every $f \in C_{0,w}(\mathbb{R}^2)$.

Indeed, from (3.5.10), we get

$$\|M_n^k\| \le \left(1 + \frac{C}{n} \right)^k \le e^C$$

with respect to the norm in $C_w^{(b)}(\mathbb{R}^2)$ and hence the mean operators $\sum_{k=0}^{n-1} M_n^k/n$ are equibounded when acting on the space $C_w^{(b)}(\mathbb{R}^2)$. Since $C_w^{2,(b)}(\mathbb{R}^2)$ is dense in $C_{0,w}(\mathbb{R}^2)$, Proposition 3.5.7 can be applied to every $f \in C_{0,w}(\mathbb{R}^2)$. □

Finally, we can establish a Voronovskaja-type formula with respect to the weighted uniform norm.

Theorem 3.5.9 (Voronovskaja-type formula) *Assume that $a, b \in C_0(\mathbb{R}^2)$ and consider the second-order partial differential operator $A : C^2(\mathbb{R}^2) \to C(\mathbb{R}^2)$ defined by*

$$A(f) := \frac{1}{6} \left(\alpha \frac{\partial^2 f}{\partial x^2} + \beta \frac{\partial^2 f}{\partial y^2} + \gamma \frac{\partial^2 f}{\partial x \partial y} \right) \,, \qquad f \in C^2(\mathbb{R}^2) \,, \tag{3.5.19}$$

where the coefficients $\alpha, \beta, \gamma : \mathbb{R}^2 \to \mathbb{R}$ *are given by*

$$\alpha := a^2 \cos^2\theta + b^2 \sin^2\theta\ ,\ \ \beta := a^2 \sin^2\theta + b^2 \cos^2\theta\ ,\ \ \gamma := (a^2 - b^2)\sin(2\theta)\ .$$

Then, for every $f \in C_w^{2,(b)}(\mathbb{R}^2)$ *with uniformly continuous second-order partial derivatives, we have*

$$\lim_{n\to\infty} \|n\,(S_n f - f) - A(f)\|_w = 0\ .$$

PROOF. Let $f \in C_w^{2,(b)}(\mathbb{R}^2)$ have uniformly continuous second-order partial derivatives. For every $x, y, s, t \in \mathbb{R}$, we can write

$$\begin{aligned}
f(s,t) - f(x,y) &= \frac{\partial f}{\partial x}(x,y)(s-x) + \frac{\partial f}{\partial y}(t-y)\\
&+\frac{\partial^2 f}{\partial x^2}(x,y)\frac{(s-x)^2}{2} + \frac{\partial^2 f}{\partial y^2}(x,y)\frac{(t-y)^2}{2} + \frac{\partial^2 f}{\partial x \partial y}(s-x)(t-y)\\
&+\eta(s,t,x,y)\left((s-x)^2 + (t-y)^2\right)
\end{aligned}$$

where $\eta : \mathbb{R}^4 \to \mathbb{R}$ satisfies $\lim_{(s,t)\to(x,y)} \eta(s,t,x,y) = 0$ uniformly with respect to $(x,y) \in \mathbb{R}^2$. Then

$$\begin{aligned}
n\,(S_n f(x,y) - f(x,y)) &= n\,S_n(f - f(x,y))(x,y) \qquad (3.5.20)\\
&= n\,\frac{\partial f}{\partial x}(x,y) S_n(\mathrm{pr}_1 - x)(x,y) + n\,\frac{\partial f}{\partial y}(x,y) S_n(\mathrm{pr}_1 - y)(x,y)\\
&+\frac{n}{2}\,\frac{\partial^2 f}{\partial x^2}(x,y) S_n((\mathrm{pr}_1 - x)^2)(x,y)\\
&+\frac{n}{2}\,\frac{\partial^2 f}{\partial y^2}(x,y) S_n((\mathrm{pr}_2 - y)^2)(x,y)\\
&+n\,\frac{\partial^2 f}{\partial x \partial y} S_n((\mathrm{pr}_1 - x)(\mathrm{pr}_2 - y))(x,y)\\
&+n\,S_n\left(\eta(\mathrm{pr}_1, \mathrm{pr}_2, x, y)\left((\mathrm{pr}_1 - x)^2 + (\mathrm{pr}_2 - y)^2\right)\right)(x,y)\ .
\end{aligned}$$

We observe that the first two addends in (3.5.20) vanishes.

Moreover, we have

$$\begin{aligned}
\frac{n}{2}\,S_n\left((\mathrm{pr}_1 - x)^2\right)(x,y) &= n\left(S_n(\mathrm{pr}_1^2)(x,y) - 2x S_n(\mathrm{pr}_1)(x,y) + x^2\right)\\
&= \frac{1}{6n}\sum_{k=0}^{n-1} M_n^k(a^2 \cos^2\theta + b^2 \sin^2\theta)(x,y)\\
&= \frac{1}{6n}\sum_{k=0}^{n-1} M_n^k(\alpha)(x,y)\ ,
\end{aligned}$$

$$\begin{aligned}\frac{n}{2}\,S_n\left((\mathrm{pr}_2-y)^2\right)(x,y) &= n\left(S_n(\mathrm{pr}_2^2)(x,y)-2yS_n(\mathrm{pr}_2)(x,y)+y^2\right)\\ &= \frac{1}{6n}\sum_{k=0}^{n-1} M_n^k(a^2\,\sin^2\theta+b^2\,\cos^2\theta)(x,y)\\ &= \frac{1}{6n}\sum_{k=0}^{n-1} M_n^k(\beta)(x,y)\ ,\end{aligned}$$

$$\begin{aligned} & n\,S_n((\mathrm{pr}_1-x)(\mathrm{pr}_2-y))(x,y)\\ &\quad = n(S_n(\mathrm{pr}_1\mathrm{pr}_2)(x,y)-yS_n(\mathrm{pr}_1)-xS_n(\mathrm{pr}_2)(x,y)+xy)\\ &\quad = \frac{1}{6n}\sum_{k=0}^{n-1} M_n^k\left((a^2-b^2)\,\sin(2\theta)\right)(x,y)\\ &\quad = \frac{1}{6n}\sum_{k=0}^{n-1} M_n^k\,(\gamma)\,(x,y)\ .\end{aligned}$$

Our assumptions on the functions a and b ensure that all the functions α, β and γ are in $C_{0,w}(\mathbb{R}^2)$ and hence from Proposition 3.5.7 (see also Remark 3.5.8), it follows

$$\begin{aligned}\lim_{n\to+\infty}\frac{n}{2}\,S_n\left((\mathrm{pr}_1-x)^2\right)(x,y) &= \frac{1}{6}\,\alpha(x,y)\\ \lim_{n\to+\infty}\frac{n}{2}\,S_n\left((\mathrm{pr}_2-y)^2\right)(x,y) &= \frac{1}{6}\,\beta(x,y)\\ \lim_{n\to+\infty} nS_n((\mathrm{pr}_1-x)(\mathrm{pr}_2-y))(x,y) &= \frac{1}{6}\,\gamma(x,y)\end{aligned}$$

uniformly with respect to the weighted uniform norm in $C_w^{(b)}(\mathbb{R}^2)$.

Finally, we have only to show that the last addend in (3.5.20) converges to 0 with respect to the weighted uniform norm. To this end, let $\varepsilon > 0$ and consider $\delta > 0$ such that $|\eta(s,t,x,y)| \le \varepsilon$ whenever $(s-x)^2+(t-y)^2 \le \delta^2$. Moreover, take $M > 0$ such that $|\eta(s,t,x,y)| \le M$ for every $(x,y),(s,t) \in \mathbb{R}^2$. We have

$$|\eta(s,t,x,y)|\,((s-x)^2+(t-y)^2) \le \varepsilon((s-x)^2+(t-y)^2)$$

if $(s-x)^2+(t-y)^2 \le \delta^2$ and

$$|\eta(s,t,x,y)|\,((s-x)^2+(t-y)^2) \le \frac{M}{\delta^2}\,((s-x)^2+(t-y)^2)^2\ ,$$

whenever $(s-x)^2+(t-y)^2 > \delta^2$. In any case

$$|\eta(s,t,x,y)|\,((s-x)^2+(t-y)^2) \le \varepsilon((s-x)^2+(t-y)^2)+\frac{M}{\delta^2}((s-x)^2+(t-y)^2)^2$$

and hence

$$\begin{aligned}
&\left|n\,S_n\left(\eta(\mathrm{pr}_1,\mathrm{pr}_2,x,y)\left((\mathrm{pr}_1-x)^2+(\mathrm{pr}_2-y)^2\right)\right)(x,y)\right|\\
&\quad\le n\,S_n\left(|\eta(s,t,x,y)|\left((\mathrm{pr}_1-x)^2+(\mathrm{pr}_2-y)^2\right)\right)(x,y)\\
&\quad\le n\,S_n\left(\varepsilon((\mathrm{pr}_1-x)^2+(\mathrm{pr}_2-y)^2)\right)(x,y)\\
&\qquad\quad+n\,S_n\left(\frac{M}{\delta^2}((\mathrm{pr}_1-x)^2+(\mathrm{pr}_2-y)^2)^2\right)(x,y)\\
&\quad\le \varepsilon n\,S_n\left((\mathrm{pr}_1-x)^2+(\mathrm{pr}_2-y)^2\right)(x,y)\\
&\qquad\quad+\frac{M}{\delta^2}n\,S_n\left(\left((\mathrm{pr}_1-x)^2+(\mathrm{pr}_2-y)^2\right)^2\right)(x,y)\,.
\end{aligned}$$

Observe that

$$\lim_{n\to+\infty} n\,S_n((\mathrm{pr}_1-x)^2+(\mathrm{pr}_2-y)^2)(x,y)=\frac{1}{3}(a^2(x,y)+b^2(x,y))$$

uniformly with respect to the weighted norm and hence, from the arbitrarily of ε, it remains only to show that

$$\lim_{n\to+\infty}\left\|n\,S_n\left(\left((\mathrm{pr}_1-x)^2+(\mathrm{pr}_2-y)^2\right)^2\right)\right\|_w=0\,. \tag{3.5.21}$$

Indeed, a straightforward calculation yields

$$\begin{aligned}
&M_n\left(\left((\mathrm{pr}_1-x)^2+(\mathrm{pr}_2-y)^2\right)^2\right)=((\mathrm{pr}_1-x)^2+(\mathrm{pr}_2-y)^2)^2\\
&\quad+\frac{1}{n^4}\left(\frac{1}{5}\,a^4+\frac{2}{9}\,a^2\,b^2+\frac{1}{5}\,b^4\right)+\frac{2}{n^2}\,\alpha\left((\mathrm{pr}_1-x)^2+\frac{1}{3}(\mathrm{pr}_2-y)^2\right)\\
&\quad+\frac{2}{n^2}\,\beta\left(\frac{1}{3}(\mathrm{pr}_1-x)^2+(\mathrm{pr}_2-y)^2\right)+\frac{4}{3n^2}\,\gamma\,(\mathrm{pr}_1-x)(\mathrm{pr}_2-y)\,,
\end{aligned}$$

and taking n iterations of the above formula, we have

$$\begin{aligned}
&n\,S_n\left(\left((\mathrm{pr}_1-x)^2+(\mathrm{pr}_2-y)^2\right)^2\right)=((\mathrm{pr}_1-x)^2+(\mathrm{pr}_2-y)^2)^2\\
&\quad+\frac{1}{n^2}\left(\frac{1}{n}\sum_{k=0}^{n-1}M_n^k\left(\frac{1}{5}\,a^4+\frac{2}{9}\,a^2\,b^2+\frac{1}{5}\,b^4\right)\right)\\
&\quad+\frac{1}{n}\sum_{k=0}^{n-1}M_n^k\left(2\alpha\left((\mathrm{pr}_1-x)^2+\frac{1}{3}(\mathrm{pr}_2-y)^2\right)\right)\\
&\quad+\frac{1}{n}\sum_{k=0}^{n-1}M_n^k\left(2\beta\left(\frac{1}{3}(\mathrm{pr}_1-x)^2+(\mathrm{pr}_2-y)^2\right)\right)\\
&\quad+\frac{1}{n}\sum_{k=0}^{n-1}M_n^k\left(\frac{4}{3}\gamma(\mathrm{pr}_1-x)(\mathrm{pr}_2-y)\right)\,.
\end{aligned}$$

Now, we discuss the convergence in $C_w^{(b)}(\mathbb{R}^2)$ of the preceding addends evaluated at (x,y). The first addend $((\mathrm{pr}_1 - x)^2 + (\mathrm{pr}_2 - y)^2)^2$ vanishes identically at (x,y). As regards to the second addend, we observe that the assumptions on a and b ensure that $a^4, b^4, a^2b^2 \in C_{0,w}(\mathbb{R}^2)$ and hence, from Proposition 3.5.7 and Remark 3.5.8, we have

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{k=0}^{n-1} M_n^k \left(\frac{1}{5}\, a^4 + \frac{2}{9}\, a^2\, b^2 + \frac{1}{5}\, b^4 \right) = \frac{1}{5}\, a^4 + \frac{2}{9}\, a^2\, b^2 + \frac{1}{5}\, b^4$$

for the weighted uniform norm. Therefore the second addend converges to 0 uniformly with respect to the weighted norm due to the factor $1/n^2$. Finally, the same argument can be applied to the last three addends and we find that they converge in $C_w^{(b)}(\mathbb{R}^2)$ respectively to the functions

$$2\alpha \left((\mathrm{pr}_1 - x)^2 + \frac{1}{3}(\mathrm{pr}_2 - y)^2 \right) , \qquad 2\beta \left(\frac{1}{3}(\mathrm{pr}_1 - x)^2 + (\mathrm{pr}_2 - y)^2 \right) ,$$

and

$$\frac{4}{3}\, \gamma\, (\mathrm{pr}_1 - x)(\mathrm{pr}_2 - y)$$

which vanish identically at (x,y).

Hence (3.5.21) has been established and the proof is complete. □

Under additional assumptions on the functions a and b, we can state the Voronovskaja-type formula in the space $C_0(\mathbb{R}^2)$ with respect to the uniform norm.

We begin with the analogous of Proposition 3.5.7 in the space $C_0(\mathbb{R}^2)$.

Proposition 3.5.10 *For every $f \in C^2(\mathbb{R}^2)$ having bounded second-order partial derivatives, we have*

$$\lim_{n\to+\infty} \left\| \frac{1}{n} \sum_{k=0}^{n-1} M_n^k(f) - f \right\| = 0\ . \tag{3.5.22}$$

Proof. Indeed, since the operators M_n are positive contractions, from (3.5.15) we have

$$\begin{aligned} \left\| \frac{1}{n} \sum_{k=0}^{n-1} M_n^k(f) - f \right\| &\le \frac{1}{n} \sum_{k=0}^{n-1} \left\| M_n^k(f) - f \right\| \\ &\le \frac{1}{n}\, \|M_n(f) - f\| \sum_{k=0}^{n-1} k \le \frac{1}{n}\, \frac{C_f(\|a^2\| + \|b^2\|)}{n^2}\, \frac{n(n-1)}{2}\ . \end{aligned}$$

Taking the limit as $n \to +\infty$, we have the validity of (3.5.22). □

Theorem 3.5.11 (Voronovskaja-type formula in $C_0(\mathbb{R}^2)$) *Assume that*

$$a(\mathbf{1} + \mathrm{pr}_1 + \mathrm{pr}_2) \in C_0(\mathbb{R}^2)\ , \qquad b(\mathbf{1} + \mathrm{pr}_1 + \mathrm{pr}_2) \in C_0(\mathbb{R}^2)$$

and consider the differential operator A defined by (3.5.19)*.*

Then, for every $f \in C_0^2(\mathbb{R}^2)$, we have

$$\lim_{n\to\infty} \|n\,(S_n f - f) - A(f)\| = 0\ .$$

PROOF. We observe that Proposition 3.5.10 continues to hold for every function in $C_0(\mathbb{R}^2)$ by a density argument, and that our assumptions ensure that α, β and γ and all the functions involved in the remainder estimates are in $C_0(\mathbb{R}^2)$. Using this remark in place of Proposition 3.5.7, we can proceed exactly as in the proof of Theorem 3.5.9. □