Chapter 2

Applications to classical sequences of operators

2.1 Infinite-dimensional setting

2.1.1 Application to Schnabl-type operators

In the preceding chapter we have stated a general result concerning with the approximation of a C_0-semigroup in an abstract setting (Theorem 1.1.2). The general setting is motivated by some recent applications in population genetics involving Bernstein-Schnabl operators in an infinite-dimensional setting (see, e.g., [2]). Moreover starting with this sequence of operators, other sequences such as Stancu and Lototsky operators were considered in the same setting.

In this section, we consider all these operators and in each case we study the consequences of the quantitative estimates in the preceding chapter.

We start with the Bernstein-Schnabl operators and we state the quantitative estimates of the convergence of their iterates to the associated semigroup. These operators have been introduced and studied in [2, 24] and a unified treatment of these operators can be found in [9, Chapter 6] together with supplementary references.

The result in this section are collected in [37].

Consider a metrizable convex compact subset K of some locally convex space and let $T : C(K) \to C(K)$ be a positive projection on $C(K)$ such that its range $H := T(C(K))$ contains the subspace $A(K)$ of $C(K)$ consisting of all affine continuous real functions on K and is invariant under convex translation, in the sense that the function $x \mapsto h(tx + (1-t)z)$ belongs to H whenever $h \in H$, $t \in [0,1]$ and $z \in K$.

Now, for every $x \in K$ consider the probability Radon measure $\mu_x^T \in \mathcal{M}^+(K)$ defined by $\mu_x^T(f) = Tf(x)$ for every $f \in C(K)$.

For every $n \geq 1$, the n-th Bernstein-Schnabl operator $B_n : C(K) \to$

$C(K)$ associated with the projection T is defined by setting, for every $f \in C(K)$ and $x \in K$,

$$B_n f(x) := \int_K \cdots \int_K f\left(\frac{x_1 + \cdots + x_n}{n}\right) d\mu_x^T(x_1) \ldots d\mu_x^T(x_n) \,. \qquad (2.1.1)$$

Lototsky-Schnabl operators are defined by considering a strictly positive function $\gamma \in C(K)$ with values in the interval $]0,1]$ and by substituting the measures μ_x^T with the probability Radon measures $\nu_x^T := \gamma(x)\,\mu_x^T + (1 - \gamma(x))\,\varepsilon_x \in \mathcal{M}^+(K)$, where ε_x denotes the Dirac measure at $x \in K$. Hence, for every $n \geq 1$, the n-th Lototsky-Schnabl operator $L_{n,\gamma} : C(K) \to C(K)$ is defined by

$$L_{n,\gamma} f(x) := \int_K \cdots \int_K f\left(\frac{x_1 + \cdots + x_n}{n}\right) d\nu_x^T(x_1) \ldots d\nu_x^T(x_n) \qquad (2.1.2)$$

for every $f \in C(K)$ and $x \in K$.

Finally, in order to introduce the Stancu-Schnabl operators, we first define the polynomial

$$p_n(a) := \prod_{j=0}^{n-1} (1 + j\,a) \,, \qquad a \in \mathbb{R} \,;$$

moreover, we use the convention to write $|v|_k = n$ for $v = (v_1, \ldots, v_k) \in \mathbb{N}^k$ satisfying $v_1, \ldots, v_k \geq 1$ and $\sum_{i=1}^k v_i = n$.

Now, we fix a sequence $(a_n)_{n\geq 1}$ of positive functions in $C(K)$ such that $(n\,a_n)_{n\geq 1}$ uniformly converges to $b \in C(K)$; as observed in [24] the result concerning a sequence $(a_n)_{n\in\mathbb{N}}$ of real numbers in [22] and [9] remain unchanged in the case where $(a_n)_{n\in\mathbb{N}}$ is a sequence of real continuous functions.

The n-th Stancu-Schnabl operator $S_{n,a_n} : C(K) \to C(K)$ is defined by setting, for every $f \in C(K)$ and $x \in K$,

$$\begin{aligned} S_{n,a_n} f(x) \quad := \quad & \frac{1}{p_n(a_n(x))} \sum_{k=1}^{n} \frac{n!}{k!}\, a_n^{n-k}(x) \sum_{|v|_k = n} \frac{1}{v_1 \cdots v_k} \qquad (2.1.3) \\ & \times \int_K \cdots \int_K f\left(\frac{v_1\,x_1 + \cdots + v_k\,x_k}{n}\right) d\mu_x^T(x_1) \ldots d\mu_x^T(x_k) \,. \end{aligned}$$

Observe that the Bernstein-Schnabl operators can be obtained as a particular case of both Lototsky-Schnabl operators taking $\gamma = 1$ and of Stancu-Schnabl operators taking $a_n = 0$ for every $n \geq 1$.

Now, denote by $A_\infty(K)$ the subalgebra of $C(K)$ consisting of all functions in $C(K)$ which are finite products of elements of $A(K)$ and define the operator $L_T : A_\infty(K) \to C(K)$ by setting, for every $f = h_1 \cdots h_m \in A_\infty(K)$,

$$L_T(h_1 \cdots h_m) := \begin{cases} 0\,, & m = 1\,, \\ T(h_1\,h_2) - h_1\,h_2\,, & m = 2\,, \\ \displaystyle\sum_{1 \le i < j \le m} (T(h_i\,h_j) - h_i\,h_j) \prod_{\substack{r=1 \\ r \ne i,j}}^{m} h_r\,, & m \ge 3\,. \end{cases} \tag{2.1.4}$$

Observe that $A_\infty(K)$ is dense in $C(K)$ by the Stone-Weierstrass theorem.

Moreover, if K is a compact convex subset of $\mathbb{R}^d$ then $A_\infty(K) \subset C^2(K)$ and for every $f \in A_\infty(K)$ we have (see [9, Theorem 6.2.5, p. 433])

$$L_T(f) = \frac{1}{2} \sum_{i,j=1}^{d} a_{ij} \frac{\partial^2 f}{\partial x_i\,\partial x_j}\,,$$

where $a_{ij}(x) := T((\mathrm{pr}_i - x_i)(\mathrm{pr}_j - x_j))(x) = T(\mathrm{pr}_i\,\mathrm{pr}_j)(x) - x_i\,x_j$ and pr_i denotes the canonical i-th projection.

In order to apply the results in Chapter 1, we recall that

$$\|L_{n,\gamma}\| \le 1\,, \quad \|S_{n,a_n}\| \le 1\,, \qquad n \ge 1\,;$$

moreover, for every $f \in A_\infty(K)$, from [9, Section 6.2, pp. 427–429] we easily obtain

$$\|n(L_{n,\gamma} f - f) - \gamma\,L_T(f)\| \le \frac{1}{n} \sum_{i \in I} \|L_i(f)\| \tag{2.1.5}$$

$$\|n(L_{n,\gamma} f - f)\| \le \|\gamma\,L_T(f)\| + \frac{1}{n} \sum_{i \in I} \|L_i(f)\|\,, \tag{2.1.6}$$

and further

$$\|n(S_{n,a_n} f - f) - (1+b)\,L_T(f)\| \le \left\| \frac{n a_n - b}{1 + a_n} \right\| \|L_T(f)\| + \frac{1}{n} \sum_{i \in I} \|L_i(f)\|\,, \tag{2.1.7}$$

$$\|n(S_{n,a_n} f - f)\| \le \left\| \frac{1 + n a_n}{1 + a_n} \right\| \|L_T(f)\| + \frac{1}{n} \sum_{i \in I} \|L_i(f)\|\,, \tag{2.1.8}$$

where I is a set of indices and, for every $i \in I$, $L_i : A_\infty(K) \to C(K)$ is a linear map such that, for every $f = h_1 \cdots h_m \in A_\infty(K)$, $L_i(h_1 \cdots h_m)$ belongs to the linear subspace generated by

$$\{h_1 \cdots h_m, T(h_1 h_2) h_3 \cdots h_m, \ldots, T(h_1 h_2 h_3) h_4 \cdots h_m, \ldots, T(h_1 \cdots h_m)\}$$

and is different from 0 only for a finite set of indices.

We have the following result.

Theorem 2.1.1 *Assume that $T(h_1 h_2) \in A(K)$ for every $h_1, h_2 \in A(K)$. Then*

1) **Lototsky operators**
The closure of the operator $(\gamma\, L_T, A_\infty(K))$ *generates a* C_0*-semigroup* $(T_\gamma(t))_{t\geq 0}$ *of positive contractions on* $C(K)$ *and, for every* $t \geq 0$, $(k(n))_{n\geq 1}$ *sequence of positive integers and* $f \in A_\infty(K)$, *we have*

$$\|T_\gamma(t)f - L_{n,\gamma}^{k(n)} f\| \leq \frac{t}{n} \sum_{i\in I} \|L_i(f)\| + \left(\left| \frac{k(n)}{n} - t \right| + \sqrt{\frac{2}{\pi}}\, \frac{\sqrt{k(n)}}{n} \right) \left(\|\gamma\, L_T(f)\| + \frac{1}{n} \sum_{i\in I} \|L_i(f)\| \right) \tag{2.1.9}$$

and in particular, taking $k(n) = [nt]$

$$\|T_\gamma(t)f - L_{n,\gamma}^{[nt]} f\| \leq \frac{t}{n} \sum_{i\in I} \|L_i(f)\| + \frac{1}{\sqrt{n}} \left(\frac{1}{\sqrt{n}} + \sqrt{\frac{2t}{\pi}} \right) \left(\|\gamma\, L_T(f)\| + \frac{1}{n} \sum_{i\in I} \|L_i(f)\| \right) . \tag{2.1.10}$$

Moreover, for every $\lambda \in \mathbb{C}$ *such that* $\operatorname{Re}\lambda > 0$ *and* $n \geq 1$, *consider the operator* $L_{\lambda,n,\gamma} : C(K) \to C(K)$ *defined by*

$$L_{\lambda,n,\gamma} f := \int_0^{+\infty} e^{-\lambda\, t} L_{n,\gamma}^{[n\,t]} f \, dt \,, \qquad f \in C(K)$$

and let $R(\lambda, \gamma\, L_T)$ *be the resolvent operator of the closure of* $(\gamma\, L_T, A_\infty(K))$. *Then, for every* $n \geq 1$ *and* $f \in A_\infty(K)$ *we have*

$$\|R(\lambda, \gamma\, L_T)f - L_{\lambda,n,\gamma} f\| \leq \frac{1}{n(\operatorname{Re}\lambda)^2} \sum_{i\in I} \|L_i(f)\| + \frac{1}{\sqrt{n}\operatorname{Re}\lambda} \left(\frac{1}{\sqrt{n}} + \frac{1}{\sqrt{2\operatorname{Re}\lambda}} \right) \left(\|\gamma\, L_T(f)\| + \frac{1}{n} \sum_{i\in I} \|L_i(f)\| \right) . \tag{2.1.11}$$

2) **Stancu operators**
The closure of the operator $((1{+}b)\, L_T, A_\infty(K))$ *generates a* C_0*-semigroup* $(T_{1+b}(t))_{t\geq 0}$ *of positive contractions on* $C(K)$ *and, for every* $t \geq 0$, $(k(n))_{n\geq 1}$ *sequence of positive integers and* $f \in A_\infty(K)$, *we have*

$$\begin{aligned} \|T_{1+b}(t)f - S_{n,a_n}^{k(n)} f\| &\leq t \left(\left\| \frac{na_n - b}{1+a_n} \right\| \|L_T(f)\| + \frac{1}{n} \sum_{i\in I} \|L_i(f)\| \right) \\ &+ \left(\left| \frac{k(n)}{n} - t \right| + \sqrt{\frac{2}{\pi}}\, \frac{\sqrt{k(n)}}{n} \right) \\ &\times \left(\left\| \frac{1+na_n}{1+a_n} \right\| \|L_T(f)\| + \frac{1}{n} \sum_{i\in I} \|L_i(f)\| \right) \end{aligned} \tag{2.1.12}$$

and in particular, taking $k(n) = [nt]$,

$$\|T_{1+b}(t)f - S_{n,a_n}^{[nt]} f\| \leq t \left(\left\| \frac{na_n - b}{1 + a_n} \right\| \|L_T(f)\| + \frac{1}{n} \sum_{i \in I} \|L_i(f)\| \right)$$
$$+ \frac{1}{\sqrt{n}} \left(\frac{1}{\sqrt{n}} + \sqrt{\frac{2t}{\pi}} \right)$$
$$\times \left(\left\| \frac{1 + na_n}{1 + a_n} \right\| \|L_T(f)\| + \frac{1}{n} \sum_{i \in I} \|L_i(f)\| \right) . \tag{2.1.13}$$

Moreover, for every $\lambda \in \mathbb{C}$ *such that* $\operatorname{Re} \lambda > 0$ *and* $n \geq 1$*, consider the operator* $S_{\lambda,n,a_n} : C(K) \to C(K)$ *defined by*

$$S_{\lambda,n,a_n} f := \int_0^{+\infty} e^{-\lambda t} S_{n,a_n}^{[n\,t]} f \, dt \; , \qquad f \in C(K) \; .$$

If we denote by $R(\lambda, (1 + b)\, L_T)$ *the resolvent operator of the closure of* $((1 + b)\, L_T, A_\infty(K))$*, for every* $n \geq 1$ *and* $f \in A_\infty(K)$ *we have*

$$\|R(\lambda, (1 + b)\, L_T)f - S_{\lambda,n,a_n} f\| \tag{2.1.14}$$
$$\leq \frac{1}{(\operatorname{Re} \lambda)^2} \left(\left\| \frac{na_n - b}{1 + a_n} \right\| \|L_T(f)\| + \frac{1}{n} \sum_{i \in I} \|L_i(f)\| \right)$$
$$+ \frac{1}{\sqrt{n} \operatorname{Re} \lambda} \left(\frac{1}{\sqrt{n}} + \frac{1}{\sqrt{2 \operatorname{Re} \lambda}} \right) \tag{2.1.15}$$
$$\times \left(\left\| \frac{1 + na_n}{1 + a_n} \right\| \|L_T(f)\| + \frac{1}{n} \sum_{i \in I} \|L_i(f)\| \right) .$$

3) **Bernstein operators**
In the particular case of Bernstein-Schnabl operators the preceding estimates become

$$\|T(t)f - B_n^{k(n)} f\| \leq \frac{t}{n} \sum_{i \in I} \|L_i(f)\| \tag{2.1.16}$$
$$+ \left(\left| \frac{k(n)}{n} - t \right| + \sqrt{\frac{2}{\pi}} \frac{\sqrt{k(n)}}{n} \right) \left(\|L_T(f)\| + \frac{1}{n} \sum_{i \in I} \|L_i(f)\| \right)$$

and, taking $k(n) = [nt]$,

$$\|T(t)f - B_n^{[nt]} f\| \leq \frac{t}{n} \sum_{i \in I} \|L_i(f)\| \tag{2.1.17}$$
$$+ \frac{1}{\sqrt{n}} \left(\frac{1}{\sqrt{n}} + \sqrt{\frac{2t}{\pi}} \right) \left(\|L_T(f)\| + \frac{1}{n} \sum_{i \in I} \|L_i(f)\| \right)$$

and further

$$\|R(\lambda, L_T)f - B_{\lambda,n}f\| \leq \frac{1}{n(\operatorname{Re}\lambda)^2} \sum_{i\in I} \|L_i(f)\| \tag{2.1.18}$$
$$+\frac{1}{\sqrt{n}\operatorname{Re}\lambda}\left(\frac{1}{\sqrt{n}}+\frac{1}{\sqrt{2\operatorname{Re}\lambda}}\right)\left(\|L_T(f)\|+\frac{1}{n}\sum_{i\in I}\|L_i(f)\|\right),$$

where $(T(t))_{t\geq 0}$ *is the* C_0*-semigroup generated by the closure of* $(L_T, A_\infty(K))$, $B_{\lambda,n} : C(K) \to C(K)$ *is defined by*

$$B_{\lambda,n}f := \int_0^{+\infty} e^{-\lambda t} B_n^{[n\,t]} f\, dt\,, \qquad f \in C(K)\,,$$

and $R(\lambda, L_T)$ *is the resolvent operator of the closure of* $(L_T, A_\infty(K))$.

Proof. The existence of the C_0-semigroups generated by the closures of the operators $(\gamma\, L_T, A_\infty(K))$ and $((1 + b)\, L_T, A_\infty(K))$ is a consequence of [9, Theorem 6.2.6, p. 436]. Moreover, from (2.1.6)–(2.1.5) we can apply Theorem 1.1.2 and Theorem 1.2.1 considering the seminorms $\varphi_n : A_\infty(K) \to \mathbb{R}$ and $\psi_n : A_\infty(K) \to \mathbb{R}$ defined by

$$\varphi_n(f) := \|\gamma\, L_T(f)\|+\frac{1}{n}\sum_{i\in I}\|L_i(f)\|\,,\ \psi_n(f) := \frac{1}{n}\sum_{i\in I}\|L_i(f)\|\,, \quad f \in A_\infty(K)\,,$$

and we get (2.1.9) and (2.1.11). The particular case $k(n) = [nt]$ follows from (1.1.11).

Analogously, if we define

$$\begin{aligned}\varphi_n(f) \quad &:= \quad \left\|\frac{1+na_n}{1+a_n}\right\| \|L_T(f)\| + \frac{1}{n}\sum_{i\in I}\|L_i(f)\|\,,\\ \psi_n(f) \quad &:= \quad \left\|\frac{na_n-b}{1+a_n}\right\| \|L_T(f)\| + \frac{1}{n}\sum_{i\in I}\|L_i(f)\|\,,\end{aligned}$$

from (2.1.8)–(2.1.7), Theorem 1.1.2, Theorem 1.2.1 and (1.1.11) we get (2.1.12), (2.1.13) and (2.1.14).

Finally, the case of Bernstein-Schnabl operators is obtained taking $\gamma = 1$ in (2.1.9)–(2.1.11) (or $a_n = 0$ in (2.1.12)–(2.1.14)). □

2.2 Finite dimensional setting

2.2.1 Best order of convergence in $C^{2,\alpha}(K)$

In this section we consider a domain K of $\mathbb{R}^d$ and apply Theorem 1.1.2 and 1.2.1 to some classical sequences of linear operators connected with some second-order differential operators. In order to describe the rate of convergence in the Voronovskaja-type formula we restrict our attention to the class $C^{2,\alpha}(K)$ of twice differentiable functions with α-Hölder continuous second-order derivative, and give a general quantitative estimate in terms of the α-Hölder constant defined by

$$L_{f''} := \sup_{\substack{x,y\in K\\ x\neq y}} \frac{1}{|x-y|^\alpha} |f''(x) - f''(y)| \,. \tag{2.2.1}$$

This result can be easily applied to a wide range of linear operators, by simply evaluating them at the functions $(\mathrm{pr}_i - x_i)$, $(\mathrm{pr}_i - x_i)(\mathrm{pr}_j - x_j)$, this will determinate the coefficient of the differential operator associated with the Voronovskaja formula, and at the function $(\mathrm{pr}_i - x_i)^2(\mathrm{pr}_j - x_j)^2$ which affects the rate of convergence.

In the next theorem we consider a linear operator L on $C(K)$, and its associated differential operator

$$\begin{aligned}\mathcal{A}_L f(x) := \frac{1}{2} \sum_{i,j=1}^{d} \frac{\partial^2 f}{\partial x_i\, \partial x_j}\,(x)\, L\left((\mathrm{pr}_i - x_i)(\mathrm{pr}_j - x_j)\right)(x) \\ + \sum_{i=1}^{d} \frac{\partial f}{\partial x_i}\,(x)\, L(\mathrm{pr}_i - x_i)(x)\end{aligned} \tag{2.2.2}$$

defined for every $f \in C^2(K)$ and $x \in K$. This operator is strictly related to the differential operator associated with the Voronovskaja-type formula when we shall consider a sequence of linear operators.

Theorem 2.2.1 *Let $K \subset \mathbb{R}^d$, and $L : C(K) \to C(K)$ a linear positive operator. For every $x \in K$ denote by $\psi_x \colon K \to \mathbb{R}$ the real function defined by $\psi_x(y) := |y - x|$ for every $y \in K$. Then for every $f \in C^{2,\alpha}(K)$ we have*

$$\begin{aligned}|L(f)(x) - f(x) - \mathcal{A}_L(x)| &\le |f(x)||L\mathbf{1}(x) - 1| \\ &+ \frac{L_{f''}}{2}\left(L(\psi_x^2)(x)\right)^{\alpha/2}\left((L(\psi_x^2)(x))^2 L(\mathbf{1})(x) + L(\psi_x^4)(x)\right)^{1/2} \,.\end{aligned} \tag{2.2.3}$$

Proof. Let $f \in C^{2,\alpha}(K)$ and $x = (x_1, \ldots, x_d) \in K$. For every $y = (y_1, \ldots, y_d) \in K$, there exists $\xi(y)$ in the segment joining x and y such that

$$
\begin{aligned}
f(y) &- f(x) \\
&= \sum_{i=1}^{d} \frac{\partial f}{\partial x_i}(x)\,(y_i - x_i) + \frac{1}{2} \sum_{i,j=1}^{d} \frac{\partial^2 f}{\partial x_i\,\partial x_j}(\xi(y))\,(y_i - x_i)(y_j - x_j) \\
&= \sum_{i=1}^{d} \frac{\partial f}{\partial x_i}(x)\,(y_i - x_i) + \frac{1}{2} \sum_{i,j=1}^{d} \frac{\partial^2 f}{\partial x_i\,\partial x_j}(x)\,(y_i - x_i)(y_j - x_j) \\
&\quad + \frac{1}{2} \sum_{i,j=1}^{d} \left(\frac{\partial^2 f}{\partial x_i\,\partial x_j}(\xi(y)) - \frac{\partial^2 f}{\partial x_i\,\partial x_j}(x) \right) (y_i - x_i)(y_j - x_j)\,.
\end{aligned}
$$

Hence

$$
\begin{aligned}
f &- f(x) \cdot \mathbf{1} \\
&= \sum_{i=1}^{d} \frac{\partial f}{\partial x_i}(x)\,(\mathrm{pr}_i - x_i) + \frac{1}{2} \sum_{i,j=1}^{d} \frac{\partial^2 f}{\partial x_i\,\partial x_j}(x)\,(\mathrm{pr}_i - x_i)(\mathrm{pr}_j - x_j) \\
&+ \frac{1}{2} \sum_{i,j=1}^{d} \left(\frac{\partial^2 f}{\partial x_i\,\partial x_j} \circ \xi - \frac{\partial^2 f}{\partial x_i\,\partial x_j}(x) \right) (\mathrm{pr}_i - x_i)(\mathrm{pr}_j - x_j)\,,
\end{aligned}
$$

and evaluating L of both sides at the point x we get

$$
\begin{aligned}
L(f)(x) - f(x) + f(x) - f(x) \cdot L(\mathbf{1})(x) &= \sum_{i=1}^{d} \frac{\partial f}{\partial x_i}(x)\,L(\mathrm{pr}_i - x_i) \\
+ \frac{1}{2} \sum_{i,j=1}^{d} \frac{\partial^2 f}{\partial x_i\,\partial x_j}(x)\,L\big((\mathrm{pr}_i - x_i)(\mathrm{pr}_j - x_j)\big)(x) \\
+ \frac{1}{2} \sum_{i,j=1}^{d} L\left(\left(\frac{\partial^2 f}{\partial x_i\,\partial x_j} \circ \xi - \frac{\partial^2 f}{\partial x_i\,\partial x_j}(x) \right) (\mathrm{pr}_i - x_i)(\mathrm{pr}_j - x_j)\right)(x)\,.
\end{aligned}
$$

Taking into account that L is positive we can write

$$
\begin{aligned}
|L(f)(x) - f(x) - \mathcal{A}f(x)| &\le |f(x)||L\mathbf{1}(x) - 1| \\
+ \frac{1}{2} \sum_{i,j=1}^{d} L\left(\left(\frac{\partial^2 f}{\partial x_i\,\partial x_j} \circ \xi - \frac{\partial^2 f}{\partial x_i\,\partial x_j}(x) \right) (\mathrm{pr}_i - x_i)(\mathrm{pr}_j - x_j)\right)(x)\,.
\end{aligned}
\tag{2.2.4}
$$

Since $f \in C^{2,\alpha}(K_d)$ we can estimate the last term as follows

$$
\sum_{i,j=1}^{d} \left| \frac{\partial^2 f}{\partial x_i\,\partial x_j}(\xi(y)) - \frac{\partial^2 f}{\partial x_i\,\partial x_j}(x) \right| \le L_{f''}|y - x|^{\alpha}\,,
$$

where $L_{f''}$ is the Lipschitz constant of f given by (2.2.1). Moreover, using the inequalities $|(y_i - x_i)(y_j - x_j)| \le |y - x|^2$ from (2.2.4) we get

$$|L(f)(x) - f(x) - \mathcal{A}f(x)| \le |f(x)||L\mathbf{1}(x) - 1| + \frac{L_{f''}}{2} L\left(\psi_x^{2+\alpha}\right)(x) \, .$$

At this point using the Cauchy-Schwartz inequality (see, e.g., [9, Section 1.2, p. 21]) we obtain

$$|L(f)(x)-f(x)-\mathcal{A}f(x)| \le |f(x)||L\mathbf{1}(x)-1|+\frac{L_{f''}}{2}\sqrt{L\left(\psi_x^2\right)(x)}\sqrt{L\left(\psi_x^{2+2\alpha}\right)(x)} \, .$$

Observe that for every $\delta > 0$, we have

$$\psi_x(y)^{2+2\alpha} \le \left(\delta^2 + \frac{\psi_x(y)^4}{\delta^2}\right)\delta^{2\alpha} \; ;$$

indeed if $|y - x| \le \delta$ we obviously have $|y - x|^{2+2\alpha} \le \delta^{2+2\alpha}$ and otherwise if $|y-x| > \delta$ then $1 \le \left(\frac{|y-x|}{\delta}\right)^{2-2\alpha}$ and $|y-x|^{2+2\alpha} \le |y-x|^{2+2\alpha}\left(\frac{|y-x|}{\delta}\right)^{2-2\alpha} = \frac{|y-x|^4}{\delta^2}\delta^{2\alpha}$. Therefore

$$\begin{aligned}
|L(f)(x) - f(x) - \mathcal{A}f(x)| &\le |f(x)||L\mathbf{1}(x) - 1| \\
&\quad + \frac{L_{f''}}{2}\sqrt{L\left(\psi_x^2\right)(x)}\sqrt{\delta^{2\alpha}\left(\delta^2 L(\mathbf{1})(x) + \frac{1}{\delta^2}L(\psi_x^4)(x)\right)} \\
&= |f(x)||L\mathbf{1}(x) - 1| + \frac{L_{f''}}{2}\delta^{\alpha}\sqrt{\delta^2 L(\psi_x^2)(x)L(\mathbf{1})(x) + \frac{L(\psi_x^2)(x)}{\delta^2}L(\psi_x^4)(x)} \, ,
\end{aligned}$$

and choosing $\delta^2 = L(\psi_x^2)(x)$ we obtain

$$\begin{aligned}
|L(f)(x) - f(x) - \mathcal{A}f(x)| &\le |f(x)||L\mathbf{1}(x) - 1| \\
&\quad + \frac{L_{f''}}{2}\left(L(\psi_x^2)(x)\right)^{\alpha/2}\left((L(\psi_x^2)(x))^2 L(\mathbf{1})(x) + L(\psi_x^4)(x)\right)^{1/2} \, .
\end{aligned}$$

□

In concrete applications we have a sequence of linear operators $(L_n)_{n\in\mathbb{N}}$, and a sequence of positive real numbers $(h_n)_{n\in\mathbb{N}}$ converging to zero, such that the operator

$$\frac{\mathcal{A}_{L_n}}{h_n} := \mathcal{A}_n$$

converges to a second-order differential operator

$$Af(x) := \sum_{i,j}^{d} a_{i,j}(x)D_{i,j}f(x) + \sum_{i=1}^{d} b_i(x)D_i f(x)$$

where $a_{i,j}, b_i$ are bounded, positive, continuous functions on $\overset{\circ}{K}$. The link between the linear operators L_n and A is given by a Voronovskaja-type formula:

Theorem 2.2.2 *Let K be a set of $\mathbb{R}^d$, let $(h_n)_{n\in\mathbb{N}}$ be a sequence of positive real number converging to 0 such that for every $x \in K$ and $f \in C^2(K)$, with uniformly continuous and bounded second-order partial derivatives,*

1. $\lim_{n\to\infty} \mathcal{A}_n f(x) = Af(x)$
2. $\lim_{n\to\infty} \dfrac{L_n\mathbf{1}(x) - 1}{h_n} = 0$
3. $\lim_{n\to\infty} \dfrac{L_n(\psi_x^4)(x)}{h_n} = 0.$

Then

$$\lim_{n\to\infty} \frac{L_n f(x) - f(x)}{h_n} = Af(x)\ , \quad \textit{for every } x \in K \textit{ and } f \in C^2(K)\ . \tag{2.2.5}$$

Proof. Let $f \in C^2(K)$ with uniformly continuous and bounded second-order partial derivatives. We can still apply (2.2.4) to the operator L_n and dividing by h_n we get, for every $x \in K$,

$$\begin{aligned}
&\left|\frac{L_n f(x) - f(x)}{h_n} - Af(x)\right| \le \frac{1}{h_n}|L_n f(x) - f(x) - \mathcal{A}f(x)| + |\mathcal{A}_n f(x) - Af(x)| \\
&\quad \le |f(x)|\frac{|L_n\mathbf{1}(x) - 1|}{h_n} \\
&\qquad + \left|\frac{1}{2h_n}\sum_{i,j=1}^{d} L_n\left(\left(\frac{\partial^2 f}{\partial x_i\,\partial x_j}\circ\xi - \frac{\partial^2 f}{\partial x_i\,\partial x_j}(x)\right)(\mathrm{pr}_i - x_i)(\mathrm{pr}_j - x_j)\right)(x)\right| \\
&\qquad + |\mathcal{A}_n f(x) - Af(x)|\ .
\end{aligned}$$

Since f'' is uniformly continuous for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|t - x| \le \delta$ implies that $\left|\frac{\partial^2 f}{\partial x_i\,\partial x_j}(t) - \frac{\partial^2 f}{\partial x_i\,\partial x_j}(x)\right| \le \varepsilon$. Then

$$\begin{aligned}
&\left|\frac{\partial^2 f}{\partial x_i\,\partial x_j}(\xi(t)) - \frac{\partial^2 f}{\partial x_i\,\partial x_j}(x)\right| |(t_i - x_i)(t_j - x_j)| \\
&\qquad \le \varepsilon|(t_i - x_i)(t_j - x_j)| + 2\left\|\frac{\partial^2 f}{\partial x_i\,\partial x_j}\right\|\frac{(t-x)^4}{\delta^2}\ ;
\end{aligned}$$

indeed if $|t - x| \le \delta$ we have $|\xi(t) - x| \le |t - x|$, and

$$\left|\frac{\partial^2 f}{\partial x_i\,\partial x_j}(\xi(t)) - \frac{\partial^2 f}{\partial x_i\,\partial x_j}(x)\right| \le \varepsilon\ ; \tag{2.2.6}$$

conversely if $|t-x|>\delta$ we have that $1<\frac{|t-x|}{\delta}$ and using the inequality $|(t_i-x_i)(t_j-x_j)|\le|t-x|^2$ we have again the validity of (2.2.6). Finally

$$\begin{aligned}
&\left|\frac{L_nf(x)-f(x)}{h_n}-Af(x)\right| \\
&\quad\le|f(x)|\frac{|L_n\mathbf{1}(x)-1|}{h_n}+\varepsilon\frac{1}{2}\sum_{i,j=1}^{d}\frac{L_n((\mathrm{pr}_i-x_i)(\mathrm{pr}_j-x_j))(x)}{h_n} \\
&\quad\quad+\|D^2f\|\frac{L_n(\psi_x^4)(x)}{\delta^2h_n}+|\mathcal{A}_nf(x)-Af(x)|\ ,
\end{aligned}$$

which converges to $\varepsilon\sum_{i,j=1}^{d}a_{i,j}(x)$ as $n\to\infty$. Since ε is arbitrary the proof is complete. □

Remark 2.2.3 If the hypotheses 1., 2. and 3. in Theorem 2.2.2 hold with respect a uniform norm (or with respect a weighted uniform norm) then (2.2.5) holds uniformly (or with respect to the weighted uniform norm) as well. □

2.2.2 Application to Bernstein operators

In this section we consider the particular case of the standard simplex of $\mathbb{R}^d$

$$K_d:=\left\{x=(x_1,\dots,x_d)\in\mathbb{R}^d\mid x_1,\dots,x_d\ge0\ ,\ \sum_{k=1}^{d}x_k\le1\right\}\ ,$$

and the classical sequences of multi-dimensional Bernstein, Lototsky and Stancu operator on K_d. These operators coincide with the Bernstein-Schnabl, Lototsky-Schnabl and Stancu-Schnabl investigated in the preceding section considering the particular projection $T_d:C(K_d)\to C(K_d)$ defined by

$$T_df(x):=\sum_{i=0}^{d}x_if(\delta_{i1},\dots,\delta_{id})\ ,\quad f\in C(K_d)\ ,\ x=(x_1,\dots,x_d)\in K_d\ ,$$

which maps any continuous function f into the affine functions which interpolates f at the vertices of K_d (see [9, Section 6.3.3, p. 450]).

Observe that the projection T_d satisfies all our general assumptions and also those in Theorem 2.1.1, and therefore we already have an estimate of the convergence of the iterates of our operators to the associated semigroup on the subspace $A_\infty(K_d)$. However, here, we want to point out some additional information on a larger subspace.

The results in this section have been published in [36] in a preliminary version and are stated in [37] in the definitive version.

We consider the case of Bernstein operators, which are explicitly given by

$$B_n f(x_1,\dots,x_d) := \sum_{h_1+\dots+h_d\le n} \frac{n!}{h_0!\,h_1!\dots h_d!}\, x_0^{h_0}\, x_1^{h_1}\dots x_d^{h_d}\, f\left(\frac{h_1}{n},\dots,\frac{h_d}{n}\right)$$

for every $f\in C(K_d)$ and $(x_1,\dots,x_d)\in K_d$, where $x_0 := 1-x_1-\dots-x_d$ and $h_0 := n-h_1-\dots-h_d$.

In our situation, the operator L_T, defined by (2.1.4), coincides on $A_\infty(K_d)$ with the differential operator $A : C^2(K_d)\to C(K_d)$ defined by

$$Af(x) = \sum_{i,j=1}^{d} \frac{x_i(\delta_{ij}-x_j)}{2}\frac{\partial^2 f}{\partial x_i\,\partial x_j}(x) \tag{2.2.7}$$

whenever $f\in C^2(K_d)$ and $x=(x_1,\dots,x_d)\in K_d$.

It is well-known that the closure of $(A, C^2(K_d))$ generates a C_0-semigroup of positive contractions on $C(K_d)$ and that $C^2(K_d)$ is a core for this closure (see e.g. [9, Theorem 6.2.6, p. 436] or also [49]).

From Theorem 2.1.1 the estimates required in Theorems 1.1.2 and 1.2.1 are already available in $A_\infty(K_d)$. However our aim is to investigate the validity of similar estimates in the space $C^{2,\alpha}(K_d)$.

Taking $L = B_n$ in (2.2.2) we get the differential operator

$$\begin{aligned}\mathcal{A}_{B_n} f(x) := {}& \frac{1}{2}\sum_{i,j=1}^{d} \frac{\partial^2 f}{\partial x_i\,\partial x_j}(x)\, B_n\left((\mathrm{pr}_i - x_i)(\mathrm{pr}_j - x_j)\right)(x)\\ &+ \sum_{i=1}^{d} \frac{\partial f}{\partial x_i}(x)\, B_n(\mathrm{pr}_i - x_i)\ ,\end{aligned}$$

and in order to apply Theorem 2.2.1 we need to evaluate the operator B_n at the function $(\mathrm{id}-x)^4$, where id denote the identity function defined by $\mathrm{id}(x) := x$ for every $x\in\mathbb{R}$.

Proposition 2.2.4 *For every $x\in K_d$, we have*

$$\begin{aligned}B_n((\mathrm{id}-x)^4)(x) = {}& \frac{1}{n^2}\left(\psi(x)^2 + 2\sum_{i,j=1}^{d} x_i^2(\delta_{i,j}-x_j)^2\right) \\ &+ \frac{1}{n^3}\left(\sum_{i,j=1}^{d} x_i x_j(x_i+x_j-3x_ix_j) + \sum_{i=1}^{d} x_i(1-2x_i)^2\right.\\ &\left. -\psi(x)^2 - 2\sum_{i,j=1}^{d} x_i^2(\delta_{i,j}-x_j)^2\right)\ ,\end{aligned} \tag{2.2.8}$$

where $\psi(x) = \sum_{i=1}^{d} x_i(1-x_i)$.

PROOF. We have

$$B_n((\mathrm{id}-x)^4)(x) = \sum_{i,j=1}^{d} B_n((\mathrm{pr}_i - x_i)(\mathrm{pr}_j - x_j))(x) \ ;$$

in order to compute $B_n((\mathrm{pr}_i - x_i)(\mathrm{pr}_j - x_j))(x)$ we use the fact that B_n coincides with the n-th Bernstein-Schnabl operator associated to the projection T_d, that is

$$B_n f(x) = \int_{K_d} \dots \int_{K_d} f\left(\frac{x_1 + \cdots + x_n}{n}\right) d\mu_x^{T_d}(x_1) \, \dots \, d\mu_x^{T_d}(x_n) \ . \quad (2.2.9)$$

We fix $x \in K_d$ and consider two affine function h_1, h_2 such that $h_1(x) = h_2(x) = 0$. Consider the function $f := h_1^2 h_2^2$; then we have

$$h_1^2 h_2^2 \left(\frac{x_1 + \cdots + x_n}{n}\right) = \frac{1}{n^4} \sum_{i,j,k,l=1}^{n} h_1(x_i) h_1(x_j) h_2(x_k) h_2(x_l) \ ,$$

and consequently

$$\begin{aligned}
&B_n(h_1^2 h_2^2)(x) \\
&\quad = \frac{1}{n^4} \sum_{i,j,k,l=1}^{n} \int_{K_d} \dots \int_{K_d} h_1(x_i) h_1(x_j) h_2(x_k) h_2(x_l) \, d\mu_x^{T_d}(x_1) \, \dots \, d\mu_x^{T_d}(x_n) \\
&\quad = \frac{1}{n^4} \sum_{\substack{i,j,k,l=1 \\ i=j=k=l}}^{n} T_d(h_1^2 h_2^2)(x) + \frac{1}{n^4} \sum_{\substack{i,j,k,l=1 \\ i=j,k=l,i\neq l}}^{n} T_d(h_1^2)(x) T_d(h_2^2)(x) \\
&\qquad + \frac{1}{n^4} \sum_{\substack{i,j,k,l=1 \\ i=k,j=l,i\neq j}}^{n} (T_d(h_1 h_2)(x))^2 + \frac{1}{n^4} \sum_{\substack{i,j,k,l=1 \\ i=l,j=k,i\neq j}}^{n} (T_d(h_1 h_2)(x))^2 \\
&\quad = \frac{1}{n^3} T_d(h_1^2 h_2^2)(x) + \frac{n(n-1)}{n^4} T_d(h_1^2)(x) T_d(h_2^2)(x) + 2\frac{n(n-1)}{n^4} (T_d(h_1 h_2)(x))^2 \\
&\quad = \frac{1}{n^2} \left[T_d(h_1^2)(x) T_d(h_2^2)(x) + 2\left(T_d(h_1 h_2)(x)\right) \right] \\
&\qquad + \frac{1}{n^3} \left[T_d(h_1^2 h_2^2)(x) - T_d(h_1^2)(x) T(h_2^2)(x) - 2(T_d(h_1 h_2)(x))^2 \right] \ .
\end{aligned}$$

Now let $1 \le i, j \le d$ and take $h_1 = \mathrm{pr}_i - x_i$ and $h_2 = \mathrm{pr}_j - x_j$; we obtain

$$T_d((\mathrm{pr}_i - x_i)(\mathrm{pr}_j - x_j))(x) = x_i(\delta_{i,j} - x_j)$$

and

$$T_d((\mathrm{pr}_i - x_i)^2 (\mathrm{pr}_j - x_j)^2)(x) = x_i x_j (x_i + x_j - 3 x_i x_j) + \delta_{i,j} x_i (1 - 2x_i)^2 \ .$$

Hence

$$
\begin{aligned}
&B_n((\mathrm{pr}_i - x_i)(\mathrm{pr}_j - x_j))(x) \\
&\quad = \frac{1}{n^2}(x_i(1-x_i)x_j(1-x_j) + 2x_i^2(\delta_{i,j} - x_j)^2) \\
&\qquad + \frac{1}{n^3}\big(x_i x_j(x_i + x_j - 3x_i x_j) + \delta_{i,j} x_i (1-2x_i)^2 \\
&\qquad - x_i(1-x_i)x_j(1-x_j) - 2x_i^2(\delta_{i,j} - x_j)^2\big) \ ,
\end{aligned}
$$

and this completes the proof. □

Remark 2.2.5 We have

$$
\left|B_n((\mathrm{id} - x)^4)(x)\right| \le \frac{1}{n^2}\left(\psi(x)^2 + 2\sum_{i,j=1}^{d} x_i^2(\delta_{i,j} - x_j)^2\right) + \frac{3}{n^3}\ . \qquad (2.2.10)
$$

Indeed since $x \in K_d$, we have $0 \le \sum_{i=1}^{d} x_i^2 \le 1$ and $(1-2x_i)^2 \le 1$ and therefore

$$
\begin{aligned}
&\sum_{i,j=1}^{d} x_i x_j (x_i + x_j - 3x_i x_j) + \sum_{i=1}^{d} x_i(1-2x_i)^2 - \psi(x)^2 - 2\sum_{i,j=1}^{d} x_i^2(\delta_{i,j} - x_j)^2 \\
&\quad \le 2\sum_{i,j=1}^{d} x_i x_j^2 + \sum_{i=1}^{d} x_i(1-2x_i)^2 \le 2\sum_{i=1}^{d} x_i \sum_{j=1}^{d} x_i^2 + \sum_{i=1}^{d} x_i \le 3\ .
\end{aligned}
$$

Using the above inequalities, (2.2.10) directly follows from (2.2.8). □

Now we can establish a quantitative version of the Voronovskaja-type formula for the Bernstein operators in the space $C^{2,\alpha}([0,1])$.

Theorem 2.2.6 (Quantitative Voronovskaja's formula for Bernstein operators) *Consider the Bernstein operators on $C(K_d)$ and the differential operator* (2.2.7)*. For every $f \in C^{2,\alpha}(K_d)$ and $x \in K_d$ we have*

$$
|n(B_n(f)(x) - f(x)) - Af(x)| \le L_{f''}\left(\frac{1}{2} - \frac{1}{2d} + \frac{3}{4n}\right)^{1/2}\left(\frac{\psi(x)}{n}\right)^{\alpha/2}
$$

if $d > 1$ and

$$
|n(B_n(f)(x) - f(x)) - Af(x)| \le L_{f''}\left(\frac{1}{16} + \frac{3}{4n}\right)^{1/2}\left(\frac{\psi(x)}{n}\right)^{\alpha/2}
$$

if $d = 1$, where $\psi(x) = \sum_{i=1}^{d} x_i(1-x_i)$.

PROOF. Recalling that, for every $i, j = 1, \dots, d$,

$$B_n\mathbf{1} = \mathbf{1}\ , \qquad B_n\mathrm{pr}_i = \mathrm{pr}_i\ , \qquad B_n(\mathrm{pr}_i\,\mathrm{pr}_j) = \mathrm{pr}_i\,\mathrm{pr}_j + \frac{1}{n}\,\mathrm{pr}_i\,(\delta_{ij} - \mathrm{pr}_j)\ ,$$

we have

$$B_n(\mathrm{pr}_i - x_i)(x) = 0\,, \quad B_n\left((\mathrm{pr}_i - x_i)(\mathrm{pr}_j - x_j)\right)(x) = \frac{x_i(\delta_{ij} - x_j)}{n},$$

and therefore

$$n\mathcal{A}_{B_n} f = Af$$

and $B_n(\psi_x^2)(x) = \frac{1}{n}\sum_{i=1}^n x_i(1 - x_i) = \frac{1}{n}\psi(x)$. From Theorem 2.2.1 and (2.2.10) we have

$$\begin{aligned}
&|n(B_n(f)(x) - f(x)) - Af(x)| \\
&\quad\le \frac{L_{f''}}{2}\frac{1}{n^{\alpha/2}}(\psi(x))^{\alpha/2} n\left(\frac{\psi(x)^2}{n^2} + B_n((\mathrm{id} - x)^4)(x)\right)^{1/2} \\
&\quad= \frac{L_{f''}}{2}\frac{1}{n^{\alpha/2}}(\psi(x))^{\alpha/2} n\left(\frac{2}{n^2}\psi(x)^2 + \frac{2}{n^2}\sum_{i,j=1}^{d} x_i^2(\delta_{i,j} - x_j)^2 + \frac{3}{n^3}\right)^{1/2} \\
&\quad\le \frac{L_{f''}}{2}\frac{1}{n^{\alpha/2}}(\psi(x))^{\alpha/2}\left[2\left(\psi(x)^2 + \sum_{i,j=1}^{d} x_i^2(\delta_{i,j} - x_j)^2\right) + \frac{3}{n}\right]^{1/2}.
\end{aligned}$$

The function $g(x) = \psi(x)^2 + \sum_{i,j=1}^d x_i^2(\delta_{i,j} - x_j)^2$ attains its maximum in K_d at the point $\overline{x} = (1/d, \dots, 1/d)$ if $d > 1$ and at $\overline{x} = 1/2$ if $d = 1$; then

$$\begin{aligned}
g(\overline{x}) &= \left(\sum_{i=1}^{d}\frac{1}{d}\left(1 - \frac{1}{d}\right)\right)^2 + \sum_{i=1}^{d}\left(\frac{1}{d}\right)^2\left(1 - \frac{1}{d}\right)^2 + \sum_{i\neq j}\left(\frac{1}{d}\right)^4 \\
&= \left(1 - \frac{1}{d}\right)^2 + \frac{1}{d}\left(1 - \frac{1}{d}\right)^2 + \frac{d(d-1)}{d^4} = 1 - \frac{1}{d}
\end{aligned}$$

if $d > 1$ and $g(\overline{x}) = 1/8$ if $d = 1$. Therefore

$$|n(B_n(f)(x) - f(x)) - Af(x)| \le \frac{L_{f''}}{2}\frac{1}{n^{\alpha/2}}(\psi(x))^{\alpha/2}\left(2\left(1 - \frac{1}{d}\right) + \frac{3}{n}\right)^{1/2}$$

if $d > 1$ and

$$|n(B_n(f)(x) - f(x)) - Af(x)| \le \frac{L_{f''}}{2}\frac{1}{n^{\alpha/2}}(\psi(x))^{\alpha/2}\left(\frac{1}{4} + \frac{3}{n}\right)^{1/2}$$

if $d = 1$. $\square$

Remark 2.2.7 Taking into account that $\psi(x) \le 1-\frac{1}{d}$ if $d>1$ and $\psi(x) \le \frac{1}{4}$ if $d=1$, we have, for $n>1$

$$\|n(B_n(f)-f)-Af\| \le L_{f''}\frac{1}{n^{\alpha/2}}\ . \tag{2.2.11}$$

□

The following last result is a consequence of Theorem 1.1.2, 1.2.1 and (1.1.11) by means of (2.2.11)

Theorem 2.2.8 *Consider the Bernstein operators on $C(K_d)$ and the differential operator (2.2.7). Then, the closure of $(A, C^2(K_d))$ generates a C_0-semigroup $(T(t))_{t\ge 0}$ on $C(K_d)$ such that, for every $t \ge 0$, $(k(n))_{n\ge 1}$ sequence of positive integers and $f \in C^{2,\alpha}(K_d)$, we have*

$$\begin{aligned}\|T(t)f - B_n^{k(n)}f\| & \qquad\qquad (2.2.12)\\ \le \frac{L_{f''}\,t}{n^{\alpha/2}} + \left(\left|\frac{k(n)}{n}-t\right| + \sqrt{\frac{2}{\pi}}\,\frac{\sqrt{k(n)}}{n}\right)\left(\|Af\|+\frac{L_{f''}}{n^{\alpha/2}}\right)\ ,\end{aligned}$$

and in particular, taking $k(n)=[nt]$,

$$\|T(t)f - B_n^{[nt]}f\| \le \frac{L_{f''}\,t}{n^{\alpha/2}} + \frac{1}{\sqrt{n}}\left(\frac{1}{\sqrt{n}}+\sqrt{\frac{2t}{\pi}}\right)\left(\|Af\|+\frac{L_{f''}}{n^{\alpha/2}}\right)\ . \tag{2.2.13}$$

Moreover, for every $\lambda \in \mathbb{C}$ such that $\operatorname{Re}\lambda > 0$ and $n \ge 1$, consider the operator $B_{\lambda,n} : C(K_d) \to C(K_d)$ defined by

$$B_{\lambda,n}f := \int_0^{+\infty} e^{-\lambda t} B_n^{[n\,t]} f\,dt\ , \qquad f \in C(K)\ .$$

If $R(\lambda, A)$ denotes the resolvent operator of the closure of $(A, C^2(K_d))$, for every $n>1$ and $f \in C^{2,\alpha}(K_d)$ we have

$$\begin{aligned}\|R(\lambda,A)f - B_{\lambda,n}f\| &\le \frac{1}{(\operatorname{Re}\lambda)^2}\,\frac{L_{f''}}{n^{\alpha/2}} \qquad\qquad (2.2.14)\\ &+\frac{1}{\sqrt{n}\operatorname{Re}\lambda}\left(\frac{1}{\sqrt{n}}+\frac{1}{\sqrt{2\operatorname{Re}\lambda}}\right)\left(\|Af\|+\frac{L_{f''}}{n^{\alpha/2}}\right)\ .\end{aligned}$$

In the one dimensional case we have a partial converse result.

First we recall some needed notions related to the smoothness of a function f.

The *divided difference operator* $\Delta_h(f,x)$ is defined by

$$\Delta_h(f,x) := f(x+h) - f(x)\ , \qquad h \ge 0\ .$$

If $r \geq 2$, the *r-th order divided difference* Δ_h^r is defined as the r-fold composition of Δ_h with itself.

If $f \in C([a,b])$, then the *modulus of continuity of* f is defined by

$$\omega(f,\delta) := \sup_{0 \leq h \leq \delta} \|\Delta_h(f,\cdot)\|_{[a,b-h]} \,, \qquad 0 \leq \delta \leq b-a \,.$$

Accordingly, if $r \geq 2$ the *r-th order modulus of smoothness* is given by

$$\omega_r(f,\delta) := \sup_{0 \leq h \leq \delta} \|\Delta_h^r(f,\cdot)\|_{[a,b-rh]} \,, \qquad 0 \leq \delta \leq \frac{b-a}{r} \,.$$

If $0 < \alpha \leq 1$, we e denote by $\mathrm{Lip}\,\alpha$ the set of all functions $f \in C([a,b])$ for which there exists $M > 0$ such that $\omega(f,\delta) \leq M\delta^\alpha$, i.e. $\mathrm{Lip}\,\alpha$ is the Lipschitz α-class. If $0 < \alpha \leq 2$ we denote by $\mathrm{Lip}^*\,\alpha$ the set of all functions $f \in C([a,b])$ for which there exists $M > 0$ such that $\omega_2(f,\delta) \leq M\delta^\alpha$. It is noteworthy that if $0 < \alpha < 1$ the class $\mathrm{Lip}^*\,\alpha$ and $\mathrm{Lip}\,\alpha$ coincide (see [45, p. 6 (1.3.5)]).

Proposition 2.2.9 *Consider the Bernstein operators on* $C([0,1])$ *and the differential operator* (2.2.7)*. Let* $f \in C^2([0,1])$ *and assume that there exist a constant* $C > 0$ *and* $\alpha \in]0,1[$ *such that*

$$\|n(B_n f - f) - Af\| \leq \frac{C}{n^{\alpha/2}} \,. \tag{2.2.15}$$

then $f \in C^{2,\alpha}_{\mathrm{loc}}(0,1)$.

Proof. First we explicitly evaluate the differential operator (2.2.7) which, in our situation, becomes

$$Af(x) = \frac{x(1-x)}{2} f''(x) \,.$$

Now let $f \in C^{2,\alpha}_{\mathrm{loc}}([0,1])$; since $f \in C^2([0,1])$, for every $x \in [0,1]$ and $y \in [0,1]$, there exists $\xi(y)$ in the segment joining x and y such that

$$f(y) - f(x) = f'(x)\,(y-x) + \frac{1}{2}\, f''(x)\,(y-x)^2 + \eta(y,x)\,(y-x)^2 \,, \tag{2.2.16}$$

where $\eta(t,x) := \frac{1}{2}\,(f''(\xi(t)) - f''(x))$. Then we can write

$$\begin{aligned} B_n(f)(x) - f(x) &= B_n(\mathrm{id}-x)(x) + \frac{1}{2} f''(x) B_n(\mathrm{id}-x)^2(x) \\ &\quad + B_n((\mathrm{id}-x)^2 \eta(\mathrm{id},x))(x) \,. \end{aligned}$$

Taking into account that

$$B_n(\mathrm{id}-x)(x) = 0 \,, \quad B_n(\mathrm{id}-x)^2(x) = \frac{x(1-x)}{n} \,,$$

we have

$$B_n(f)(x) - f(x) = \frac{1}{n} Af(x) + B_n((\mathrm{id} - x)^2 \eta(\mathrm{id}, x))(x) ,$$

and from (2.2.15) it follows that

$$\left| B_n \left((\mathrm{id} - x)^2 \eta(\mathrm{id}, x) \right) (x) \right| \leq \frac{C}{n} \frac{1}{n^{\alpha/2}} . \tag{2.2.17}$$

Now, we consider a linear combination of Bernstein polynomials introduced by Butzer [20] and recursively defined by

$$\begin{aligned} &B_{n,0} := B_n \\ &(2^r - 1) B_{n,r} = 2^r B_{2n,r-1} - B_{n,r-1} . \end{aligned}$$

A result of Ditzian [47] states that

$$\|B_{n,r}(f) - f\| = O\left(\frac{1}{n^{\beta/2}}\right) \iff \|\varphi^\beta \Delta_h^{2r} f\|_{[rh, 1-rh]} = O(h^\beta) \tag{2.2.18}$$

for $\beta < 2r$ and $\varphi^2(x) := x(1-x)$.
If we set $g_x(y) := (y - x)^2 \eta(y, x)$ and take $r = 2$ we get

$$B_{n,2} = \frac{8}{3} B_{4n} - 2B_{2n} + \frac{1}{3} B_n$$

and from (2.2.17) it follows that $\|B_{n,2}(g_x)\| = O\left(\frac{1}{n^{1+\alpha/2}}\right)$. Therefore we take (2.2.18) with $\beta = 2 + \alpha$, then for every $\delta > 2h$ exists $C = C(\delta) > 0$ such that

$$\left| \Delta_h^4 g_x(y) \right| \leq C h^{2+\alpha} , \qquad y \in [\delta, 1 - \delta] . \tag{2.2.19}$$

Now we evaluate $\Delta_h^4 g_x(y)$ at the point $y = x$; we have

$$\Delta_h^4 g_x(x) = g_x(x + 4h) - 4g_x(x + 3h) + 6g_x(x + 2h) - 4g_x(x + h) + g_x(x) .$$

Taking into account that, from (2.2.16), for every $s \geq 0$

$$\begin{aligned} g_x(x + s) &= f(x + s) - f(x) - f'(x)(x + s - x) - \frac{1}{2} f''(x)(x + s - x)^2 \\ &= f(x + s) - f(x) - f'(x)s - \frac{1}{2} f''(x) s^2 , \end{aligned}$$

we have

$$\begin{aligned} \Delta_h^4 g_x(x) &= f(x + 4h) - 4f(x + 3h) + 6f(x + 2h) - 4f(x + h) \\ &\quad - f(x)(1 - 4 + 6 - 4) \\ &\quad - f'(x)h(4 - 12 + 12 - 4) - f''(x)h^2(16 - 36 + 24 - 4) \\ &= \Delta_h^4 f(x) . \end{aligned}$$

Now we evaluate the 4-*th* order finite-difference of f in terms of the second-order derivative of f,

$$\begin{aligned}\Delta_h^4 f(x) &= \int_0^h \Delta_h^3 f'(x+t)dt = \int_0^h \int_0^h \Delta_h^2 f''(x+t+s)dsdt \\ &= h^2 \Delta_h^2 f''(x+\xi(x)) \end{aligned} \tag{2.2.20}$$

where $\xi(x) \in [x, x+2h]$. Now we consider the function $z(x) := x + \xi(x)$ for $x \in [\delta, 1-\delta]$, we have that $z(\delta) \leq \delta + 2h$ and $1-\delta \leq z(1-\delta)$, then $z^{-1}([\delta + 2h, 1-\delta]) \subset [\delta, 1-\delta]$. Consequently, since the function z is continuous, for every $h \geq 0$ and $\overline{z} \in [\delta + 2h, 1-\delta]$ we can take $x \in [\delta, 1-\delta]$ such that $z(x) = x + \xi(x) = \overline{z}$, and from (2.2.19) and (2.2.20) we can write

$$\left| h^2 \Delta_h^2 f''(z) \right| = |\Delta_h^4 g_x(x)| \leq C h^{2+\alpha} , \qquad z \in [\delta + 2h, 1-\delta] ,$$

and hence, since $\delta > 2h$,

$$\|\Delta_h^2 f''\|_{[2\delta, 1-\delta]} = O(h^\alpha) .$$

The last expression yields $\omega_2(f,h) \leq C h^\alpha$, i.e. $f'' \in \mathrm{Lip}^*(\alpha)$ locally in $(0,1)$. Since $0 < \alpha < 1$, we have that f is also in $\mathrm{Lip}(\alpha)$. □

For the sake of brevity we do not investigate the analogous results for Lototsky-Schnabl and Stancu-Schnabl operators in this setting. In the next section we give some details on a particular class of Stancu operators.

2.2.3 Application to Stancu operators

In this section we consider some quantitative estimates of the convergence of suitable combinations of iterates of Stancu operators to the associated C_0-semigroup and the resolvent operator of its generator in the context of spaces of continuous functions on the d-dimensional simplex.
Stancu operators were introduced by D. D. Stancu in [66, 67] in the context of spaces of continuous functions on the interval $[0,1]$; if $a \in \mathbb{R}$, the n-th Stancu operator $Q_{n,a} : C([0,1]) \to C([0,1])$ is defined by setting

$$Q_{n,a}f(x) := \sum_{k=0}^{n} f\left(\frac{k}{n}\right) q_{nk}(x,a), \quad f \in C([0,1]),\ x \in [0,1],$$

where

$$q_{nk}(x,a) := \binom{n}{k} \frac{\Phi_k(x,a)\,\Phi_{n-k}(1-x,a)}{\Phi_n(1,a)}, \quad \Phi_k(x,a) := \prod_{j=0}^{k-1}(x+ja).$$

In this setting these operators have been studied by Mühlbach [60, 61]. Further generalizations were considered by Felbecker [51] and by Campiti [22, 23]; in these last papers also connections with the representation of a suitable C_0-semigroups have been considered.

The results in this section have been published in [39].

First, we consider the standard simplex K_d of $\mathbb{R}^d$ and the Stancu operators $S_{n,a_n} : C(K_d) \to C(K_d)$ on K_d, which are associated with a sequence $(a_n)_{n\ge1}$ of positive real numbers and are defined by setting, for every $f \in C(K_d)$ and $x = (x_1,\dots,x_d) \in K_d$,

$$S_{n,a_n}f(x_1,\dots,x_d) := \frac{1}{p_n(a_n)} \sum_{h_1+\cdots+h_d\le n} f\left(\frac{h_1}{n},\dots,\frac{h_d}{n}\right) \times \frac{n!}{h_0!\,h_1!\dots h_d!} \prod_{i=0}^{d} \Phi_{h_i}(x_i,a_n), \tag{2.2.21}$$

for every $f \in C(K_d)$ and $(x_1,\dots,x_d) \in K_d$, where as usual $x_0 := 1 - x_1 - \cdots - x_d$, $h_0 := n - h_1 - \cdots - h_d$ and

$$p_n(a) := \prod_{j=0}^{n-1}(1 + j\,a), \qquad a \in \mathbb{R}.$$

In the sequel we assume that the sequence $(na_n)_{n\ge1}$ converges to $b \ge 0$ and consider the differential operator $A : C^2(K_d) \to C(K_d)$ defined by

$$Af(x) = (1+b) \sum_{i,j=1}^{d} \frac{x_i(\delta_{ij} - x_j)}{2} \frac{\partial^2 f}{\partial x_i\,\partial x_j}(x)\ , \tag{2.2.22}$$

whenever $f \in C^2(K_d)$ and $x = (x_1, \dots, x_d) \in K_d$.

It is well known that the closure of the operator $(A, C^2(K_d))$ generates a C_0-semigroup of positive contractions on $C(K_d)$, and that $C^2(K_d)$ is a core for this closure (see e.g. [9, Theorem 6.2.6, p. 436]).

Moreover, the operator A is connected with Stancu operators by means of the following Voronovskaja's formula established in [51, 23]:

$$\lim_{n\to+\infty} n(S_{n,a_n}(f) - f) = A(f), \qquad f \in C^2(K_d). \tag{2.2.23}$$

Now we establish a quantitative version of (2.2.23)

Proposition 2.2.10 *Consider the Stancu operators (2.2.21) on $C(K_d)$ and the differential operator (2.2.22). There exists a constant $C > 0$ such that for every $f \in C^{2,\alpha}(K_d)$ we have*

$$\|n(S_{n,a_n}(f) - f) - A(f)\| \le C\left(\frac{1}{n^{\alpha/2}} + |na_n - b|\right) M_f \ ,$$

where M_f is the seminorm on $C^{2,\alpha}(K_d)$ defined by

$$M_f := L_{f''} + \|D^2 f\| \ . \tag{2.2.24}$$

Proof. Let $f \in C^{2,\alpha}(K_d)$ we have

$$\begin{aligned} &|n(S_{n,a_n}(f) - f)(x) - Af(x)| \qquad (2.2.25)\\ &\quad \le |n(S_{n,a_n}f(x) - f(x) - \mathcal{A}_{S_{n,a_n}}f(x))| + |n\mathcal{A}_{S_{n,a_n}}f(x) - Af(x)| \ , \end{aligned}$$

where $\mathcal{A}_{S_{n,a_n}}$ is the operator (2.2.2) obtained by taking $L = S_{n,a_n}$. In order to write an explicit expression of S_{n,a_n}, we recall that, for every $i, j = 1, \dots, d$,

$$S_{n,a_n}(\mathbf{1}) = \mathbf{1}, \qquad S_{n,a_n}(\mathrm{pr}_i) = \mathrm{pr}_i,$$

$$S_{n,a_n}(\mathrm{pr}_i\,\mathrm{pr}_j) = \mathrm{pr}_i\,\mathrm{pr}_j + \frac{1+na_n}{n(1+a_n)}\,\mathrm{pr}_i\,(\delta_{ij} - \mathrm{pr}_j) \ ,$$

and consequently

$$S_{n,a_n}((\mathrm{pr}_i - x_i)(\mathrm{pr}_j - x_j))(x) = \frac{1+na_n}{n(1+a_n)}\,x_i\,(\delta_{ij} - x_j) \tag{2.2.26}$$

and

$$S_{n,a_n}(\mathrm{pr}_i - x_i)(x) = 0 \ .$$

Hence the operator $\mathcal{A}_{S_{n,a_n}}$ becomes

$$\mathcal{A}_{S_{n,a_n}}f(x) = \frac{1+na_n}{n(1+a_n)}\sum_{i,j=1}^{d}\frac{x_i\,(\delta_{ij} - x_j)}{2}\frac{\partial^2 f}{\partial x_i \partial x_j}(x) \ .$$

In regard to the first term in (2.2.25), we apply Theorem 2.2.1 with $L = S_{n,a_n}$, and we get

$$
\begin{aligned}
&|n(S_{n,a_n} f(x) - f(x) - \mathcal{A}_{S_{n,a_n}} f(x))| \\
&\qquad \le n \frac{L_{f''}}{2} \left(S_{n,a_n}(\psi_x^2)(x) \right)^{\alpha/2} \left((S_{n,a_n}(\psi_x^2)(x))^2 L(\mathbf{1})(x) + L(\psi_x^4)(x) \right)^{1/2} \ .
\end{aligned}
$$

From (2.2.26) we have

$$
S_{n,a_n}(\psi_x^2)(x) = \frac{1 + n a_n}{n(1 + a_n)} \psi(x) \left(\sum_{i=1}^{d} x_i (1 - x_i) \right)
$$

and therefore

$$
\left| S_{n,a_n}(\psi_x^2)(x) \right| \le \frac{1 + n a_n}{n(1 + a_n)} \left(1 - \frac{1}{d} \right) \le \frac{1 + n a_n}{n(1 + a_n)} \ .
$$

Moreover from [9, Lemma 6.2.2, p. 429], we obtain the existence of a constant $C_1 > 0$ such that

$$
\left| S_{n,a_n} \left((\mathrm{pr}_i - x_i)^2 (\mathrm{pr}_j - x_j)^2 \right) (x) \right| \le \frac{C_1}{n^2}
$$

and hence

$$
S_{n,a_n}(\psi_x^4)(x) \le \frac{d^2 C_1}{n^2} \ .
$$

The first term in (2.2.25) can be estimated as follows

$$
\begin{aligned}
&|n(S_{n,a_n} f(x) - f(x) - \mathcal{A}_{S_{n,a_n}} f(x))| \\
&\qquad \le \frac{L_{f''}}{2} \left(\frac{1 + n a_n}{n(1 + a_n)} \right)^{\alpha/2} n \left(\frac{1 + n a_n}{n(1 + a_n)} + \frac{d\sqrt{C_1}}{n} \right) \ .
\end{aligned}
$$

In regard to the second term in (2.2.25) we have

$$
\begin{aligned}
|n\mathcal{A}_{S_{n,a_n}} f(x) - A f(x)| &\le \left| \frac{1 + n a_n}{1 + a_n} - (b+1) \right| \left| \sum_{i,j=1}^{d} \frac{x_i (\delta_{ij} - x_j)}{2} \frac{\partial^2 f}{\partial x_i \partial x_j}(x) \right| \\
&\le \left| \frac{n a_n - b - a_n(1+b)}{1 + a_n} \right| \|D^2 f\| \\
&\le \left(|n a_n - b| + \frac{1}{n} n a_n (b+1) \right) \|D^2 f\| \\
&\le \left(|n a_n - b| + \frac{C_2}{n} \right) \|D^2 f\| \ .
\end{aligned}
$$

Finally from the above inequalities it follows

$$\begin{aligned}
&|n(S_{n,a_n}(f)-f)(x)-Af(x)| \\
&\quad\le \frac{L_{f''}}{2}\left(\frac{1+na_n}{n(1+a_n)}\right)^{\alpha/2} n\left(\frac{1+na_n}{n(1+a_n)}+\frac{d\sqrt{C_1}}{n}\right) \\
&\qquad + \left(|na_n-b|+\frac{C_2}{n}\right)\|D^2 f\| \\
&\quad\le C_3\frac{1}{n^{\alpha/2}}L_{f''}+\left(|na_n-b|+\frac{C_2}{n}\right)\|D^2 f\| \\
&\quad\le C\left(\frac{1}{n^{\alpha/2}}+|na_n-b|\right)M_f\ ,
\end{aligned}$$

where M_f is the seminorm defined by (2.2.24). □

The preceding result allows us to get the quantitative estimate obtained in Theorem 1.1.2 in the particular case where the growth bound of the semigroup is equal to 0.

Theorem 2.2.11 *Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of positive real numbers such that $(na_n)_{n\in\mathbb{N}}$ converges to $b\in\mathbb{R}$ and consider the Stancu operators on $C(K_d)$ and the differential operator (2.2.22). Then, the closure of $(A, C^2(K_d))$ generates a C_0-semigroup $(T(t))_{t\ge 0}$ on $C(K_d)$ such that, for every $t\ge 0$, $(k(n))_{n\in\mathbb{N}}$ sequence of positive integers and $f\in C^{2,\alpha}(K_d)$, we have*

$$\begin{aligned}
\|T(t)f-S_{n,a_n}^{k(n)}f\| &\le CM_f\,t\left(\frac{1}{n^{\alpha/2}}+|na_n-b|\right) \\
&+\left(\left|\frac{k(n)}{n}-t\right|+\sqrt{\frac{2}{\pi}}\frac{\sqrt{k(n)}}{n}\right)\left(\|A(f)\|+CM_f\left(\frac{1}{n^{\alpha/2}}+|na_n-b|\right)\right)
\end{aligned}\tag{2.2.27}$$

and taking $k(n):=[nt]$,

$$\begin{aligned}
\|T(t)f-S_{n,a_n}^{[nt]}f\| &\le CM_f\,t\left(\frac{1}{n^{\alpha/2}}+|na_n-b|\right) \\
&+\frac{1}{\sqrt{n}}\left(\frac{1}{\sqrt{n}}+\sqrt{\frac{2t}{\pi}}\right)\left(\|A(f)\|+CM_f\left(\frac{1}{n^{\alpha/2}}+|na_n-b|\right)\right).
\end{aligned}\tag{2.2.28}$$

In the particular case where $a_n:=b/n$, estimate (2.2.27) becomes

$$\|T(t)f-S_{n,b/n}^{k(n)}f\|\le\frac{CM_f\,t}{n^{\alpha/2}}+\left(\left|\frac{k(n)}{n}-t\right|+\sqrt{\frac{2}{\pi}}\frac{\sqrt{k(n)}}{n}\right)\left(\|A(f)\|+\frac{CM_f}{n^{\alpha/2}}\right)\ ,$$

and if $k(n):=[nt]$, from (2.2.28) we get

$$\|T(t)f-S_{n,b/n}^{[nt]}f\|\le\frac{CM_f\,t}{n^{\alpha/2}}+\frac{1}{\sqrt{n}}\left(\frac{1}{\sqrt{n}}+\sqrt{\frac{2t}{\pi}}\right)\left(\|A(f)\|+\frac{CM_f}{n^{\alpha/2}}\right).$$

In order to approximate the resolvent operator of the closure of $(A, C^2(K_d))$, Let $(s_n)_{n\geq 1}$ be a sequence of positive integers tending to $+\infty$ and for every $n \geq 1$, consider the linear operator $P_{\lambda,s_n,n,a_n} : C(K_d) \to C(K_d)$ defined by

$$P_{\lambda,s_n,n,a_n}(u) := \frac{1}{n} \sum_{k=0}^{s_n} e^{-\lambda k/n} S^k_{n,a_n}(u), \qquad u \in C(K_d).$$

We are now in a position to state the following result.

Theorem 2.2.12 *For every $n \geq 1$ and $f \in C^{2,\alpha}(K_d)$, we have*

$$\begin{aligned}
&\|P_{\lambda,s_n,n,a_n}(f) - R(\lambda, A)f\| \\
&\quad \leq \frac{1}{(\operatorname{Re}\lambda)^2} \left(\|A(f)\| + CM_f \left(\frac{1}{n^{\alpha/2}} + |na_n - b| \right) \right) \\
&\quad\quad + \frac{1}{\sqrt{n}} \left(\frac{1}{\sqrt{n}\operatorname{Re}\lambda} + \frac{1}{\sqrt{2}\,(\operatorname{Re}\lambda)^{3/2}} \right) \left(CM_f \left(\frac{1}{n^{\alpha/2}} + |na_n - b| \right) \right) \\
&\quad\quad + \frac{e^{-(\operatorname{Re}\lambda)s_n/n} + \frac{|\lambda|^{3/2}}{n\,|\operatorname{Re}\sqrt{\lambda}|}}{\left(1 - \frac{\operatorname{Re}\lambda}{n}\right)\operatorname{Re}\lambda} \|f\|.
\end{aligned}$$

Hence, if we assume that

$$\lim_{n\to+\infty} \frac{s_n}{n} = +\infty,$$

then the sequence $(P_{\lambda,s_n,n,a_n})_{n\geq 1}$ strongly converges to $R(\lambda, A)$ on $C(K_d)$.

If we take in particular $a_n := b/n$, then

$$\begin{aligned}
\|P_{\lambda,s_n,n,b/n}(f) - R(\lambda, A)f\| &\leq \frac{1}{(\operatorname{Re}\lambda)^2} \left(\|A(u)\| + \frac{CM_f}{n^{\alpha/2}} \right) \\
&\quad + \frac{1}{\sqrt{n}} \left(\frac{1}{\sqrt{n}\operatorname{Re}\lambda} + \frac{1}{\sqrt{2}\,(\operatorname{Re}\lambda)^{3/2}} \right) \frac{CM_f}{n^{\alpha/2}} \\
&\quad + \frac{e^{-(\operatorname{Re}\lambda)s_n/n} + \frac{|\lambda|^{3/2}}{n\,|\operatorname{Re}\sqrt{\lambda}|}}{\left(1 - \frac{\operatorname{Re}\lambda}{n}\right)\operatorname{Re}\lambda} \|f\|.
\end{aligned}$$

For the sake of simplicity, we have associated Stancu operators with a sequence $(a_n)_{n\geq 1}$ of real numbers; as observed in [24], all estimates concerning Stancu operators remain valid if we take a sequence $(a_n)_{n\in\mathbb{N}}$ of continuous functions on K_d such that $(na_n)_{n\in\mathbb{N}}$ uniformly converges to a function $b \in C(K_d)$ and consequently also the results in this section are true in this more general context. We explicitly observe that in this case the differential operator A is more general, as well as the estimates on the semigroup and the resolvent operators.