Chapter II

Preliminary and auxiliary results

In this chapter we collect some brief recalls on the main topics involved in this work. These recalls are only intended to fix some notation and references for the subsequent chapters and not to furnish a complete treatment of the subject.

II.1 Recalls on semigroup's theory

For the sake of completeness we collect here some basic definitions and results on semigroup's theory which will be frequently used in the sequel. We refer to the monographs [48] and [64] for a complete introduction to the subject.

Let E be a Banach space over the field $\mathbb{K}$. We shall denote by $\mathcal{L}(E)$ the space of all bounded linear operators in E.

A *semigroup* (or a *one-parameter semigroup*) of bounded linear operators on E is a family $(T(t))_{t\geq 0}$ of elements of $\mathcal{L}(E)$ such that

1. $T(0) = I$,
2. $T(s+t) = T(s)T(t)$ for every $s, t \geq 0$,

where I denotes the identity operator on E.

A semigroup is said to be *strongly continuous* (or a C_0*-semigroup*) if for every $t_0 \geq 0$ and $f \in E$

$$\lim_{t \to t_0} \|T(t)f - T(t_0)f\| = 0 \ .$$

The *growth bound* of the semigroup is defined by

$$\omega_0 := \inf \{ \, \omega \in \mathbb{R} \mid \text{there exists } M \geq 0 \text{ such that } \|T(t)\| \leq M \exp(\omega t) \text{ for every } \ t \geq 0 \} \ .$$

The *generator* of a C_0-semigroup is a linear operator $A : D(A) \to E$ defined by

$$Af := \lim_{t\to 0^+} \frac{T(t)f - f}{t} ,$$

on the linear subspace

$$D(A) := \left\{ f \in E \mid \text{there exists } \lim_{t\to 0^+} \frac{T(t)f - f}{t} \in E \right\} .$$

A semigroup is said to by *uniformly continuous* if for every $t_0 \geq 0$

$$\lim_{t\to t_0} \|T(t) - T(t_0)\| = 0 .$$

In this case we have $D(A) = E$ and A is bounded; conversely, if A is bounded, we can set

$$T(t) := \exp(tA) := \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n \quad (t \geq 0) ,$$

and we obtain a uniformly continuous semigroup whose generator is A.

A linear operator $A : D(A) \to E$ is said to be *closed* if $D(A)$ endowed with the *graph norm*

$$\|f\|_A := \|f\| + \|A(f)\| , \quad f \in D(A) ,$$

becomes a Banach space. In other words, the graph $\{(f, Af) \mid f \in D(A)\}$ is closed in $E \times E$.

We say that a linear operator $B : D(B) \to E$ is *an extension* of A if $D(A) \subset D(B)$ and $Af = Bf$ for every $f \in D(A)$.

A linear operator $A : D(A) \to E$ is *closable* if there exists a closed extension of A. The smallest closed extension $\overline{A} : D(\overline{A}) \to E$ of A is called the *closure* of A.

A *core* for a linear operator $A : D(A) \to E$ is a linear subspace D_0 of $D(A)$ which is dense in $D(A)$ with respect to the graph norm.

If $A : D(A) \to E$ is a closed operator, we shall denote by $\rho(A)$ its *resolvent set*, i.e.,

$$\rho(A) := \{\lambda \in \mathbb{K} \mid \lambda I - A \text{ is invertible}\} .$$

The *spectrum* $\sigma(A)$ is defined as

$$\sigma(A) := \mathbb{K} \setminus \rho(A) .$$

If $\lambda \in \rho(A)$ we shall denote by $R(\lambda, A)$ the inverse of $\lambda I - A$.

If A is the generator of a C_0-semigroup $(T(t))_{t\geq 0}$ then $\rho(A) \neq \varnothing$. Moreover, if ω_0 denotes the growth bound of $(T(t))_{t\geq 0}$ we have, for every $\lambda \in \mathbb{K}$ such that $\operatorname{Re}\lambda > \omega_0$ and $f \in E$

$$R(\lambda, A)f = \int_0^{\infty} \exp(-\lambda t) T(t) f \, dt .$$

Now we recall a generation and approximation theorem which is very important for our purposes.

Theorem II.1.1 (Trotter's approximation theorem) *Let $(L_n)_{n\in\mathbb{N}}$ be a sequence of bounded linear operators on E and let $(\rho_n)_{n\in N}$ be a decreasing sequence of positive real numbers tending to 0. Suppose that there exists $M \geq 0$ and $\omega \in \mathbb{R}$ such that*

$$\|L_n^k\| \leq Me^{\omega\,\rho_n\,k}\ , \qquad \text{for every } k,\, n \in \mathbb{N}\ . \tag{II.1.1}$$

Moreover, assume that $\mathcal{D}$ is a dense subspace of E and for every $f \in \mathcal{D}$ the following Voronovskaja-type formula holds

$$Af := \lim_{n\to\infty} \frac{L_n(f) - f}{\rho_n}\ .$$

If $(\lambda - A)(\mathcal{D})$ is dense in E for some $\lambda > \omega$, then the closure of A generates a C_0-semigroup $(T(t))_{t\geq 0}$ and for every $f \in E$ and every sequence $(k(n))_{n\in\mathbb{N}}$ of positive integers satisfying $\lim_{n\to+\infty} k(n)/\rho_n = t$, we have

$$T(t)f = \lim_{n\to\infty} L_n^{k(n)}(f)\ . \tag{II.1.2}$$

II.2 Feller's classification and generation results

In different applications we need to consider suitable domains where a second-order differential operator generates a C_0-semigroup. In this context, the classification of Feller may help us to decide the right choice of the domain of our operators (see [52, 54, 53]). After the work of Feller, it was also considered the problem of investigating the generation of a C_0-semigroup on an assigned domain. Necessary and sufficient conditions in order for A to be the generator of a C_0-semigroup in $C(\overline{I})$ have been given by Clément and Timmermans [44] when Ventcel's boundary conditions are imposed at the endpoints, and by Timmermans in [69] on the maximal domain. More recently, the existence of a C_0-semigroup has been characterized in [29] also in the case of Neumann's type boundary conditions at the endpoints. In the space $L^1(I)$ the characterization of the generation of a C_0-semigroup has been completely achieved in [19] on the adjoint maximal domain, on the adjoint Dirichlet domain and on the adjoint Neumann domain. However, all the results obtained in [44], [69], [29] and [19] are very closely related to the pioneer work by Feller [54].

In the sequel we consider an interval $I :=]r_1, r_2[$ with $-\infty \le r_1 < r_2 \le +\infty$ and two continuous real functions $\alpha, \beta : I \to \mathbb{R}$ with $\alpha > 0$ in I. We define the second-order differential operator

$$Au(x) := \alpha(x)\, u''(x) + \beta(x)\, u'(x) \;, \qquad u \in C(\overline{I}) \cap C^2(I) \;.$$

In order to state the main characterizations, we fix $x_0 \in I$ and define, for every $x \in I$,

$$W(x) := \exp\left(-\int_{x_0}^{x} \frac{\beta(t)}{\alpha(t)}\, dt\right) \tag{II.2.1}$$

and

$$Q(x) := \frac{1}{\alpha(x)W(x)} \int_{x_0}^{x} W(s)\, ds \;, \quad R(x) := W(x) \int_{x_0}^{x} \frac{1}{\alpha(s)W(s)}\, ds \;. \tag{II.2.2}$$

It is also useful to set $I_1 :=]r_1, x_0]$ and $I_2 := [x_0, r_2[$. The endpoint r_i, $i = 1, 2$ is said to be

$$\begin{array}{lll} \textit{a regular boundary} & \text{if} & Q \in L^1(I_i)\;, R \in L^1(I_i)\;; \\ \textit{an exit boundary} & \text{if} & Q \notin L^1(I_i)\;, R \in L^1(I_i)\;; \\ \textit{an entrance boundary} & \text{if} & Q \in L^1(I_i)\;, R \notin L^1(I_i)\;; \\ \textit{a natural boundary} & \text{if} & Q \notin L^1(I_i)\;, R \notin L^1(I_i)\;. \end{array} \tag{II.2.3}$$

The following generation results will be used in the sequel.

First, we consider the maximal domain of A defined as follows

$$D_M(A) = \{u \in C(\bar{I}) \cap C^2(I) \mid Au \in C(\bar{I})\} \;. \tag{II.2.4}$$

Theorem II.2.1 (Feller [54], Timmermans [69]) *The linear operator $(A, D_M(A))$ generates a C_0-semigroup in $C(\overline{I})$ if and only if r_1 ed r_2 are both entrance or natural endpoints. The semigroup is of positive contractions whenever it exists.*

If we consider Ventcel's boundary condition, we get the following domain of A

$$D_V(A) = \{u \in C(\overline{I}) \cap C^2(I) \mid \lim_{x \to r_1, r_2} Au(x) = 0\} \tag{II.2.5}$$

and we have the following generation result.

Theorem II.2.2 (Feller [54], Clément-Timmermans [44]) *The linear operator $(A, D_V(A))$ generates a C_0-semigroup in $C(\overline{I})$ if and only if both r_1 and r_2 are not entrance boundary points. The semigroup is of positive contractions whenever it exists.*

II.3 Cosine functions

In this section we briefly collect some definitions and properties of cosine functions. We limit ourselves to those notions which are strictly necessary for the sequel. An organic treatment of the subject can be found in the monographs [50, 17] and in the paper by Sova [68].

A family $(C(t))_{t\in\mathbb{R}}$ of linear operators on a Banach space E is a *cosine function* if the following conditions are satisfied

1. $C(0) = I$,
2. $C(s+t) + C(s-t) = 2\,C(s)C(t)$ for every $s, t \geq 0$.

A cosine function is said to be *strongly continuous* (or a C_0-*cosine function*) if for every $t_0 \geq 0$ and $f \in E$

$$\lim_{t\to t_0} \|C(t)f - C(t_0)f\| = 0\ .$$

The infinitesimal generator of a strongly continuous cosine function is the linear operator $A : D(A) \to E$ defined by

$$A(f) := \lim_{t\to 0} \frac{C(t)u - 2u + C(-t)u}{h^2}$$

on

$$D(A) := \left\{ u \in E \mid \lim_{t\to 0} \frac{C(t)u - 2u + C(-t)u}{h^2} \in E \right\}\ .$$

If a linear operator $A : D(A) \to E$ generates a strongly continuous cosine function $(C(t))_{t\in\mathbb{R}}$, then the solution of the second-order hyperbolic problem

$$\begin{cases} \dfrac{\partial^2}{\partial t^2} u(t,x) = Au(t,x)\ , & t \in \mathbb{R}\ ; \\ u(0,x) = u_0(x)\ , & x \in \mathbb{R}\ ; \\ \dfrac{\partial}{\partial t} u(t,x)|_{t=0} = u_1(x)\ , & x \in \mathbb{R}\ . \end{cases}$$

is given by

$$u(t,x) = C(t)u_0(x) + \int_0^t C(s)u_1(x)\,ds\ .$$

If $(C(t))_{t\in\mathbb{R}}$ is a cosine function on E, there exist $\omega \in \mathbb{R}$ and $C_0 \geq 0$ such that

$$\|C(t)\| \leq C_0\, e^{\omega\,|t|}.$$

For every $\mathrm{Re}\,\lambda > \omega$, we have $\lambda^2 \in \rho(A)$ and

$$R(\lambda^2, A)u = \frac{1}{\lambda} \int_0^{+\infty} e^{-\lambda\, t}\, C(t)u\, dt\ , \qquad u \in E\ .$$