Chapter 5

Parabolic Schrödinger operators

In this chapter we consider the parabolic Schrödinger operator

$$\mathcal{A} = \partial_t - \Delta + V \quad \text{on} \quad \mathbb{R}^{N+1}$$

where $V = V(x,t)$ is a nonnegative potential which belongs to the parabolic Reverse Hölder class B_p for some $p > 1$. Examples of such potentials are all polynomials but also singular functions like $\max\{|x|, t^{\frac{1}{2}}\}^\alpha$ for $\alpha > -\frac{N+2}{p}$. We prove the L^p boundedness of the operators $D^2(\partial_t - \Delta + V)^{-1}$, $V(\partial_t - \Delta + V)^{-1}$ and $\partial_t(\partial_t - \Delta + V)^{-1}$, thus characterizing the domain of the operator $\mathcal{A}$ on $L^p(\mathbb{R}^{N+1})$.
The wide literature on the characterization of the domain of (elliptic) Schrödinger operator can be divided in two classes, concerning the assumptions on the potential V. The equality $D(-\Delta + V) = D(-\Delta) \cap D(V)$ holds in $L^p(\mathbb{R}^N)$, $1 < p < \infty$ either assuming an oscillation condition like $|\nabla V| \leq cV^{3/2}$, see [37], or assuming that V belongs to suitable Reverse Hölder classes. The two conditions are incomparable but one find easily examples of polynomials (which satisfy a reverse Hölder inequality) for which the oscillation condition above fails.
In [41] Shen proved the L^p boundedness of $D^2(-\Delta+V)^{-1}$ on $\mathbb{R}^N$ for $1 < p < \infty$, assuming $V \in B_p$ and under the restrictions $N \geq 3$, $p \geq \frac{N}{2}$, introducing an auxiliary function $m(x, V)$, which is well defined for $p \geq \frac{N}{2}$ and allows to estimate the fundamental solution.
In a recent work, P. Auscher and B. Ali , see [3], extended Shen's result removing the original restrictions on the space dimension and on p. In their proof they use a criterion to prove L^p boundedness of operators in absence of kernels, see [42, Theorem 3.1], [2, Theorem 3.14], and weighted mean value inequalities for nonnegative subharmonic functions, with respect to Muckenhoupt weights. Following Shen's approach, W. Gao and Y. Jiang extended the results to the parabolic case. In [18], they consider the parabolic operator $\partial_t - \Delta + V$ where

$V \in B_p$ is a nonnegative potential depending only on the space variables and, under the assumptions $N \geq 3$ and $p > (N+2)/2$, they prove the boundedness of $V(\partial_t - \Delta + V)^{-1}$ in L^p.
We obtain the L^p boundedness of $V\mathcal{A}^{-1}$ (and consequently of $\partial_t \mathcal{A}^{-1}$ and $D^2\mathcal{A}^{-1}$) if $0 \leq V \in B_p$ for $1 < p < \infty$, without any restriction on the space dimension; moreover, our potentials may also depend on the time variable. Our approach is similar to that of [3]. We use a more general version of the boundedness criterion in absence of kernels in homogeneous spaces (see Theorem D.1.1) and the Harnack inequality for subsolutions of the heat equation. A crucial role is played by some properties of the B_p weights, originally proved in the classical case when $\mathbb{R}^N$ is equipped with the Lebesgue measure and the Euclidean distance. Since we need parabolic cylinders instead of balls of $\mathbb{R}^N$, we use the more general theory of B_p weights in homogeneous spaces, as treated in [48, Chapter I].
The chapter is organized as follows.
In Section 5.1 we introduce the reverse Hölder classes B_p and the Muckenhoupt classes A_p. We state some properties satisfied by these weights and we establish a relation between the two classes.
In Section 5.2 we define the parabolic Schrödinger operator in $L^p(\mathbb{R}^{N+1})$ and we prove some properties, in particular invertibility and consistency of the resolvent operators.
We start the last section by observing that $V\mathcal{A}^{-1}$ is always bounded in L^1. Then, using the Harnack inequality for subsolutions of the heat equation and an approximation procedure, we prove a weighted mean value inequality for positive solutions of the equation $\mathcal{A}u = 0$ with respect to B_p weights which allows us to apply Shen's interpolation theorem and deduce the boundedness of $V\mathcal{A}^{-1}$ in L^p.

For the whole chapter we fix the following notation.
Notation
Given $X_0 = (x_0^1, ..., x_0^N, t_0)$, $R > 0$, with parabolic cylinder of center $X_0 = (x_0, t_0)$ and radius R we mean the set

$$K = K(X_0, R) = \{(x^1, ..., x^N, t) \in R^{N+1} : |x^i - x_0^i| < R,\ |t - t_0| < R^2\}.$$

5.1 The parabolic reverse Hölder classes

The classical theory about Muckenhoupt and reverse Hölder classes has been originately formulated for weights in $\mathbb{R}^N$ endowed with the euclidean distance, see for example [47, Chapter V]. We will consider however potentials satisfying the "Reverse Hölder Property" with respect to cylinders rather than Euclidean balls. Many properties remain true in this setting. A theory on these classes of weights in homogeneous spaces (like $\mathbb{R}^{N+1}$ with the parabolic distance) is presented for example in [48, Chapter I] to which we refer for the proofs of the results stated in this Section and needed in what follows.

Definition 5.1.1. *Let $1 < p \leq \infty$. We say that $\omega \in B_p$, the class of the reverse Hölder weights of order p, if $\omega \in L^p_{loc}$, $\omega > 0$ a.e. and there exists a positive constant C such the inequality*

$$\left(\frac{1}{|K|} \int_K \omega(x,t)^p \, dx \, dt \right)^{\frac{1}{p}} \leq \frac{C}{|K|} \int_K \omega(x,t) \, dx \, dt \tag{5.1}$$

holds, for every parabolic cylinder K. If $p = \infty$, the left hand side of the inequality above has to be replaced by the essential supremum of ω on K. The smallest positive constant C such that (5.1) holds is the B_p constant of ω.

Observe that $B_q \subset B_p$ if $p < q$. An important feature of the B_p weights is the following self improvement property due to Gehring.

Proposition 5.1.2. *Assume that $\omega \in B_p$ for some $p < \infty$. Then there exists $\varepsilon > 0$, depending on the B_p constant of ω, such that $\omega \in B_{p+\varepsilon}$.*

The following property connects B_p weights with Muckenhoupt classes. In particular it implies that B_p weights induce doubling measures.

Definition 5.1.3. *Let $1 < p < \infty$. We say that $\omega \in A_p$ if it is nonnegative and it satisfies the inequality*

$$\frac{1}{|K|} \int_K \omega(x,t) dx \, dt \left[\frac{1}{|K|} \int_K \omega(x,t)^{-\frac{p'}{p}} \right] \leq A < \infty$$

for all K parabolic cylinders and some positive constant A.
The space A_1 consists of nonnegative functions ω such that

$$\frac{1}{|K|} \int_K \omega(x,t) dx \, dt \leq A\omega(x,t)$$

for almost every $(x,t) \in K$, for all K parabolic cylinders and some positive constant A.
In both cases, the smallest constant for which the inequality holds is called the A_p bound of ω.

Proposition 5.1.4. *If $\omega \in B_p$ for some $p > 1$, then there exists $1 \leq t < \infty$ and $c > 0$, depending on p and the B_p constant of ω, such that the inequality*

$$\left(\frac{1}{|K|} \int_K g \right)^t \leq \frac{c}{\omega(K)} \int_K g^t \omega \tag{5.2}$$

holds for all nonnegative functions g and all parabolic cylinders K. Here $\omega(K) = \displaystyle\int_K \omega$.

Remark 5.1.5. It is possible to prove that ω satisfies (5.2) is equivalent to say that $\omega \in A_t$ (see [47, Chapter V, 1.4]).

It is not hard to see that all polynomials belong to the reverse Hölder classes. The idea is that the space of all polynolmials of a fixed degree is a finite dimension space. Therefore all the norms are equivalent and the reverse Hölder inequality holds with a constant depending only on the degree of the polynomial and on N for all the cylinders with unitary radius. Up a rescaling the inequality follows for all the cylinders in $\mathbb{R}^{N+1}$. Also singular functions like $\max\{|x|, t^{\frac{1}{2}}\}^\alpha$ for $\alpha > -\frac{N+2}{p}$ belong to B_p. Here we give a proof.

Example 5.1.6. *The functions* $\max\{|x|, t^{\frac{1}{2}}\}^\alpha$ *belong to* B_p *for* $\alpha > -\frac{N+2}{p}$.

Proof. Observe that it is sufficient to prove the inequality for parabolic cylinders of unitary radius. A change of variables provides the estimate in the general case.
The hypothesis $\alpha > -\frac{N+2}{p}$ insures integrability near 0. Note that $f(x,t) = \max\{|x|, t^{\frac{1}{2}}\}^\alpha = d(x,0)^\alpha$ where d is the parabolic distance. Let $K(X_0, 1)$ be a parabolic cylinder of center X_0 and radius 1. Set

$$M = \max\left\{ \left(\int_{K(X_0,1)} f(X)^p \right)^{\frac{1}{p}} \left(\int_{K(X_0,1)} f(X) \right)^{-1}, \, X_0 : d(X_0, 0) \leq 2 \right\}.$$

Suppose $d(X_0, 0) > 2$. If $X \in K(X_0, 1)$ we have

$$\frac{d(X,0)}{d(X_0,0)} \leq \frac{d(X - X_0, 0)}{d(X_0, 0)} + \frac{d(X_0,0)}{d(X_0,0)} \leq 1 + \frac{1}{d(X_0,0)} \leq \frac{3}{2}$$

and

$$\frac{d(X,0)}{d(X_0,0)} \geq \frac{d(X_0,0)}{d(X_0,0)} - \frac{d(X - X_0, 0)}{d(X_0,0)} \geq 1 - \frac{1}{2} = \frac{1}{2}.$$

Therefore if $d(X_0, 0) > 2$

$$\frac{1}{2} \leq \frac{d(X,0)}{d(X_0,0)} \leq \frac{3}{2}$$

and

$$\left(\int_{K(X_0,1)} f(X)^p \right)^{\frac{1}{p}} \leq \left(\frac{3}{2} d(X_0, 0) \right)^\alpha = \left(\frac{3}{2} \right)^\alpha \int_{K(X_0,1)} f(X_0)$$
$$\leq 3^\alpha \int_{K(X_0,1)} f(X).$$

The reverse Hölder inequality is true with B_p constant given by the maximum between M and 3^α. □

5.2 Definition of the operator and some properties

In this section we assume that $0 \leq V \in L^p_{loc}$ for some $1 \leq p \leq \infty$ and consider the parabolic operator

$$\mathcal{A} = \partial_t - \Delta + V$$

in L^p, endowed with the maximal domain

$$D_p(\mathcal{A}) = \{u \in L^p \ : \ Vu \in L^1_{loc},\ \mathcal{A}u \in L^p\}.$$

Observe that C_c^∞ is contained in $D_p(\mathcal{A})$, since $V \in L^p_{loc}$. In some results, however, we shall only assume $0 \le V \in L^1_{loc}$.
We shall prove that $\mathcal{A}_p := (\mathcal{A}, D_p(\mathcal{A}))$ is a closed operator, that C_c^∞ is a core and that $\lambda + \mathcal{A}$ is invertible for positive λ. We follow Kato's strategy, see [19], where these results are obtained in the elliptic case.
Our main result is the following.

Theorem 5.2.1. *For every $\lambda > 0$ the operator $\lambda + \mathcal{A}_p$ is invertible and $\|(\lambda + \mathcal{A})^{-1}\|_p \le \frac{1}{\lambda}$. Moreover, if $1 \le p < \infty$, C_c^∞ is a core for $\mathcal{A}_p$*

The basic tool is a distributional inequality proved by Kato for the laplacian (see [39, Theorem X.2]). For completeness we provide here a short proof in the parabolic case.

Lemma 5.2.2 (Parabolic Kato's inequality)**.** *Let $u \in L^1_{loc}$ be such that $(\partial_t - \Delta)u \in L^1_{loc}$. Define*

$$\operatorname{sign}(u) = \begin{cases} 0 & if \quad u(x) = 0 \\ \overline{u(x)}/|u(x)| & if \quad u(x) \neq 0. \end{cases}$$

Then $|u|$ satisfies the following distributional inequality

$$(\partial_t - \Delta)|u| \le \operatorname{Re}[\operatorname{sign}(u)(\partial_t - \Delta)u].$$

Proof. We first suppose that $u \in C^\infty$. Define

$$u_\varepsilon(x) = \sqrt{|u|^2 + \varepsilon^2} \tag{5.3}$$

so that $u_\varepsilon \in C^\infty$. Since

$$u_\varepsilon \nabla u_\varepsilon = \operatorname{Re}[\overline{u} \nabla u]. \tag{5.4}$$

and $u_\varepsilon \ge |u|$, then (5.4) implies that

$$|\nabla u_\varepsilon| \le |\overline{u}||u_\varepsilon|^{-1}|\nabla u| \le |\nabla u|. \tag{5.5}$$

Taking the divergence of (5.4) we obtain

$$u_\varepsilon \Delta u_\varepsilon + |\nabla u_\varepsilon|^2 = \operatorname{Re}(\overline{u}\Delta u) + |\nabla u|^2$$

so by (5.5)

$$\Delta u_\varepsilon \ge \operatorname{Re}[\operatorname{sign}_\varepsilon(u)\Delta u], \tag{5.6}$$

where $\operatorname{sign}_\varepsilon(u) = \overline{u}/u_\varepsilon$. Differentiating (5.3) with respect to t we obtain

$$\partial_t u_\varepsilon = \operatorname{Re}[\operatorname{sign}_\varepsilon(u)\partial_t u] \tag{5.7}$$

and, combining (5.6) and (5.7),

$$(\partial_t - \Delta)u_\varepsilon \leq \mathrm{Re}[\mathrm{sign}_\varepsilon(u)(\partial_t - \Delta)u]. \tag{5.8}$$

Let now $u \in L^1_{loc}$ be such that $(\Delta - \partial_t)u \in L^1_{loc}$ and let ϕ_n be an approximate identity. Since $u^n = u * \phi_n \in C^\infty$, then by (5.8)

$$(\partial_t - \Delta)(u^n)_\varepsilon \leq \mathrm{Re}[\mathrm{sign}_\varepsilon(u^n)(\partial_t - \Delta)u^n]. \tag{5.9}$$

Fix $\varepsilon > 0$ and let $n \to \infty$. Then $u^n \to u$ in L^1_{loc} and a.e. (passing to a subsequence, if necessary). Thus $\mathrm{sign}_\varepsilon(u^n) \to \mathrm{sign}_\varepsilon(u)$ a.e. Since $(\partial_t - \Delta)u^n = ((\partial_t - \Delta)u) * \phi_n$ and $(\partial_t - \Delta)u \in L^1_{loc}$, then $(\partial_t - \Delta)u^n \to (\partial_t - \Delta)u$ in L^1_{loc}, too. It is now easy to see that $\mathrm{sign}_\varepsilon(u^n)(\partial_t - \Delta)u^n$ converges in the sense of distributions to $\mathrm{sign}_\varepsilon(u)(\partial_t - \Delta)u$. Thus, letting $n \to \infty$ in (5.8) we conclude that

$$(\partial_t - \Delta)u_\varepsilon \leq \mathrm{Re}[\mathrm{sign}_\varepsilon(u)(\partial_t - \Delta)u].$$

Now taking $\varepsilon \to 0$ we obtain the desired inequality for u, since $\mathrm{sign}_\varepsilon(u) \to \mathrm{sign}(u)$ and $|\mathrm{sign}_\varepsilon(u)| \leq 1$. □

Remark 5.2.3. Changing t with $-t$ one obtains that if u, $(\partial_t + \Delta)u \in L^1_{loc}$, then

$$(\partial_t + \Delta)|u| \leq \mathrm{Re}[\mathrm{sign}(u)(\partial_t + \Delta)u].$$

The following results are easy consequences of Kato's inequality.

Lemma 5.2.4. *Let $0 \leq V \in L^1_{loc}$. Assume that $u, (\partial_t - \Delta)u, Vu \in L^1_{loc}$ and set, for $\lambda \geq 0$, $f = (\lambda + \mathcal{A})u$. Then*

$$(\lambda + \partial_t - \Delta + V)|u| \leq |f|. \tag{5.10}$$

Proof. The claim immediately follows by Lemma 5.2.2. Indeed

$$(\lambda + \partial_t - \Delta + V)|u| \leq \mathrm{Re}[\mathrm{sign}(u)((\partial_t - \Delta)u + \lambda u + Vu)] = \mathrm{Re}[f\,\mathrm{sign}(u)] \leq |f|.$$

□

Lemma 5.2.5. *For every positive $\lambda > 0$ the operator $(\lambda + \partial_t - \Delta)^{-1}$ is a positive map of $\mathcal{S}'$ onto itself.*

Proof. Since $\lambda - \partial_t - \Delta$ is invertible from $\mathcal{S}$ onto $\mathcal{S}$, its adjoint operator $\lambda + \partial_t - \Delta$ is invertible from $\mathcal{S}'$ into itself. Let now $0 \leq \psi \in \mathcal{S}'$ and let $\phi \in \mathcal{S}'$ be such that $0 \leq \psi = (\lambda + \partial_t - \Delta)\phi$. If $0 \leq u \in \mathcal{S}$, then

$$\langle \phi, u \rangle = \langle (\lambda + \partial_t - \Delta)^{-1}(\lambda + \partial_t - \Delta)\phi, u \rangle = \langle (\lambda + \partial_t - \Delta)\phi, (\lambda - \partial_t - \Delta)^{-1}u \rangle \geq 0$$

since $(\lambda - \partial_t - \Delta)^{-1}$ is positive on $\mathcal{S}$, by the maximum principle. This proves that $\phi = (\lambda + \partial_t - \Delta)^{-1}\psi$ is positive. □

An estimate for the resolvent operator easily follows.

Proposition 5.2.6. *Let $1 \le p \le \infty$, $\lambda > 0$. Then, if $u \in D_p(\mathcal{A})$,*

$$\lambda\|u\|_p \le \|(\lambda + \mathcal{A})u\|_p. \tag{5.11}$$

PROOF. Let $u \in D_p(\mathcal{A})$, set $f = (\lambda + \mathcal{A})u \in L^p$. By (5.10)

$$(\lambda + \partial_t - \Delta)|u| \le (\lambda + \mathcal{A})|u| \le |f|$$

and Lemma 5.2.5 yields

$$|u| \le (\lambda + \partial_t - \Delta)^{-1}|f|. \tag{5.12}$$

Then

$$\|u\|_p \le \|(\lambda + \partial_t - \Delta)^{-1}|f|\|_p \le \frac{1}{\lambda}\|f\|_p.$$

□

The positivity of the resolvent is proved along the same way.

Proposition 5.2.7. *Let $0 \le V \in L^1_{loc}$ and $\lambda > 0$. If $u, (\partial_t - \Delta)u, Vu \in L^1_{loc}$ and $f = (\lambda + \mathcal{A})u \ge 0$, then $u \ge 0$.*

PROOF. Subtracting the equality $f = (\lambda + \mathcal{A})u \ge 0$ from (5.10) we obtain $(\lambda + \partial_t - \Delta + V)(|u| - u) \le 0$, hence $(\lambda + \partial_t - \Delta)(|u| - u) \le 0$. Lemma 5.2.5 implies $|u| - u \le 0$ so that $u = |u|$. □

Proposition 5.2.8. *For every $1 \le p \le \infty$, the operator $\mathcal{A}_p$ is closed. Moreover, if $\lambda > 0$, $\lambda + \mathcal{A}_p$ has closed range.*

PROOF. Let $(u_n) \subset D_p(\mathcal{A})$ such that

$$u_n \to u, \quad \mathcal{A}u_n = (\partial_t - \Delta)u_n + Vu_n = f_n \to f \text{ in } L^p.$$

We apply (5.10) to $u = u_n - u_m$, $f = f_n - f_m$ and $\lambda = 0$ obtaining

$$(\partial_t - \Delta + V)|u_n - u_m| \le |f_n - f_m|.$$

Then, for every $0 \le \phi \in C_c^\infty$

$$0 \le \langle V|u_n - u_m|, \phi\rangle \le \langle |f_n - f_m|, \phi\rangle + \langle |u_n - u_m|, (\Delta + \partial_t)\phi\rangle.$$

Letting n, m to infinity, the right hand side of the previous inequality tends to 0 and this shows that $Vu_n\phi$ is a Cauchy sequence in L^1. Since its limit is $Vu\phi$ we conclude (by the arbitrariness of ϕ) that $Vu \in L^1_{loc}$ and that $Vu_n \to Vu$ in L^1_{loc}. Then $f_n = (\partial_t - \Delta + V)u_n \to (\partial_t - \Delta + V)u$ in the sense of distributions. On the other hand $f_n \to f$ in L^p, therefore $u \in D_p(\mathcal{A})$ and $f = (\partial_t - \Delta + V)u \in L^p$. This proves the closedness of $\mathcal{A}$.
Finally, $\lambda + \mathcal{A}$ has closed range, by (5.11). □

PROOF (Theorem 5.2.1). Assume first that $1 \le p < \infty$. Since $\mathcal{A}_p$ is closed and has closed range, we have only to prove that $(\lambda + \mathcal{A})(C_c^\infty)$ is dense in L^p.

Let $u \in L^{p'}$ such that $\int(\lambda+\partial_t-\Delta+V)\phi u=0$ for every $\phi \in C_c^\infty$. We have to show that $u=0$. Evidently u satisfies $\lambda u-\partial_t u-\Delta u+Vu=0$ in the sense of distributions and, since $V \in L^p_{loc}$ and $u \in L^{p'}$, $Vu \in L^1_{loc}$. Thus $u \in D_{p'}(\mathcal{B})$ and $(\lambda+\mathcal{B})u=0$, where $\mathcal{B}=-\partial_t-\Delta+V$. The injectivity of $\lambda+\mathcal{B}$ (that follows from Proposition 5.2.6 changing t to $-t$) implies $u=0$ and proves the density of $(\lambda+\mathcal{A})(C_c^\infty)$ in L^p.

Next we consider the case where $p=\infty$. Let $0 \le f \in L^\infty$ and consider a sequence $f_n \in L^\infty \cap L^1$ such that $0 \le f_n \nearrow f$. By the first part of the proof, there are $u_n \in D_1(\mathcal{A})$ such that $(\lambda+\mathcal{A})u_n=f_n$. By Proposition 5.2.7 the sequence (u_n) is increasing and consists of nonnegative functions and, since $\lambda\|u_n\|_\infty \le \|f_n\|_\infty \le \|f\|_\infty$, its (pointwise) limit u belongs to L^∞. Moreover $Vu_n \to Vu$ in L^1_{loc} because $V \in L^\infty_{loc}$ and $u_n \to u$, $0 \le u_n \le u$. Hence $f_n=(\lambda+\mathcal{A})u_n \to (\lambda+\partial_t-\Delta)u+Vu$ in the sense of distributions. But $f_n \to f$ monotonically and then $(\lambda+\mathcal{A})u=f$. This means that $u \in D_\infty(\mathcal{A})$ and $(\lambda+\mathcal{A})u=f$. Since a general $f \in L^\infty$ is a linear combination of positive elements, the proof is complete. $\square$

Finally, we prove the consistency of the resolvent operators.

Proposition 5.2.9. *Let $1 \le p \le q$ and $0 \le V \in L^q_{loc}$. If $\lambda>0$ and $f \in L^p \cap L^q$, then $(\lambda+\mathcal{A}_p)^{-1}f=(\lambda+\mathcal{A}_q)^{-1}f$.*

Proof. Let $u=(\lambda+\mathcal{A}_p)^{-1}f$, $v=(\lambda+\mathcal{A}_q)^{-1}f$ and $w=u-v$. Then $w, Vw \in L^1_{loc}$ and $(\partial_t-\Delta)w=-(\lambda+V)w \in L^1_{loc}$. Since $(\lambda+\mathcal{A})w=0$, by Proposition 5.2.7 we deduce that $w=0$. $\square$

5.3 Characterization of the domain of $\mathcal{A}$

In this section we assume that all functions are real-valued.

5.3.1 The operator $\mathcal{A}$ on L^1.

It is easy to obtain a-priori estimates for $p=1$, leading to a (partial) description of $D_1(\mathcal{A})$. They will also play a key role in the proof of the a-priori estimates in L^p.

Lemma 5.3.1. *Assume that $0 \le V \in L^1_{loc}$. For every $u \in D_1(\mathcal{A})$ we have*

$$\|Vu\|_1 \le \|\mathcal{A}u\|_1, \quad \|(\partial_t-\Delta)u\|_1 \le 2\|\mathcal{A}u\|_1. \tag{5.13}$$

Proof. Let $h_n:\mathbb{R}\to\mathbb{R}$ be a sequence of smooth functions such that $|h_n| \le C$, $h_n'(s) \ge 0$ and $h_n(s) \to \mathrm{sign}(s)$ for $n \to \infty$ and for every $s \in \mathbb{R}$. Let H_n be such that $H_n'=h_n$ and $H_n(0)=0$. If $u \in C_c^\infty$ then, by the Lebesgue convergence Theorem, we have

$$\int_{\mathbb{R}^{N+1}} \mathrm{sign}(u)\partial_t u=\lim_n \int_{\mathbb{R}^{N+1}} h_n(u)\partial_t u=\lim_n \int_{\mathbb{R}^{N+1}} \partial_t(H_n(u))=0, \tag{5.14}$$

$$-\int_{\mathbb{R}^{N+1}} \operatorname{sign}(u)\Delta u = -\lim_n \int_{\mathbb{R}^{N+1}} h_n(u)\Delta u = \lim_n \int_{\mathbb{R}^{N+1}} |\nabla u|^2 h_n'(u) \geq 0. \tag{5.15}$$

Therefore, if $\mathcal{A}u = f$ we obtain

$$\int_{\mathbb{R}^{N+1}} V|u| \leq \int_{\mathbb{R}^{N+1}} \operatorname{sign}(u)(\partial_t - \Delta + V)u = \int_{\mathbb{R}^{N+1}} f \operatorname{sign}(u) \leq \int_{\mathbb{R}^{N+1}} |f|$$

and the first inequality is proved for $u \in C_c^\infty$. Since C_c^∞ is a core for $\mathcal{A}_1$ it is easily seen that it extends to every $u \in D_1(\mathcal{A})$.

The second inequality follows from the first, since $(\partial_t - \Delta) = \mathcal{A} - V$. □

The characterization of the domain of $\mathcal{A}_1$ is an immediate consequence of the lemma above. We refer to [50] for similar results in the elliptic case.

Proposition 5.3.2. *If $0 \leq V \in L^1_{loc}$, then*

$$D_1(\mathcal{A}) = \{u \in L^1 \ : \ Vu \in L^1, \ (\partial_t - \Delta)u \in L^1\}.$$

5.3.2 A priori estimates in $L^p(\mathbb{R}^{N+1})$.

We investigate when (5.13) holds for other values of p. We remark that (5.13) can fail even for $p = 2$ and in the elliptic case, see e.g. [31, Example 3.7].
The B_p property of the potential is a sufficient condition to characterize the domain of the operator. In fact we prove the following result.

Theorem 5.3.3. *Let $1 < p < \infty$. If $0 \leq V \in B_p$ then there exists a positive constant C depending only on p and the B_p constant of V, such that*

$$\|Vu\|_p \leq C\|\partial_t u - \Delta u + Vu\|_p \tag{5.16}$$

for all $u \in D_p(\mathcal{A})$. In particular,

$$D_p(\mathcal{A}) = \{u \in W_p^{2,1} : Vu \in L^p\}.$$

We will apply Theorem D.1.1 to the operator $T = V\mathcal{A}^{-1}|\cdot|$ with $p_0 = 1$, a suitable $q_0 > p$ and $\alpha_1 = 3$, $\alpha_2 = 4$. Therefore we have to prove that, if K is a parabolic cylinder and $f \in L_c^\infty$ has support in $\mathbb{R}^{N+1} \setminus 4K$, $u = \mathcal{A}^{-1}f$ satisfies

$$\left(\frac{1}{|K|}\int_K (V|u|)^{q_0}\right)^{\frac{1}{q_0}} \leq \frac{C}{|3K|}\int_{3K} V|u|$$

for some positive C independent of f. Observe that u satisfies the homogeneous equation

$$\mathcal{A}u = (\partial_t - \Delta + V)u = 0$$

in $4K$. As first step we prove a mean value inequality for functions u as above.

Lemma 5.3.4. *Assume that $0 < \varepsilon \leq V \in L^p_{loc}$. For every $r > 0$ there exists a positive constant $C = C(r)$ (hence independent of ε) such that*

$$\sup_K u \leq C\left(\frac{1}{|3K|}\int_{3K} u^r\right)^{\frac{1}{r}}$$

for all parabolic cylinders K, $0 \leq f \in L_c^\infty(\mathbb{R}^{N+1})$ with support in $\mathbb{R}^{N+1} \setminus 4K$ and $u = \mathcal{A}^{-1}f$.

PROOF. Let $K = K((x_0, t_0), R)$ a parabolic cylinder and $0 \leq f \in L_c^\infty(\mathbb{R}^{N+1})$ with support in $\mathbb{R}^{N+1} \setminus 4K$. By Theorem 5.2.1 there exists $u \in D_p(\mathcal{A})$ such that

$$\mathcal{A}u = f \quad \text{in } \mathbb{R}^{N+1}.$$

By Proposition 5.2.7 $u \geq 0$. We are going to use Harnack's inequality where, however, more regularity on the solutions is required and then an approximation procedure is needed. Let $\mathcal{A}_k$ be the operators with bounded potentials $V_k = V \wedge k$. For every k let $0 \leq u_k$ be such that $(\partial_t - \Delta + V_k)u_k = f$. The functions u_k are solutions of parabolic equations with bounded coefficients, then for all $k \in \mathbb{N}$ $u_k \in W_q^{2,1}(\mathbb{R}^{N+1})$ for all $1 < q < \infty$. Since f has support in $\mathbb{R}^{N+1} \setminus 4K$,

$$(\partial_t - \Delta)u_k = -V_k u_k \leq 0 \quad \text{in} \quad 4K.$$

Given a parabolic cylinder $K = K((x_0, t_0), R)$ and a positive constant $c > 0$, we denote by cK the cylinder with the same center as K and radius cR and by $\widetilde{K}$ the set $K \cap \{t < t_0\}$.
Let K_1 be the cylinder of center $(x_0, t_0 + R^2)$ and radius $\sqrt{2}R$. Obviously $K \subset \widetilde{K_1}$ and $\widetilde{2K_1} \subset 2K_1 \subset 3K \subset 4K$. It follows that

$$(\partial_t - \Delta)u_k = -V_k u_k \leq 0 \quad \text{in} \quad \widetilde{2K_1}.$$

By [24, Theorem 7.21] or see [35], for any $r > 0$ there exists $C = C(r) > 0$ such that

$$\sup_{\widetilde{K_1}} u_k \leq C \left(\frac{1}{R^{n+2}} \int_{\widetilde{2K_1}} u_k^r \right)^{\frac{1}{r}}$$

and hence

$$\begin{aligned} \sup_K u_k \leq \sup_{\widetilde{K_1}} u_k &\leq C \left(\frac{1}{R^{n+2}} \int_{\widetilde{2K_1}} u_k^r \right)^{\frac{1}{r}} \leq C \left(\frac{1}{R^{n+2}} \int_{3K} u_k^r \right)^{\frac{1}{r}} \\ &= C \left(\frac{1}{|3K|} \int_{3K} u_k^r \right)^{\frac{1}{r}}. \end{aligned} \tag{5.17}$$

Let us observe that the constant C is independent of the potential V_k. This allows us to let $k \to \infty$ in the above inequality.
Let $k,\ m \in \mathbb{N}$ with $k > m$. Then

$$\partial_t(u_k - u_m) - \Delta(u_k - u_m) + V_k(u_k - u_m) = (V_m - V_k)u_m \leq 0$$

and by Proposition 5.2.7 (or simply by the maximum principle) $u_k - u_m \leq 0$. Therefore (u_k) is decreasing and converges pointwise to a function $w \geq 0$. Moreover, by Lemma 5.3.1, $\|V_k u_k\|_1 \leq \|f\|_1$ for every $k \in \mathbb{N}$ and then, by Fatou's Lemma, $Vw \in L^1$. By Proposition 5.2.6, $\|u_k\|_q \leq C\|f\|_q$ for all $1 \leq$

$q \leq \infty$ and, since $u_k \to w$ pointwise, $w \in L^q$ for all $1 \leq q \leq \infty$.
Since for every $\phi \in C_c^\infty$

$$\int_{\mathbb{R}^{N+1}} u_k(-\partial_t \phi - \Delta\phi + V_k\phi) = \int_{\mathbb{R}^{N+1}} f\phi,$$

letting k to infinity we get

$$\int_{\mathbb{R}^{N+1}} w(-\partial_t \phi - \Delta\phi + V\phi) = \int_{\mathbb{R}^{N+1}} f\phi$$

and therefore $\mathcal{A}w = f$ in the sense of distributions. This shows that w belongs to $D_p(\mathcal{A})$ and, by Theorem 5.2.1, $w = u$, that is u_k converges to u pointwise. Since u_k is decreasing, (5.17) yields

$$\sup_K u \leq \sup_K u_k \leq C\left(\frac{1}{|3K|}\int_{3K}(u_k)^r\right)^{\frac{1}{r}}. \tag{5.18}$$

Finally, u_k is decreasing, therefore $u_k^r \leq u_1^r \in L^1$ and letting $k \to \infty$ in (5.18) we obtain the thesis by dominated convergence. □

Now we prove that Lemma 5.3.4 holds if we replace the Lebesgue measure with that induced by the density V.

Lemma 5.3.5. *Suppose* $0 < \varepsilon \leq V \in B_p$ *and fix* $0 < s < \infty$ *and* u *as in Lemma 5.3.4. Then for every cylinder* K

$$\sup_K u \leq \left(\frac{C}{V(3K)}\int_{3K} Vu^s\right)^{\frac{1}{s}}$$

where C *depends only on* s, p *and the* B_p *constant of* V *and*

$$V(3K) = \int_{3K} V.$$

Proof. Let $0 < s < \infty$ and K be a parabolic cylinder of $\mathbb{R}^{N+1}$. We fix t as in Proposition 5.1.4. By using Lemma 5.3.4 with $r = \frac{s}{t}$ and (5.2) we obtain

$$\sup_K u \leq C\left(\frac{1}{|3K|}\int_{3K} u^{\frac{s}{t}}\right)^{\frac{t}{s}} \leq C\left(\frac{1}{V(3K)}\int_{3K} Vu^s\right)^{\frac{1}{s}}.$$

□

By combining the estimate in Lemma 5.3.5 and the B_q property we deduce the following.

Corollary 5.3.6. *Let* $0 < \varepsilon \leq V \in B_p$, $0 < s < \infty$ *and* u *as in Lemma 5.3.4. Then for every cylinder* K

$$\left(\frac{1}{|K|}\int_K (Vu^s)^p\right)^{\frac{1}{p}} \leq \frac{C}{|3K|}\int_{3K} Vu^s,$$

where C *depends only on* s, p *and the* B_p *constant of* V.

Proof. By using the B_p property of V and Lemma 5.3.5 we obtain

$$\left(\frac{1}{|K|}\int_K (Vu^s)^p\right)^{\frac{1}{p}} \le \left(\frac{1}{|K|}\int_K V^p\right)^{\frac{1}{p}} \sup_K u^s \le C\left(\frac{1}{|K|}\int_K V\right)\sup_K u^s$$
$$\le \frac{C}{|3K|}\int_{3K} Vu^s.$$

□

We can now prove our main result.

Proof (Theorem 5.3.3). Suppose first that $0 < \varepsilon \le V \in B_p$ for some ε. By Proposition 5.1.2 there exists $q_0 > p$ such that $V \in B_{q_0}$.
Let K be a parabolic cylinder in $\mathbb{R}^{N+1}$ and $f \in L^\infty_c(\mathbb{R}^{N+1})$ with support in $\mathbb{R}^{N+1} \setminus 4K$. We set $T = V\mathcal{A}^{-1}|\cdot|$. Then $Tf = Vu$ and $u \ge 0$ by Proposition 5.2.7. Note that, since $V \ge \varepsilon > 0$, Proposition 5.2.9 shows that T acts in a consistent way in the L^q scale. By Corollary 5.3.6 with $s = 1$,

$$\left(\frac{1}{|K|}\int_K (Tf)^{q_0}\right)^{\frac{1}{q_0}} = \left(\frac{1}{|K|}\int_K (Vu)^{q_0}\right)^{\frac{1}{q_0}} \le \frac{C}{|3K|}\int_{3K} Vu = \frac{C}{|3K|}\int_{3K} |Tf|.$$

By Lemma 5.3.1 T is bounded on L^1 and, by Proposition 5.2.7, it is also sublinear. Choosing $p_0 = 1$ and q_0 as above in Theorem D.1.1, we deduce that

$$\|Vu\|_p = \|Tf\|_p \le C\|f\|_p \tag{5.19}$$

for every $f \in L^\infty_c$, where C depends only on p and the B_p constant of V. Since, by Proposition 5.2.7 again, the operator $V\mathcal{A}^{-1}$ preserves positivity, we have that $|V\mathcal{A}^{-1}f| \le Tf$. Therefore by 5.19 we deduce that

$$\|V\mathcal{A}^{-1}f\|_p \le C\|f\|_p$$

for every $f \in L^\infty_c$ and finally, by approximation, for every $f \in L^p$. Then the identity

$$(\partial_t - \Delta)u = f - Vu \in L^p$$

proves, by parabolic regularity, that the distribution u belongs to $W^{2,1}_p$. Then

$$D_p(\mathcal{A}) \subset \{u \in W^{2,1}_p : Vu \in L^p\}$$

and, since the opposite inclusion is obvious, the characterization of the domain is proved. Now we prove (5.16) in the general case when $V \ge 0$. Let $u \in D_p(A)$. then for every $\varepsilon > 0$ we have

$$\|(V+\varepsilon)u\|_p \le C\|\partial_t u - \Delta u + (V+\varepsilon)u\|_p.$$

Since C depends only on p and the B_p constant of $V + \varepsilon$ which is independent of $0 < \varepsilon \le 1$, letting $\varepsilon \to 0$ the proof is complete. □

Finally we show that the results of this section hold when the time variable varies in an interval, rather than in the whole space. We fix $-\infty \le S < T \le \infty$ and consider the set

$$Q(S,T) = \mathbb{R}^N \times (S,T)$$

and the operator $\mathcal{A}$ endowed with the domain

$$D_p^{S,T} = \left\{u \in W_p^{2,1}\left(Q(S,T)\right) : Vu \in L^p\left(Q(S,T)\right), \quad u(\cdot,S) = 0\right\}.$$

Clearly the initial condition $u(\cdot,S) = 0$ makes sense only when $S > -\infty$.

Proposition 5.3.7. *If $1 < p < \infty$, $0 \le V \in B_p$ and $\lambda > 0$, then the operator $\lambda + \mathcal{A}$ is invertible from $D_p^{S,T}$ to $L^p\left(Q(S,T)\right)$.*

Proof. Given $f \in L^p\left(Q(S,T)\right)$, let $g \in L^p$ be its extension by 0 outside the time interval (S,T) and $u \in D_p(\mathcal{A})$ such that $\lambda u + \mathcal{A}u = g$ in $\mathbb{R}^{N+1}$ (hence in $Q(S,T)$). Since $\lambda u + \mathcal{A}u = 0$ for $t \le S$ (when $S > -\infty$), multiplying this identity by $u|u|^{p-2}$ and integrating by parts we get $u = 0$ for $t \le S$, hence $u(\cdot,S) = 0$ and $u \in D_p^{S,T}$. Infact we have

$$\int_{Q(-\infty,S)} (\lambda + V)|u|^p + \frac{1}{p}\int_{Q(-\infty,S)} \partial_t(|u|^p) - \int_{Q(-\infty,S)} u|u|^{p-2}\Delta u = 0,$$

which implies, since $\int_{Q(-\infty,S)} u|u|^{p-2}\Delta u \le 0$ (see Appendix C),

$$\int_{Q(-\infty,S)} (\lambda + V)|u|^p + \frac{1}{p}\int_{\mathbb{R}^N}\int_{-\infty}^{S} \partial_t(|u|^p) \le 0$$

and then $u = 0$ for $t \le S$. This proves the existence part. Concerning uniqueness, assume that $v \in D_p^{S,T}$ satisfies $\lambda v + \mathcal{A}v = 0$ in $Q_{S,T}$. Multiplying by $v|v|^{p-2}$, integrating by parts as above and using the initial condition one easily shows that $v = 0$. $\square$

As usual, if the interval (S,T) is finite, the condition $\lambda > 0$ in not needed.