Appendix C

Some a priori estimates

The present appendix is devoted to the proof of some a priori estimates involving uniformly elliptic operators. More precisely, we derive a Schauder type parabolic estimate and an L^p elliptic estimate, by making use of classical methods suitably adapted for our purposes. Even though such estimates are well known, we have not found a proof for them exactly in the form we need.

C.1 A Schauder type parabolic estimate

Suppose we are given a second order differential operator

$$\Gamma = \sum_{i,j=1}^{N} a_{ij} D_{ij} + \sum_{i=1}^{N} b_i D_i + c, \tag{C.1.1}$$

whose coefficients $a_{ij} = a_{ji}, b_i, c$ belong to $C^{\frac{\alpha}{2},\alpha}(]0,T[\times\Omega)$, where $\alpha \in]0,1[$, Ω is a bounded open subset of $\mathbb{R}^N$ with $C^{2+\alpha}$ boundary and $T < +\infty$. Assume also that

$$\sum_{i,j=1}^{N} a_{ij}\xi_i\xi_j \geq \nu|\xi|^2, \tag{C.1.2}$$

for some $\nu > 0$. Then the operator $L = D_t - \Gamma$ is uniformly parabolic in $]0,T[\times\Omega$. Set

$$K = \max\left\{\|a_{ij}\|_{C^{\frac{\alpha}{2},\alpha}(]0,T[\times\Omega)}, \|b_i\|_{C^{\frac{\alpha}{2},\alpha}(]0,T[\times\Omega)}, \|c\|_{C^{\frac{\alpha}{2},\alpha}(]0,T[\times\Omega)}\right\},$$

where we recall that

$$\|v\|_{C^{\frac{\alpha}{2},\alpha}(]0,T[\times\Omega)} = \|v\|_\infty + [v]_{C^{\frac{\alpha}{2},\alpha}(]0,T[\times\Omega)}$$

$$[v]_{C^{\frac{\alpha}{2},\alpha}(]0,T[\times\Omega)} = \sup_{t\in]0,T[,\, x,y\in\Omega,\, x\neq y} \frac{|v(t,x)-v(t,y)|}{|x-y|^\alpha} + \sup_{t,s\,\in]0,T[,\, t\neq s,\, x\in\Omega} \frac{|v(t,x)-v(s,x)|}{|t-s|^{\frac{\alpha}{2}}}.$$

Classical parabolic interior Schauder estimates, (see [29, Section 8.11]), say that for every $\varepsilon > 0$ and $\Omega_1 \subset\subset \Omega_2 \subset\subset \Omega$ with $\mathrm{dist}(\Omega_1, \Omega\setminus\Omega_2) > 0$, there exists a constant $C > 0$, depending on $N, \alpha, \nu, K, \varepsilon, \mathrm{dist}(\Omega_1, \Omega\setminus\Omega_2)$, such that for every function $u \in C^{1+\frac{\alpha}{2},2+\alpha}(]0,T[\times\Omega_2)$ one has

$$\|u\|_{C^{1+\frac{\alpha}{2},2+\alpha}(]\varepsilon,T[\times\Omega_1)} \leq C\Big(\|Lu\|_{C^{\frac{\alpha}{2},\alpha}(]0,T[\times\Omega_2)} + \|u\|_{C(]0,T[\times\Omega_2)}\Big),$$

where (we do not write explicitly the domain)

$$\|u\|_{1+\frac{\alpha}{2},2+\alpha} = \|u\|_{1,2} + [u]_{1+\frac{\alpha}{2},2+\alpha}$$

$$\|u\|_{1,2} = \|u\|_\infty + \|u_t\|_\infty + \|Du\|_\infty + \|D^2u\|_\infty,$$

$$[u]_{1+\frac{\alpha}{2},2+\alpha} = [u_t]_{\frac{\alpha}{2},\alpha} + [D^2u]_{\frac{\alpha}{2},\alpha}$$

(see [30, Theorem IV.10.1]). Here, we derive interior estimates only with respect to the time variable. More precisely, we set

$$\begin{aligned} Q &= (-\infty, T) \times \Omega, \\ Q_\varepsilon &= (\varepsilon, T) \times \Omega, \\ S_\varepsilon &= (\varepsilon, T) \times \partial\Omega. \end{aligned}$$

Then, under the stated assumptions on Ω and Γ, the following theorem holds.

Theorem C.1.1 *There exists $C > 0$ depending on $N, \alpha, \nu, K, \varepsilon, \Omega$ such that for every $u \in C^{1+\frac{\alpha}{2}, 2+\alpha}(Q_\varepsilon)$ with normal derivative $\frac{\partial u}{\partial \eta}$ equal to 0 on $\partial\Omega$, one has*

$$\|u\|_{C^{1+\frac{\alpha}{2},2+\alpha}(Q_{2\varepsilon})} \leq C\Big(\|Lu\|_{C^{\frac{\alpha}{2},\alpha}(Q_\varepsilon)} + \|u\|_{C(Q_\varepsilon)}\Big).$$

The proof of the above theorem relies on the classical technique used to prove interior estimates, namely, the introduction of a sequence of suitable cut-off functions. In this case, we choose such functions depending only on t.

Proof. We recall that, given a function $v \in C^{1+\frac{\alpha}{2},2+\alpha}(Q)$, the following interpolatory estimate holds (see [29, Lemma 10.2.1])

$$\|v_t\|_\infty + \|Dv\|_\infty + \|D^2 v\|_\infty + [Dv]_{\frac{\alpha}{2},\alpha} + [v]_{\frac{\alpha}{2},\alpha} \leq \theta \|v\|_{1+\frac{\alpha}{2},2+\alpha} + M\theta^{-\gamma}\|v\|_\infty, \tag{C.1.3}$$

where γ and M are positive constants and $\theta > 0$ is arbitrarily small. Such an estimate can be deduced from the analogous one in $\mathbb{R}^{N+1}$ by using suitable extension operators (which do exist thanks to the regularity of Ω). Moreover if v has normal derivative equal to zero on $\partial\Omega$ then

$$\|v\|_{C^{1+\frac{\alpha}{2},2+\alpha}(Q)} \leq C\Big(\|Lv\|_{C^{\frac{\alpha}{2},\alpha}(Q)} + \|v\|_{C(Q)}\Big), \tag{C.1.4}$$

with $C = C(\alpha, \nu, N, K, \Omega) > 0$. Let us introduce the sequences

$$t_n = \sum_{j=0}^{n} 2^{-j}, \qquad s_n = \varepsilon(3 - t_n).$$

We observe that (s_n) is decreasing with $s_0 = 2\varepsilon$, $s_\infty = \varepsilon$ and $s_n - s_{n+1} = \varepsilon 2^{-n-1}$. Moreover, let ψ_n be a sequence of functions in $C^\infty(\mathbb{R})$ such that $\psi_n(t) = 1$ for $t \in (s_n, T)$, $\operatorname{supp}\psi_n \subset (s_{n+1}, 2T)$, $0 \leq \psi \leq 1$ and

$$\|\psi_n'\|_\infty \leq L2^n, \quad \|\psi_n''\| \leq L4^n, \tag{C.1.5}$$

for some constant $L > 0$ depending also on ε. Hence, the function $\psi_n u$ is in $C^{1+\frac{\alpha}{2},2+\alpha}(Q)$ and

$$\frac{\partial(\psi_n u)}{\partial \eta} = \psi_n \frac{\partial u}{\partial \eta} = 0, \qquad \text{on } \partial\Omega.$$

Applying estimate (C.1.4) we obtain

$$\|\psi_n u\|_{C^{1+\frac{\alpha}{2},2+\alpha}(Q)} \leq C\Big(\|L(\psi_n u)\|_{C^{\frac{\alpha}{2},\alpha}(Q)} + \|\psi_n u\|_{C(Q)}\Big), \tag{C.1.6}$$

with $C > 0$ independent of n. One has $L(\psi_n u) = \psi_n Lu + \psi_n' u$. Then, from (C.1.5) it follows that

$$\begin{aligned} \|\psi_n Lu\|_{C^{\frac{\alpha}{2},\alpha}(Q)} &\leq \|Lu\|_{C^{\frac{\alpha}{2},\alpha}(Q_\varepsilon)} + \|Lu\|_{C(Q_{n+1})}\|\psi_n\|_{C^{\frac{\alpha}{2}}(I_{n+1})} \\ &\leq \|Lu\|_{C^{\frac{\alpha}{2},\alpha}(Q_\varepsilon)} + 2^n c(\varepsilon, K)\|u\|_{C^{1,2}(Q_{n+1})}, \\ &\leq \|Lu\|_{C^{\frac{\alpha}{2},\alpha}(Q_\varepsilon)} + 4^n c(\varepsilon, K)\|u\|_{C^{1,2}(Q_{n+1})}, \end{aligned} \tag{C.1.7}$$

where $I_{n+1} = (s_{n+1}, T)$ and $Q_{n+1} = I_{n+1} \times \Omega$. Analogously,

$$
\begin{aligned}
\text{(C.1.8)} \qquad \|\psi'_n u\|_{C^{\frac{\alpha}{2},\alpha}(Q)} &\leq \|\psi'_n\|_{C(I_{n+1})}\|u\|_{C^{\frac{\alpha}{2},\alpha}(Q_{n+1})} + \|\psi'_n\|_{C^{\frac{\alpha}{2}}(I_{n+1})}\|u\|_{C(Q_{n+1})} \\
&\leq 2^n L\|u\|_{C^{\frac{\alpha}{2},\alpha}(Q_{n+1})} + 4^n L\|u\|_{C(Q_{n+1})} \\
&\leq 4^n L\|u\|_{C^{\frac{\alpha}{2},\alpha}(Q_{n+1})}.
\end{aligned}
$$

Taking (C.1.7) and (C.1.8) into account, from (C.1.6) we infer (for a possibly different C)

$$
\begin{aligned}
\|\psi_n u\|_{C^{1+\frac{\alpha}{2},2+\alpha}(Q)} &\leq C\Big(\|Lu\|_{C^{\frac{\alpha}{2},\alpha}(Q_\varepsilon)} + \|u\|_{C(Q_\varepsilon)}\Big) \\
&\quad +4^n c(K,\varepsilon)\Big(\|u\|_{C^{1,2}(Q_{n+1})} + \|u\|_{C^{\frac{\alpha}{2},\alpha}(Q_{n+1})}\Big) \\
&\leq C\Big(\|Lu\|_{C^{\frac{\alpha}{2},\alpha}(Q_\varepsilon)} + \|u\|_{C(Q_\varepsilon)}\Big) \\
&\quad +4^n c(K,\varepsilon)\Big(\|\psi_{n+1}u\|_{C^{1,2}(Q)} + \|\psi_{n+1}u\|_{C^{\frac{\alpha}{2},\alpha}(Q)}\Big),
\end{aligned}
$$

where in the last inequality we have used the fact that $\psi_{n+1} = 1$ in Q_{n+1}. Using the interpolatory estimate (C.1.3) we find that for every $\theta > 0$

$$
\begin{aligned}
\|\psi_n u\|_{C^{1+\frac{\alpha}{2},2+\alpha}(Q)} &\leq C\Big(\|Lu\|_{C^{\frac{\alpha}{2},\alpha}(Q_\varepsilon)} + \|u\|_{C(Q_\varepsilon)}\Big) + 4^n c(K,\varepsilon)\theta\|\psi_{n+1}u\|_{C^{1+\frac{\alpha}{2},2+\alpha}(Q)} \\
&\quad +4^n C(K,\varepsilon)\theta^{-\gamma}\|\psi_n u\|_{C(Q)}.
\end{aligned}
$$

Let us consider $\xi = 4^n c(K,\varepsilon)\theta$, with ξ independent of n. Choosing a small θ we may assume that $\xi < 1$. Since $\theta^{-\gamma} = \left(\frac{\xi}{C(K,\varepsilon)}\right)^{-\gamma} 4^{n\gamma}$, the last estimate becomes

$$
\begin{aligned}
\|\psi_n u\|_{C^{1+\frac{\alpha}{2},2+\alpha}(Q)} &\leq C\Big(\|Lu\|_{C^{\frac{\alpha}{2},\alpha}(Q_\varepsilon)} + \|u\|_{C(Q_\varepsilon)}\Big) \\
&\quad +\xi\|\psi_{n+1}u\|_{C^{1+\frac{\alpha}{2},2+\alpha}(Q)} + c_1(K,\varepsilon,M,\gamma)4^{(\gamma+1)n}\|u\|_{C(Q)}.
\end{aligned}
$$

Taking, if necessary, a smaller ξ in order to have $4^{\gamma+1}\xi < 1$, by multiplying by ξ^n and summing from 0 to ∞ we obtain

$$
\begin{aligned}
\sum_{n=0}^{\infty} \xi^n \|\psi_n u\|_{C^{1+\frac{\alpha}{2},2+\alpha}(Q)} &\leq \frac{C}{1-\xi}\Big(\|Lu\|_{C^{\frac{\alpha}{2},\alpha}(Q_\varepsilon)} + \|u\|_{C(Q_\varepsilon)}\Big) \\
&\quad +\sum_{n=1}^{\infty} \xi^n \|\psi_n u\|_{C^{1+\frac{\alpha}{2},2+\alpha}(Q)} + C_2\|u\|_{C(Q)}.
\end{aligned}
$$

Hence

$$
\|\psi_0 u\|_{C^{1+\frac{\alpha}{2},2+\alpha}(Q)} \leq \overline{C}(\|Lu\|_{C^{\frac{\alpha}{2},\alpha}(Q_\varepsilon)} + \|u\|_{C(Q_\varepsilon)}),
$$

with $\overline{C} = \overline{C}(\varepsilon, K, N, \nu, \alpha, \Omega)$. Since $\psi_0 = 1$ in $Q_{2\varepsilon}$, the statement follows. □

C.2 An L^p elliptic estimate

Let Γ be the operator defined in (C.1.1). Unlike the previous section, here it is sufficient to assume that the coefficients a_{ij} are uniformly continuous and bounded in Ω and that b_i, c belong to $L^\infty(\Omega)$, with Ω bounded open subset of $\mathbb{R}^N$ of class C^2. We also assume the ellipticity condition (C.1.2).

We present interior elliptic estimates, where the involved subdomains are not assumed to have compact closure in Ω, but are allowed to have a part of the boundary overlapped on $\partial\Omega$. Neumann boundary conditions are prescribed only on this part.

Theorem C.2.1 *Let $1 < p < \infty$ and let Ω_0 and Ω_1 be open subsets contained in Ω such that $\partial\Omega_0 \cap \partial\Omega \neq \emptyset$, $\partial\Omega_1 \cap \partial\Omega \neq \emptyset$ and $\operatorname{dist}(\Omega_0, \Omega \setminus \Omega_1) > 0$. Assume also that Ω_1 is of class C^2. Then there exists a constant $C > 0$, depending on $p, N, \nu, \Omega_0, \Omega_1$, the L^∞ norms of all the coefficients and the modulus of continuity of a_{ij}, such that for every function $u \in W^{2,p}(\Omega_1)$ with $\frac{\partial u}{\partial \eta} = 0$ on $\partial\Omega_1 \cap \partial\Omega$, the estimate*

$$\|u\|_{W^{2,p}(\Omega_0)} \leq C(\|\Gamma u\|_{L^p(\Omega_1)} + \|u\|_{L^p(\Omega_1)})$$

holds.

Proof. Let us consider an increasing sequence of domains Ω_n such that $\Omega_\infty = \Omega_1$ and $\operatorname{dist}(\Omega_n, \Omega \setminus \Omega_{n+1}) = O(2^{-n})$. Let θ_n be a function in $C^\infty(\mathbb{R}^N)$ such that $\theta_n = 1$ in Ω_n, $\theta_n = 0$ in an open set containing $\Omega \setminus \Omega_{n+1}$, $0 \leq \theta \leq 1$, $\frac{\partial \theta}{\partial \eta} = 0$ on $\partial\Omega$. We note that in the case where Ω is the halfspace $\{x_N > 0\}$, it is sufficient to take θ_n as an even reflection with respect to x_N in order to have $\frac{\theta_n}{\partial \eta} = 0$ when $x_N = 0$. For a regular bounded set, one can constructed such a function using the first step and local coordinates. Moreover, the first and second order derivatives of the functions θ_n satisfy the estimates

$$\|D\theta_n\|_\infty \leq L2^n, \qquad \|D^2\theta_n\|_\infty \leq L4^n.$$

Since $\theta_n u \in W^{2,p}(\Omega_1)$ and

$$\frac{\partial(\theta_n u)}{\partial \eta} = \frac{\partial \theta_n}{\partial \eta} u + \frac{\partial u}{\partial \eta}\theta_n = 0, \quad \text{on } \partial\Omega_1$$

we may apply the classical global L^p estimate (see [32, Theorem 3.11(iii)]) and we find that

$$\|\theta_n u\|_{W^{2,p}(\Omega_1)} \leq C(\|\Gamma(\theta_n u)\|_{L^p(\Omega_1)} + \|\theta_n u\|_{L^p(\Omega_1)}). \tag{C.2.1}$$

Now, it is readily seen that $\Gamma(\theta_n u) = \theta_n \Gamma u + B_n u$, where B_n is a first order differential operator, whose coefficients involve the coefficients of Γ, θ_n, $D\theta_n$ and $D^2\theta_n$. Therefore

$$\begin{aligned}\|B_n u\|_{L^p(\Omega_1)} &\leq 4^n C\|u\|_{W^{1,p}(\Omega_{n+1})} \leq 4^n C\|\theta_{n+1}u\|_{W^{1,p}(\Omega_1)} \\ &\leq 4^n C(\varepsilon\|\theta_{n+1}u\|_{W^{2,p}(\Omega_1)} + \varepsilon^{-1}\|\theta_{n+1}u\|_{L^p(\Omega_1)}),\end{aligned}$$

where we have used the interpolatory estimate $\|v\|_{W^{1,p}(\Omega_1)} \leq \varepsilon\|v\|_{W^{2,p}(\Omega_1)} + c\varepsilon^{-1}\|v\|_{L^p(\Omega_1)}$, which holds for every function $v \in W^{2,p}(\Omega_1)$ and every $\varepsilon > 0$.
Besides, we have $\|\theta_n \Gamma u\|_{L^p(\Omega_1)} \leq \|\Gamma u\|_{L^p(\Omega_1)}$. From (C.2.1) it follows that

$$\begin{aligned}\|\theta_n u\|_{W^{2,p}(\Omega_1)} \leq\ & C(\|\Gamma u\|_{L^p(\Omega_1)} + 4^n\varepsilon\|\theta_{n+1}u\|_{W^{2,p}(\Omega_1)} \\ & + 4^n\varepsilon^{-1}\|\theta_{n+1}u\|_{L^p(\Omega_1)} + \|u\|_{L^p(\Omega_1)}).\end{aligned}$$

Set $\xi = C4^n\varepsilon$. We need ξ independent of n. Then $\varepsilon^{-1} = (\xi/C)^{-1}4^n$ and the last inequality becomes

$$\|\theta_n u\|_{W^{2,p}(\Omega_1)} \leq C(\|\Gamma u\|_{L^p(\Omega_1)} + \|u\|_{L^p(\Omega_1)}) + \xi\|\theta_{n+1}u\|_{W^{2,p}(\Omega_1)} + C_1 4^{2n}\|\theta_{n+1}u\|_{L^p(\Omega_1)}.$$

Choose ε in such a way that $\xi < 1$ and $\xi 4^2 < 1$. Then multiplying by ξ^n and summing on n from 0 to $+\infty$ we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} \xi^n \|\theta_n u\|_{W^{2,p}(\Omega_1)} \leq\ & \frac{C}{1-\xi}(\|\Gamma u\|_{L^p(\Omega_1)} + \|u\|_{L^p(\Omega_1)}) \\ & + \sum_{n=1}^{\infty} \xi^n \|\theta_n u\|_{W^{2,p}(\Omega_1)} + C_2\|u\|_{L^p(\Omega_1)},\end{aligned}$$

which yields

$$\|\theta_0 u\|_{W^{2,p}(\Omega_1)} \leq C(\|\Gamma u\|_{L^p(\Omega_1)} + \|u\|_{L^p(\Omega_1)}).$$

Since $\theta_0 = 1$ in Ω_0 we get

$$\|u\|_{W^{2,p}(\Omega_0)} \leq C(\|\Gamma u\|_{L^p(\Omega_1)} + \|u\|_{L^p(\Omega_1)}),$$

and the proof is concluded. □